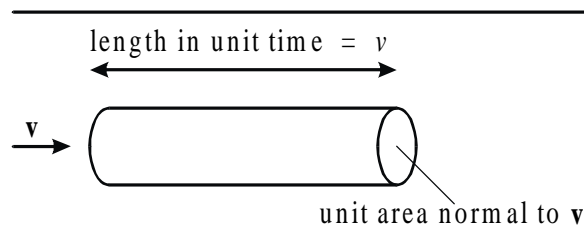# V STEADY CURRENTS

5.1 Current density vector

If N is the number of charges (of charge q) per unit volume moving with (mean) velocity v, then the charge crossing unit area per second is given by Nqv. This leads us to the definition of the *current density vector* **j**:

$$\mathbf{j} = Nq\mathbf{v}. \tag{5.1}$$

length in unit time $= v$

v

unit area normal to **v**

In other words **j** is the current crossing unit area. In the general case the charge crossing unit area per second is given by
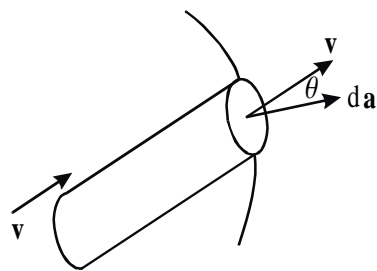
$$Nqv\,\mathrm{d}a\cos\theta$$
$$= \mathbf{j}.\mathrm{d}\mathbf{a}.$$

The total current I flowing is found by integrating over the surface:

$$I = \iint\limits_{surface} \mathbf{j}.\mathrm{d}\mathbf{a} \tag{5.2}$$

and of course the units of j and I are given by:

$$[\,j\,] = \text{amps per square metre}$$
$$[\,I\,] = \text{amps (coulombs per second)}.$$

5.2 Equation of continuity

It is an experimentally observed fact that electric charge is conserved. This means that if we

closed volume   bounding surface

consider a closed volume, then the rate of decrease of charge in the volume must be balanced by the rate of flow of charge across the bounding surface. This may be expressed mathematically by
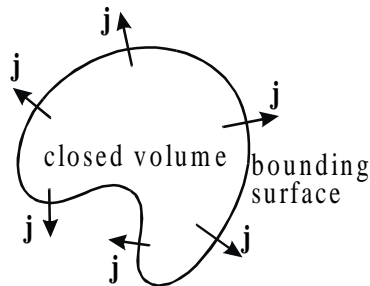
$$-\frac{\mathrm{d}}{\mathrm{d}t} \iiint\limits_{volume} \rho\,\mathrm{d}v = \oiint\limits_{\substack{bounding\\surface}} \mathbf{j}.\mathrm{d}\mathbf{a}$$

$$= \iiint\limits_{volume} \mathrm{div}\,\mathbf{j}\,\mathrm{d}v$$

where the right hand side has been transformed using Gauss's theorem (– our definition of div). Now since this result must be true for all volumes, it follows that we may remove the volume integral, to give

$$-\frac{\partial \rho}{\partial t} = \mathrm{div}\, \mathbf{j} \qquad (5.3)$$

which is known as the *Equation of continuity*. This is another one of those equations which has a very broad area of applicability. Any quantity that can flow and is conserved will obey a similar equation.

5.3 Conductivity

It is an experimentally observed fact that when a current is flowing in a (linear isotropic homogeneous) conductor, the current density **j** is proportional to the electric field **E**. This observation is equivalent to Ohm's law, as we shall see in the next section. Introducing the constant of proportionality σ, we may write

$$\mathbf{j} \;=\; \sigma\, \mathbf{E} \qquad (5.4)$$

where σ is the *electrical conductivity*. For a material which is not *linear*, higher powers of **E** will appear; for a material which is not *isotropic*, the conductivity σ is a tensor (Section 5.8); and for a material which is not *homogeneous*, the conductivity will vary with position.

Please don't confuse the conductivity with the other use of σ, namely the surface charge density. Unfortunately both σ and ρ are used in two contexts in electromagnetism – watch out!

At first glance it is a little difficult to understand the origin of the phenomenon of resistivity. After all, one would expect the *acceleration* of the charges to be proportional to the force or electric field; this is what Newton's laws tell us. However the physical content of Equation (5.4) is that the *velocity* of the charges is proportional to the electric field. The best mechanical analogy here is the *terminal velocity* of an object moving in a viscous fluid.

