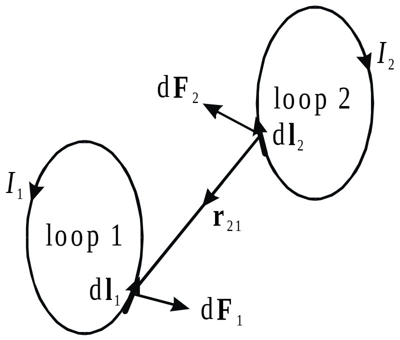
VI MAGNETIC EFFECTS OF CURRENTS

6.1 Ampère's investigations

It was Ampère who first established and quantified the force that occurs between two current-carrying conductors. This is not quite as simple as the Coulomb law for the force between charges as there are more directions to consider. As well as the separation of the force-producing objects (the current elements), in this case the directions of the two currents are important.



Here $d\mathbf{F}_1$ is the contribution to the force on the element $d\mathbf{l}_1$ of loop 1 from the element $d\mathbf{l}_2$ of loop 2.

If θ_1 is the angle between $d\mathbf{l}_1$ and the normal to the plane spanned by $\mathbf{r}_{21}$ and $d\mathbf{l}_2$, and θ_2 is the angle between $d\mathbf{l}_2$ and $\mathbf{r}_{21}$, then the conclusion of Ampère was that the magnitude of the force $d\mathbf{F}_1$ behaved as:

$$d\mathbf{F}_1 \propto \frac{I_1 I_2 dl_1 dl_2 \sin\theta_1 \sin\theta_2}{r_{12}^2}$$

The observation is that the magnetic force between two current-carrying elements $d\mathbf{l}_1$ and $d\mathbf{l}_2$ varies inversely as the square of their separation – once again we have an *inverse square law*.

Introducing the constant of proportionality $\mu_0/4\pi$, and taking account of the direction of the force by writing things in vector form we have

$$d\mathbf{F}_1 = \frac{\mu_0}{4\pi} \frac{I_1 I_2 d\mathbf{l}_1 \times (d\mathbf{l}_2 \times \mathbf{r}_{21})}{r_{12}^3} \qquad (6.1)$$

and the expression for $d\mathbf{F}_2$ is obtained by swapping the 1 and 2 in the expression.

The constant μ_0 is referred to as the *permeability of free space*. Its value is *defined* to be

$$\mu_0 = 4\pi \times 10^{-7} \text{ newtons / ampere}^2 \quad \text{or} \quad \text{henries / metre.} \qquad (6.2)$$

It might seem surprising that the value of μ_0 is defined rather than being determined experimentally. The reason is that the magnetic force formula is actually used as the *definition* of the ampère, the unit in terms of which I is measured. (And in fact the coulomb then follows as the charge transferred when a current of one ampère flows for one second). The numerical value chosen for μ_0 allows the traditional electrical units (actually predating the metric system) to be incorporated into the SI scheme.

⌈You might think that a more straightforward way of introducing magnetic forces would be to start immediately with moving point charges instead of elements of current-carrying conductors. Indeed the results of Ampère quoted above could be re-expressed by saying that a charge Q_2 moving with velocity $\mathbf{v}_2$ will exert a force $\mathbf{F}_1$ on a charge Q_1 moving with velocity $\mathbf{v}_1$ given by

$$\mathbf{F}_1 = \frac{\mu_0}{4\pi}\frac{Q_1 Q_2}{r_{12}^3}\mathbf{v}_1 \times (\mathbf{v}_2 \times \mathbf{r}_{12}).$$

This force (which only occurs when both charges are moving) is in *addition* to the Coulomb force, which applies even when the charges are stationary.

There is a distinction between this approach and that of Ampère. Ampère's results apply to elements of *conductors* carrying currents. These are electrically neutral since the background positive stationary charge cancels the negative charge of the carriers. In that case there is no Coulomb force and only the magnetic force is observed.

Although this approach might seem more straightforward, we have preferred to follow the historical argumentation of Ampère.

