III VECTORS

3.1 Vector algebra

3.1.1 Vectors

The naïve definition of a vector (in three-dimensional space) is 'a quantity which has both magnitude and direction'. While adequate in many cases, this is not strictly mathematically correct. A mathematician might well define a vector as 'any quantity which obeys the laws of vectors'. This definition certainly has the merit that it can't be wrong! Note that this discussion is not simply irrelevant mathematical rigour. Rotation is a quantity which has both a magnitude (the angle of rotation) and a direction (the axis of rotation, as specified by the ubiquitous right-handed cork-screw). However rotations about different axes, as described by vectors in this way, are not additive. And furthermore the *order* or performing the rotations is important.

The conclusion, for our purposes, is that we shall specify a vector as a quantity which a) has magnitude and direction, and b) obeys the 'parallelogram' rule of addition. Clearly this second requirement implies the associativity condition:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} .$$

In this course we shall take the laws of vector addition and subtraction as known, and proceed directly to the various products of vectors.

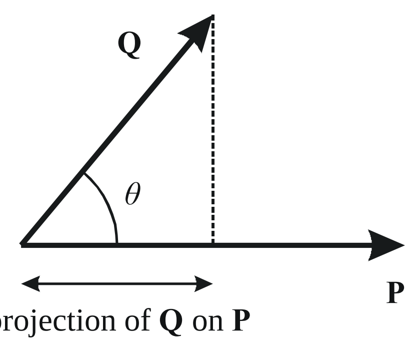
We can express a vector in terms of its components. In rectangular Cartesian co-ordinates we write

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}$$

where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ are the unit vectors in the x, y and z directions and A_x, A_y and A_z are the components of $\mathbf{A}$ in these directions.

3.1.2 The dot product

The dot product, otherwise known as the scalar product, of two vectors **P** and **Q** is written as **P.Q** and it is defined as the magnitude of one multiplied by the projection of the other upon the first.



projection of **Q** on **P**

Dot product of vectors **P** and **Q**. Since the projection of **Q** on **P** is given by $Q\cos\theta$, the dot product is

$$\mathbf{P.Q} = PQ\cos\theta. \qquad (3.1)$$

The scalar product of two perpendicular vectors is seen to be zero. Thus for the unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$:

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = 0 \qquad (3.2)$$

and

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1. \qquad (3.3)$$

If we express the vectors **P** and **Q** in rectangular co-ordinate form:

$$\mathbf{P} = P_x \hat{\mathbf{x}} + P_y \hat{\mathbf{y}} + P_z \hat{\mathbf{z}}, \quad \text{and} \quad \mathbf{Q} = Q_x \hat{\mathbf{x}} + Q_y \hat{\mathbf{y}} + Q_z \hat{\mathbf{z}},$$

then we find for the dot product, in co-ordinate form

$$\mathbf{P.Q} = P_x Q_x + P_y Q_y + P_z Q_z. \qquad (3.4)$$

Note that from this the dot product is seen to be *commutative*:

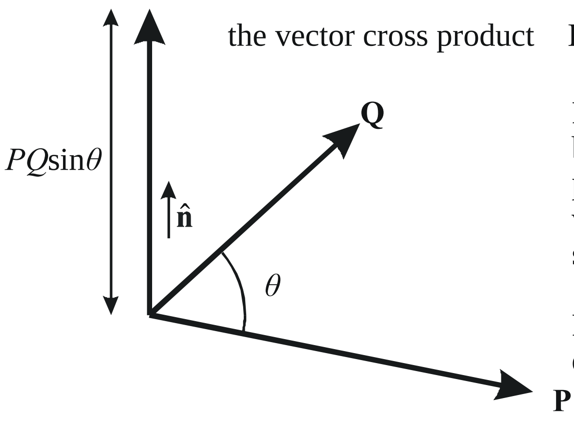
$$\mathbf{P.Q} = \mathbf{Q.P}. \qquad (3.5)$$

The dot product operates on two vectors to produce a scalar. An example of a dot product is the work W done when moving a displacement **r** in a constant force **F**:

$$W = \mathbf{r.F}.$$

3.1.3 The vector cross product

The vector cross product of two vectors **P** and **Q** is written as **P** × **Q** and it is a vector perpendicular to both **P** and **Q** in the direction of a (right handed) screw turning from **P** to **Q**. The magnitude is equal to the product of the length of the vectors multiplied by the sine of the angle between them.



the vector cross product $\quad \mathbf{P} \times \mathbf{Q} = PQ\sin\theta. \qquad (3.6)$

Physically the vector cross product may be interpreted as the area of a parallelogram of sides **P** and **Q**. This vector 'area' is pointing normal to the surface.