The fact is that in a conductor the charges are not completely free to move. They keep on bumping into things: other charges, lattice vibrations, crystal imperfections, impurities etc. We can model the effect of these various *scattering* events by saying that the charges move freely for a given time τ, after which the motion is randomised by a collision. Thus in a field **E** a charge starts from zero velocity and it accelerates for a time τ. Then there is a collision, the velocity becomes zero and the process starts all over again. The maximum velocity is $a\tau$ where a is the acceleration:

$$a \;=\; qE/m,$$

where m is the mass of the charge. The mean velocity is half the maximum velocity, giving

$$\langle v \rangle = \frac{qE\tau}{2m}$$

and since the current density is

$$j = Nq\langle v \rangle$$

we may write the (mean) value of j as:

$$j = \frac{Nq^2\tau}{2m}E \tag{5.5}$$

so that we find (according to this very simple model) that j is proportional to E and that the conductivity is given by
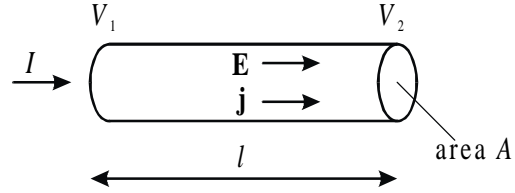
$$\sigma = \frac{Nq^2\tau}{2m} \tag{5.6}$$

This expression shows, within the limitations of this simple model, how the conductivity depends on the concentration of charge carriers, their electric charge, the collision time and the mass of the carriers.

This simple model of the mechanism for resistivity is sometimes referred to as the *Drude model*.

5.4 Ohm's law

If we now consider a uniform current density **j** in a conductor, then the current I is given by

$$I = jA.$$

But we know that the current density j is related to the electric field E through Equation (5.4) above:

$$j = \sigma E,$$

so that we may write I as

$$I = \sigma E A.$$

Observe that the two ends of the conductor are indicated as having potentials V_1 and V_2. The electric field may be expressed in terms of the difference between the potentials V:

$$E = V/l,$$

so that

$$I = \frac{\sigma A}{l}V \, . \tag{5.7}$$

In other words, we see that the current flowing in a conductor is proportional to the potential difference between its ends – Ohm's law. We denote the constant of proportionality by $1/R$, where R is the *resistance*

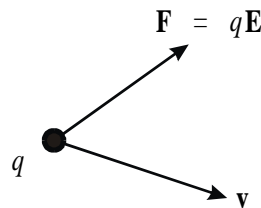$$R = \frac{\sigma A}{l} \tag{5.8}$$

which is measured in ohms.

The reciprocal of the conductivity σ is called the *resistivity*, denoted by ρ. In terms of the resistivity, the resistance of a conductor is given by

$$R = \rho \frac{l}{A} \, .$$

5.5 Power dissipation

Consider an electric charge q which is moving with velocity $\mathbf{v}$ in an electric field $\mathbf{E}$. The force on the charge is $q\mathbf{E}$ so that the rate of doing work *by* the field is

$$q\mathbf{E.v} \ .$$

Then with N charges per unit volume, the power dissipated per unit volume is:

$$
\begin{aligned}
\text{power per unit volume} \ &= \ Nq\mathbf{E.v} \\
&= \ \mathbf{j.E} \ .
\end{aligned}
\tag{5.9}
$$

Let us now apply these arguments to a cylindrical conductor (or wire!) of length l and cross section area A. The total power dissipated P is given by the power per unit volume, $\mathbf{j.E}$ multiplied by the volume:

$$P \ = \ Al\,\mathbf{j.E}$$

but since the current I is given by jA, and $E = V/l$, we obtain for the power dissipated:

$$P \ = \ I\,V. \tag{5.10}$$

The units of P are watts or joules per second.

5.6 Kirchhoff's laws

Kirchhoff's laws follow from the application of the above-established results for electrostatics to voltages and currents in electric circuits. An important idea here is that of the *steady state*. A steady state involving flowing currents is one for which there is no charge build-up. In other words

$$\frac{\partial \rho}{\partial t} = 0 \ . \tag{5.11}$$

This is the condition for the steady state.