It is also of interest to note that the above equation for the (extra) force between moving charges could be derived from the Coulomb force using special relativity on the assumption that electric charge is invariant. However, again we have opted to follow the historical route whereby the Coulomb force together with the Ampère formula may be regarded as separate experimental observations leading, ultimately, to special relativity. ⌋

6.2 Magnetic field

In an analogous manner to our introduction of the electric field, we shall break the symmetry of the magnetic force law and interpret it as one current *producing* a magnetic field **B** and the other current *responding* by experiencing a force in the field. Accordingly, we split Equation (6.1) as

$$\mathrm{d}\mathbf{B}_2 = \frac{\mu_0}{4\pi} I_2 \frac{\mathrm{d}\mathbf{l}_2 \times \mathbf{r}_{21}}{r_{21}^3} \tag{6.3}$$

and

$$\mathrm{d}\mathbf{F}_1 = I_1 \mathrm{d}\mathbf{l}_1 \times \mathrm{d}\mathbf{B}_2 \tag{6.4}$$

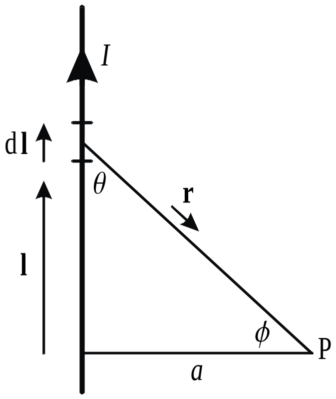
where $\mathbf{B}_2$ is the magnetic field produced *by* loop 2.

Thus Equation (6.3) describes the active aspect of a current element, while Equation (6.4) describes the passive aspect.

Active aspect of current	**Passive aspect of current**
An electric current I produces a magnetic field **B**.	An electric current i responds to a magnetic field **B**.
$\mathrm{d}\mathbf{B} = \frac{\mu_0}{4\pi} I \frac{\mathrm{d}\mathbf{l}_2 \times \mathbf{r}_{21}}{r_{21}^3}$	$\mathrm{d}\mathbf{F} = i\,\mathrm{d}\mathbf{l}_1 \times \mathrm{d}\mathbf{B}$
I is the **source** of **B**.	i **responds** to **B**.

6.3 Field of a long wire

We shall calculate the magnetic field produced by a long straight wire carrying a current of I amperes. Let us first consider the *direction* of the magnetic field at the point P. From Equation (6.3) we see that the direction information is contained in $\mathrm{d}\mathbf{l}\times\mathbf{r}$, which points into the page. Thus the lines of **B** form concentric circles around the wire. Since all contributions to the magnetic field at P point in the same direction, only this component need be considered. We then find the value of **B** by integrating the contributions from along the length of the wire.



The contribution d**B**, from the element d**l** of the wire is given, from Equation (6.3), by

$$\mathrm{d}\mathbf{B} = \frac{\mu_0}{4\pi} I \frac{\mathrm{d}\mathbf{l}\times\mathbf{r}}{r^3} = \frac{\mu_0}{4\pi} I \frac{\mathrm{d}\mathbf{l}\sin\theta}{r^2}$$

It is convenient to express everything in terms of the angle ϕ and the perpendicular distance a:

$$l = a\tan\phi$$
$$\mathrm{d}l = a\sec^2\phi\,\mathrm{d}\phi$$
$$r = a/\cos\phi\,,$$

from which we obtain

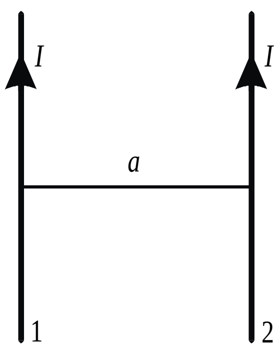
$$B = \frac{\mu_0}{4\pi} I \int_{-\pi/2}^{\pi/2} \frac{a\sec^2\phi\,\mathrm{d}\phi\cos\phi\cos^2\phi}{a^2}$$
$$= \frac{\mu_0}{4\pi}\frac{I}{a}\int_{-\pi/2}^{\pi/2}\cos\phi\,\mathrm{d}\phi$$

and on performing the integral

$$B = \frac{\mu_0 I}{2\pi a}\,. \qquad (6.5)$$

6.4 Force between two long parallel wires

The magnetic force between two long parallel wires each carrying a current I could be calculated directly from the force expression, Equation (6.1). However, since this case has a particularly simple geometry, it is more convenient to obtain the force from the magnetic field calculated in the previous section. The magnetic field B at wire 2 due to wire 1 is given, from Equation (6.5) by



$$B = \frac{\mu_0 I}{2\pi a}\,.$$

Thus the force on unit length of wire 2 is given by

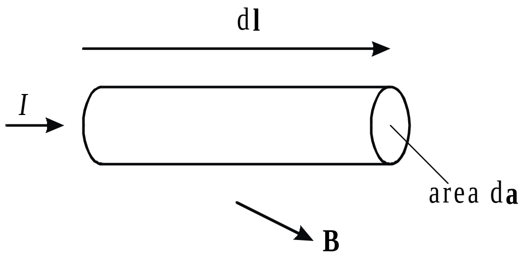
$$F = \frac{\mu_0 I^2}{2\pi a}\,. \qquad (6.6)$$