Recall we used the idea of vector area in discussing electric flux in Section 2.4.

From the definition of the cross product we see that the order of multiplication is important. The cross product is not commutative; it is *anticommutative*:

$$\mathbf{P} \times \mathbf{Q} = -\mathbf{Q} \times \mathbf{P}. \qquad (3.7)$$

From this we see that the various products of the unit vectors in rectangular Cartesian co-ordinates are

$$\hat{\mathbf{x}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0 \tag{3.8}$$

and

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} = -\hat{\mathbf{y}} \times \hat{\mathbf{x}} \tag{3.9}$$

and cyclic permutations thereof.

If **P** and **Q** are expressed in terms of their Cartesian co-ordinates then the cross product is expressed as

$$\begin{aligned}\mathbf{P} \times \mathbf{Q} &= \left(P_x\hat{\mathbf{x}} + P_y\hat{\mathbf{y}} + P_z\hat{\mathbf{z}}\right) \times \left(Q_x\hat{\mathbf{x}} + Q_y\hat{\mathbf{y}} + Q_z\hat{\mathbf{z}}\right) \\ &= \hat{\mathbf{x}}\left(P_yQ_z - P_zQ_y\right) + \hat{\mathbf{y}}\left(P_zQ_x - P_xQ_z\right) + \hat{\mathbf{z}}\left(P_xQ_y - P_yQ_x\right).\end{aligned}$$

This may be written as a determinant, a convenient form to remember the various signs in the products

$$\mathbf{P} \times \mathbf{Q} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}. \tag{3.10}$$

An example of a cross product is the moment of a force, or torque **Γ**; this is given by

$$\mathbf{\Gamma} = \mathbf{r} \times \mathbf{F} .$$

This is the torque exerted about a point by a force **F** applied a distance **r** from the point.

3.1.4 Multiple products

There are two ways in which three vectors may be multiplied together. You should satisfy yourself as to their validity. A scalar is formed by the product

$$\begin{aligned}\mathbf{A}.(\mathbf{B}\times\mathbf{C}) &= \mathbf{B}.(\mathbf{C}\times\mathbf{A}) = \mathbf{C}.(\mathbf{A}\times\mathbf{B}) \\ &= (\mathbf{A}\times\mathbf{B}).\mathbf{C} \text{ etc.}\end{aligned} \tag{3.11}$$

This is known as the scalar triple product and it has a particularly simple interpretation; it is the volume of a parallelepiped of sides **A**, **B** and **C**.

A vector is formed by the product $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$. Here the order of performing the products is important. The vector triple product can be simplified to:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}.\mathbf{C}) - \mathbf{C}(\mathbf{A}.\mathbf{B}). \tag{3.12}$$

This is the so-called ‘bac-cab’ rule.

The product **B**×**C** is perpendicular to both **B** and **C**. And then the vector product of **A** and **B**×**C** is perpendicular to **B**×**C** which means it must lie in the plane spanned by **B** and **C**. Thus the vector triple product must have the form $\beta\mathbf{B} + \gamma\mathbf{C}$ and this is indeed the structure of Eq. 3.12.

3.2 Vector calculus

3.2.1 Divergence

Gauss's law involved the idea of the *flux* of a vector through a closed surface. Recall that we wrote, in Equation (2.9):

$$\oiint_{\substack{\text{closed}\\\text{surface}}} \mathbf{E}.\mathrm{d}\mathbf{a} = \sum_i Q_i / \varepsilon_0 .$$

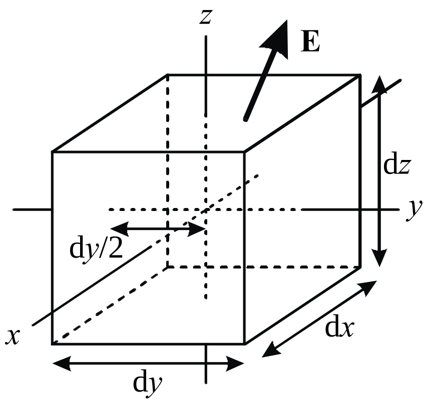
The flux of a vector out of a closed surface is an important property of the vector field; it is related to the 'production of *stuff*' within the enclosed volume – as we shall see.