5.6.1 Current law

This law follows from the application of the continuity equation, Equation (5.3). However, since we know that $\partial \rho / \partial t = 0$ in the steady state, the equation of continuity gives:

$$\text{div } \mathbf{j} \ = \ 0$$

or, writing this in integral form:

$$\oiint_{\substack{\text{closed} \\ \text{surface}}} \mathbf{j.da} = 0$$

This is telling us that the total current leaving a point in a circuit (called a *node*), is zero.
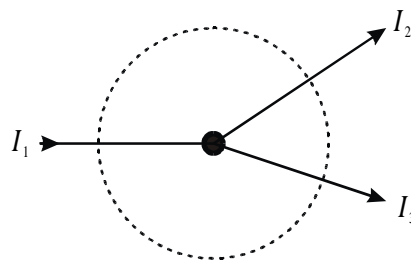
Applying this to the region enclosed by the dotted line, the implication is that:

$$I_1 = I_2 + I_3 \tag{5.12}$$

Obviously it is important to take account of the direction of the current. This result is known as Kirchhoff's *current* law.

5.6.2 Voltage law

Kirchhoff's voltage law follows directly from the fact that the **E** field is conservative. Recall that we saw the work done in moving a charge around a closed loop was zero, and that this led us to the concept of the electric potential *V*, a single-valued function of position. The change in *V* in traversing a closed circuit is thus zero, and that is the essence of Kirchhoff's voltage law.

As an exercise in manipulating vector expressions we shall derive the result in reverse order starting from the curl expression for **E**, namely curl**E** = 0. In integral form this becomes

$$\oint_{closed\ loop} \mathbf{E}.\mathbf{dl} = 0$$

But we can write **E** as −grad*V*, so that

$$\oint_{closed\ loop} \mathrm{grad}V.\mathbf{dl} = 0 \,.$$

However we have encountered an expression like grad*V*.**dl** before. In Equation (3.24) we saw that this gives the small change in a quantity *V* when one moves through a displacement d**l**, that is:

$$\mathrm{grad}V.\mathbf{dl} \;=\; \mathrm{d}V.$$

Thus our integral expression, following from curl**E** = 0, gives us

$$\oint_{closed\ loop} \mathrm{d}V = 0 \tag{5.13}$$

or, the change in *V* in going around a closed loop is zero.

In a circuit *V* can change for a number of different reasons. If a current is flowing across a resistive component then there will be a difference of potential between its terminals. This may be referred to as a *voltage drop*. Considering an electric *cell*, or a *battery* of cells, the internal chemical processes result in a force on charges, called an *electromotive force*, abbreviated EMF. This will also cause a difference of potentials between the terminals of the cell or battery.
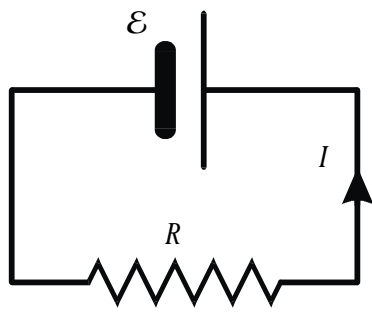
Kirchhoff's voltage law states that the sum of voltage drops and EMFs around a closed loop is zero. In this circuit this means that

$$\mathcal{E} - IR = 0.$$

The generalisation to more complex circuits is quite obvious.

5.7 Laplace's equation (in conductors)

When there are currents flowing in a conductor then the electric field (and the electric potential) within the conductor will be determined partly by the external conditions - which cause the currents to flow in the first place - and partly by the flowing charges themselves. We shall see that under certain conditions the electric potential in a conductor nevertheless still obeys Laplace's equation.

We are restricting our consideration to what we have already specified as the *steady state*. In that case, from the equation of continuity, we know that **j** has zero divergence:

$$\mathrm{div}\ \mathbf{j} = 0.$$

If we now also restrict consideration to linear, isotropic, homogeneous conductors then we can express **j** in terms of **E** as:

$$\mathbf{j} = \sigma\mathbf{E},$$

from which it follows that the divergence of **E** in this case is zero,

$$\mathrm{div}\mathbf{E} = 0.$$

And then, just as in the case of free space, since **E** may be written as

$$\mathbf{E} = -\mathrm{grad}\ V,$$

we have

$$\mathrm{div}\ \mathrm{grad}\ V = 0$$

or

$$\nabla^2 V = 0 \qquad\qquad (5.14)$$

Thus we see that Laplace's equation is obeyed in conductors *if* the above-mentioned set of conditions is satisfied.

It is important to appreciate the status of this result. The fact that Laplace's equation (or Poisson's equation) is obeyed in free space in the presence of fixed charges, is a fundamentally true result. But the derivation of Laplace's equation in conductors relies on the *phenomenological* result that the current density is proportional to the **E** field; this is only true for LIH systems. It is not a *fundamental* result.