This expression is the basis of the definition of the Ampère. Recall that the numerical value of μ_0 is $4\pi\times10^{-7}$. The force between two long wires, one metre apart, carrying one Ampère, will be 2×10^{-7} Newtons per metre of length; this defines the Ampère.

We now consider the *direction* of the force. From the previous section we saw that the **B** field at 2 pointed into the page. Now the direction of the force goes as **l** × **B**, which points to the *left*. Thus the force between wires carrying current in the same direction pushes them *together*. One concludes: *like currents attract*, *unlike currents repel*. This is the converse of the rule for charges.

6.5 The Lorentz force

Thus far, in treating magnetic effects, we have considered the force on an electric current. Now we shall examine the force on a single electric charge moving in a magnetic field.



The force on the element of wire due to the applied magnetic field **B** is, from Equation (6.4)

$$\mathrm{d}\mathbf{F} = I\,\mathrm{d}\mathbf{l}\times\mathbf{B}\,.$$

Now we can write the current I as **j . d a** or, from the expression for **j**, Nq**v . da**. The force d**F** is then

$$\mathrm{d}\mathbf{F} \;=\; Nq(\mathbf{v}\,.\,\mathrm{d}\mathbf{a})\,\mathrm{d}\mathbf{l}\times\mathbf{B}.$$

Note that **v** and d**l** are parallel, so that they may swapped in the above equation, giving

$$\mathrm{d}\mathbf{F} \;=\; Nq(\mathrm{d}\mathbf{l}\,.\,\mathrm{d}\mathbf{a})\,\mathbf{v}\times\mathbf{B}.$$

But d**l . d a** is the volume of the element of wire, and N is the number of charges *per unit volume* and since q is the magnitude of each charge, it follows that Nq(d**l .** d**a**) is the total charge Q in the element. Thus we conclude that a charge Q moving with a velocity **v** in a magnetic field **B** experiences a force **F** given by

$$\mathbf{F} \;=\; Q\,\mathbf{v}\times\mathbf{B}. \qquad (6.7)$$

We know that a charge in an electric field **E** experiences a force

$$\mathbf{F} \;=\; Q\,\mathbf{E},$$

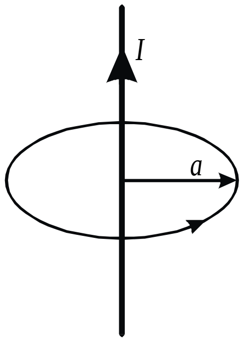
and now we have seen that if the charge is moving and if there is a magnetic field, then there will be a magnetic force given by Equation (6.7). In the presence of both an electric field and a magnetic field, it follows that a charge q moving with a velocity **v** will experience a total force given by

$$\mathbf{F} \;=\; q\{\mathbf{E}+\mathbf{v}\times\mathbf{B}\}. \qquad (6.8)$$

This force is called the Lorentz force.

6.6 Ampère's law

Ampère's law deals with the line integral of **B** around a closed loop. There are essentially two reasons for being interested in this. Firstly it will provide a means for calculating magnetic fields, although only in cases of high symmetry. Secondly, from the line integral of **B** we will be able to evaluate its curl: part of our programme of obtaining Maxwell's equations.



Let us evaluate the line integral of **B** around the indicated circular loop. In Equation (6.5) we saw that the magnitude of the field a distance *a* from a wire carrying a current *I* is

$$B = \frac{\mu_0 I}{2\pi a}.$$

And we saw that the *direction* of **B** is along the loop. Since **B** is constant around the circular loop, whose length is $2\pi a$, the line integral is given by

$$\oint_{\text{closed loop}} \mathbf{B.}\mathrm{d}\mathbf{l} = \mu_0 I\ . \qquad (6.9)$$

Observe that the *a*, the radius of the loop, has vanished. This indicates that the radius of the circle for evaluating the line integral is not important. But more significant than this, we can deform the contour of the integral just as we did in evaluating the curl of **E** from which we conclude that the result of Equation (6.9) holds for a closed loop of arbitrary shape.