In this section we introduce a function of a vector field, the *divergence*, which measures the flux emerging through the surface surrounding an infinitesimal volume. The divergence of an arbitrary vector **E**, denoted by div**E** is defined by

$$\operatorname{div}\mathbf{E} = \frac{1}{\text{volume}} \oiint_{\substack{\text{closed}\\\text{surface}}} \mathbf{E}.\mathrm{d}\mathbf{a} \qquad (3.13)$$

where the limit is taken as the volume shrinks to zero. The divergence is a scalar quantity. Note that this definition is independent of any co-ordinate system, and that it measures a *physical* property of the vector **E**.

We shall now *calculate* the divergence of a vector in rectangular Cartesian co-ordinates.



Calculation of the divergence of vector **E** through the cubic volume d*x*d*y*d*z*.

We shall evaluate the flux of **E** through each of the six faces of the cube. Assuming the value of **E** is known at the centre of the cube, we can find the value of the relevant components at the centre of each face using a Taylor expansion. Because the cube is assumed to be infinitesimally small, we can approximate the value of **E** over the face by the value at the face's centre.

Considering the face pointing in the +*x* direction, here we need to know the value **E** perpendicular to the face, that is, the value of E_x. Since the face is a distance d*x*/2 from the origin, at the centre of the face we have:

$$E_x = E_x^0 + \frac{\partial E_x}{\partial x}\frac{\mathrm{d}x}{2} .$$

In a similar way we can evaluate the perpendicular component of **E** at the centre of each face, giving

$$
\begin{aligned}
&+x\,\text{face}: E_x = E_x^0 + \frac{\partial E_x}{\partial x}\frac{\mathrm{d}x}{2} && -x\,\text{face}: E_x = E_x^0 - \frac{\partial E_x}{\partial x}\frac{\mathrm{d}x}{2}\\
&+y\,\text{face}: E_y = E_y^0 + \frac{\partial E_y}{\partial y}\frac{\mathrm{d}y}{2} && -y\,\text{face}: E_y = E_y^0 - \frac{\partial E_y}{\partial y}\frac{\mathrm{d}y}{2}\\
&+z\,\text{face}: E_z = E_z^0 + \frac{\partial E_z}{\partial z}\frac{\mathrm{d}z}{2} && -z\,\text{face}: E_z = E_z^0 - \frac{\partial E_z}{\partial z}\frac{\mathrm{d}z}{2}
\end{aligned}
$$

Each of these components of **E** must be multiplied by the area of the face to give the flux **E**.d**a** through that face. And for each of the three directions we must subtract the flux through the – face from that through the + face: we consider the total flux out of the surface. Thus for the x direction we have

$$
\begin{aligned}
\mathbf{E}.\mathrm{d}\mathbf{a} &= \left(E_x^0 + \frac{\partial E_x}{\partial x}\frac{\mathrm{d}x}{2}\right)\mathrm{d}y\mathrm{d}z - \left(E_x^0 - \frac{\partial E_x}{\partial x}\frac{\mathrm{d}x}{2}\right)\mathrm{d}y\mathrm{d}z\\
&= \frac{\partial E_x}{\partial x}\mathrm{d}x\mathrm{d}y\mathrm{d}z.
\end{aligned}
$$

observe the occurrence of dxdydz, the volume of the element. With similar calculations in the y and z directions, we find

$$
\begin{aligned}
x\,\text{direction}:\ \mathbf{E}.\mathrm{d}\mathbf{a} &= \frac{\partial E_x}{\partial x}\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
y\,\text{direction}:\ \mathbf{E}.\mathrm{d}\mathbf{a} &= \frac{\partial E_y}{\partial y}\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
z\,\text{direction}:\ \mathbf{E}.\mathrm{d}\mathbf{a} &= \frac{\partial E_z}{\partial z}\mathrm{d}x\mathrm{d}y\mathrm{d}z
\end{aligned}
$$

so that for the entire closed surface of the elemental cube the sum (integral) of the flux of **E** is given by

$$
\oiint_{\substack{\text{closed}\\\text{surface}}} \mathbf{E}.\mathrm{d}\mathbf{a} = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{3.14}
$$