5.8 Anisotropic media and tensors

In an anisotropic substance there are properties which depend on *orientation*. We are considering properties of *crystalline* materials; clearly amorphous and polycrystalline materials are isotropic. In the case of anisotropic conductivity one finds, in general, that the current density **j** is not along the same direction as the applied electric field **E**. We are, however, still considering the system to be linear so that the components of the current density and the electric field must be related by an expression of the form:

$$
\begin{array}{rcl}
j_x & = & \sigma_{xx}E_x \ + \ \sigma_{xy}E_y \ + \ \sigma_{xz}E_z \\[4pt]
j_y & = & \sigma_{yx}E_x \ + \ \sigma_{yy}E_y \ + \ \sigma_{yz}E_z \\[4pt]
j_z & = & \sigma_{zx}E_x \ + \ \sigma_{zy}E_y \ + \ \sigma_{zz}E_z
\end{array}
\qquad (5.15)
$$

where the $\sigma_{\alpha\beta}$ are the elements of the conductivity *tensor*. Since this tensor has two indices it is referred to as a tensor of *rank two*. The above relation may be written

$$\mathbf{j} = \sigma \, \mathbf{E}$$

Many other physical quantities can be represented as rank two tensors, such as electric susceptibility, magnetic susceptibility, elasticity, etc. etc. Other quantities may be represented by higher order tensors. A vector may be regarded as a rank one tensor; a scalar has rank zero.

A vector, or rank one tensor, is a physical quantity that may be represented by a column vector: a (3×1) matrix, once a coordinate system has been chosen. Thus, for example, we would write

$$\mathbf{j} \equiv \begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix}$$

If we change to a different rectangular cartesian coordinate system (by rotation) then the components of the vector **j** will be different. The components of **j** in the new frame are found by the matrix product

$$
\begin{pmatrix} j_{x'} \\ j_{y'} \\ j_{z'} \end{pmatrix} =
\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}
\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix}
$$

or

$$\mathbf{j}' = \mathsf{X}\,\mathbf{j}$$

where X is a real matrix whose elements are the direction cosines connecting the coordinate frames. The vector **E** will transform according to the same transformation rule

$$\mathbf{E}' = \mathsf{X}\,\mathbf{E}$$

and from this we may find how the conductivity tensor transforms.

In the transformed frame the relation between **j** and **E** is given by

$$\mathbf{j}' = \sigma' \mathbf{E}'$$

so using the expressions for the transformed $\mathbf{j}$ and $\mathbf{E}$, we write this as

$$\mathsf{X} \mathbf{j} = \sigma' \mathsf{X} \mathbf{E},$$

or

$$\mathbf{j} = \mathsf{X}^{-1} \sigma' \mathsf{X} \mathbf{E}.$$

But the proportionality between $\mathbf{j}$ and $\mathbf{E}$ gives the un-transformed conductivity tensor

$$\sigma = \mathsf{X}^{-1} \sigma' \mathsf{X}$$

so that σ transforms as

$$\sigma' = \mathsf{X} \sigma \mathsf{X}^{-1}.$$

A rank two tensor is a physical quantity that can be represented as a two-dimensional matrix (3×3) and which transforms according to the above rules. A scalar is a rank zero tensor; its value is independent of the coordinate system used. For this reason scalars, such as the electric potential V, are so convenient to work with.

When you have completed this chapter you should:

- appreciate that moving charges give rise to electric currents;

- understand that electric current density is a vector whereas electric current is a scalar;

- know how the equation of continuity applies to charge and current and appreciate the general nature of this equation – that it applies to any conserved quantity;

- know that many materials (conductors) have the property of conductivity and that they are observed to obey the phenomenological Ohm's law;

- understand the concepts of linear, isotropic and homogeneous;

- know of the quantities conductivity, resistivity, conductance and resistance;

- appreciate that the conductivity of a LIH system is a scalar;

- understand the Drude model for resistivity;

- realise that the work done when charges move in an electric field involves the dissipation of power and be able to calculate the power;

- understand how Kirchhoff's current law follows from the conservation of charge;

- understand how Kirchhoff's voltage law follows from the conservative nature of the electric field;

- appreciate how Laplace's equation applies to conductors, but that here it is a phenomenological result reliant upon Ohm's law;

- understand that in an anisotropic material the conductivity must be described by a (rank 2) tensor;

- have an elementary understanding of the properties of tensors and vectors and their transformation properties.