The above result is known as Ampère's law, which may be stated as: *The line integral of* **B** *around an arbitrary closed loop is given by* μ_0 *times the total current through the enclosed area*.

6.7 The curl of B

The curl of **B** follows immediately from Equation (6.9) above, using the definition of the curl:

$$\mathrm{curl}\,\mathbf{B} = \frac{1}{\text{area}} \oint_{\text{closed loop}} \mathbf{B.}\mathrm{d}\mathbf{l}$$

as the area shrinks to zero. From Equation (6.9) the curl of **B** is thus given by $\mu_0 I$/area, and the direction is along the current flow. Now let us consider a distributed distribution of current, described by a *current density* **j**. We know that the total current threading a (perpendicularly oriented) loop of area *a* is given by *ja*, from which it follows that curl**B** is given by

$$\mathrm{curl}\mathbf{B} \ = \ \mu_0\mathbf{j}. \qquad (6.10)$$

This is almost a Maxwell equation. It was Maxwell's genius that led him to realise that this equation was incomplete, and he made a crucial modification, which we will encounter shortly.

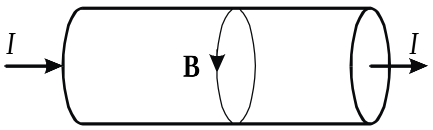
6.8 The divergence of B

There is a practical aspect and a philosophical aspect to the question of the divergence of **B**. The magnetic flux Φ penetrating a surface is defined as the integral of the normal component of **B** over the surface:

$$\Phi = \iint_{\text{surface}} \mathbf{B}.d\mathbf{a} \,. \tag{6.11}$$

This definition is similar to that for the electric flux Φ_E in Equation (2.6). The magnetic flux does not need a subscript; whenever Φ is written it is understood that it refers to magnetic flux.

The divergence of **B** is related to the flux of **B** through a closed surface enclosing a given volume. It is convenient to choose a shape that reflects the symmetry of our system. So let us take a cylinder around a straight wire carrying a current *I*. Since the lines of **B** form concentric circles, **B** is normal to the enclosing surface everywhere: on the curved side *and* on the ends. Thus there is no flux of **B** through a closed surface. This argument clearly generalises to arbitrary geometry.



From the definition of the divergence it then follows directly that the divergence of **B** (the flux through a closed surface divided by the enclosed volume) is zero:

$$\text{div}\mathbf{B} = 0. \tag{6.12}$$

The philosophical question relates to what the *source* of a magnetic field might be. The conclusion of Equation (6.12) above is that for magnetic fields *produced by moving electric charges* the divergence is zero. But are there any other sources of magnetic fields? In particular, is it possible to have a *magnetic monopole*, a magnetic charge: an isolated north or south pole. In the early days of electromagnetism, there was a branch called *magnetostatics* which parallelled electrostatics. In terms of this, if ρ_m is the density of magnetic charge then there would be an analogue of the div**E** equation stating:

$$\text{div}\mathbf{B} = \mu_0\rho_m \,.$$

The existence of magnetic monopoles is still an open question. They are required by the Grand Unified theories, but thus far there has been no *convincing* evidence of their existence. There is also a theory due to Dirac whereby the discrete nature of electric charge is connected to the existence of magnetic charge. If e is the fundamental entity of electric charge and g is the fundamental entity of magnetic charge then Dirac found the relation

$$eg = h$$

where h is Planck's constant.

The equation div**B** = 0, which is one of the Maxwell equations, is often interpreted as being equivalent to the statement that magnetic monopoles do not exist. Of more practical importance, however, is the point that lines of **B** have no beginning or end; they close on themselves.

When you have completed this chapter you should:

- know the physical phenomena contained in Ampère's formula;
- know the units of the physical quantities in Ampère's formula;
- be familiar with idea of the magnetic field;
- appreciate the active and passive aspects of electric currents;
- be able to calculate magnetic field of a long straight wire and an arrangement of currents;
- be able to calculate the force between two long parallel wires and relate this to the definition of the ampere;
- be able to calculate force on a moving charge in a given magnetic field;
- calculate the force on a moving charge in an arbitrary combination of electric and magnetic fields;
- be familiar with Ampère's law for the line integral of **B** around a current-carrying wire.
- understand how Ampère's law relates to the curl of **B**;
- be familiar with concept of magnetic flux and the connection with the divergence of **B**;
- understand that div**B** = 0 implies there are no magnetic monopoles.