But since dxdydz is the volume of the element, recalling the definition of the divergence of **E** in Equation (3.13):

$$
\operatorname{div}\mathbf{E} = \frac{1}{\text{volume}}\oiint_{\substack{\text{closed}\\\text{surface}}} \mathbf{E}.\mathrm{d}\mathbf{a}
$$

as the volume shrinks to zero, we have, in rectangular co-ordinates, the expression for div**E** as

$$
\operatorname{div}\mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \tag{3.15}
$$

3.2.2 The divergence in electrostatics

Introduction of the divergence was motivated by the property of the **E** field which was expressed in Gauss's law, Equation (2.9):

$$\oiint_{\substack{\text{closed}\\\text{surface}}} \mathbf{E}.\mathrm{d}\mathbf{a} = \sum_i Q_i / \varepsilon_0 \ .$$

If we divide both sides of this equation by the volume enclosed within the surface, then using the definition of the divergence, we may write

$$\operatorname{div}\mathbf{E} = \rho/\varepsilon_0 \tag{3.16}$$

where ρ is the *density* of electric charge:

$$\rho = Q/\text{volume}, \tag{3.17}$$

as the volume shrinks to zero.

We see that Equation (3.16) is another statement of Gauss's law, this one in a slightly more concise form.

3.2.3 Curl

We saw that the electric field **E** is a *conservative* field, and that that property could be expressed mathematically in terms of the *line integral* property in Equation (2.13):

$$\oint_{\text{closed loop}} \mathbf{E}.\mathrm{d}\mathbf{r} = 0 \ .$$

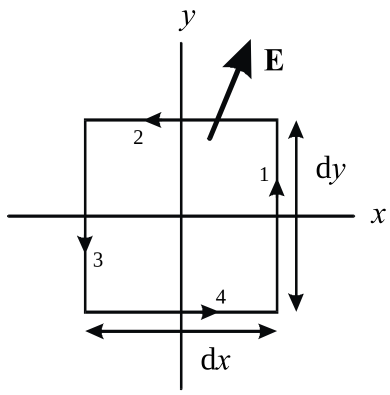
The line integral of a vector around a closed loop is another important property of a vector field. We now introduce a function of a vector field, the *curl*, which measures the line integral around a loop bounding an infinitesimal surface area. The curl of an arbitrary vector **E** , denoted by curl**E** is defined by

$$\operatorname{curl}\mathbf{E} = \frac{1}{\text{area}} \oint_{\text{closed loop}} \mathbf{E}.\mathrm{d}\mathbf{r} \tag{3.18}$$

where the limit is taken as the area shrinks to zero. The curl is a vector quantity; it points in the direction of the surface normal. Note that just as with the definition of the divergence, the curl as defined here is independent of any co-ordinate system, and that it measures a *physical* property of the vector **E**.

Our continental brethren sometimes use the designation *rotation* for the curl and you might find books referring to rot **E**. This is simply another name for curl.

We shall now *calculate* the curl of a vector in rectangular Cartesian co-ordinates.



Calculation of the curl of vector **E** around the square area dxdy.

Note that the curl is a vector quantity. As the square here is in the x–y plane we are calculating the z component of the curl of **E**.

We shall evaluate the line integral of **E** along each of the four sides of the square. Assuming the value of **E** is known at the centre of the square, we can find the value of the relevant components at the centre of each side using a Taylor expansion. Because the square is assumed to be infinitesimally small, we can approximate the value of **E** over the edge by the value at the edge's centre.

$$\text{side 1:}\quad \mathbf{E}.\mathrm{d}\mathbf{r} = \left(E_y^0 + \frac{\partial E_y}{\partial x}\frac{\mathrm{d}x}{2}\right)\mathrm{d}y \qquad \text{side 2:}\quad \mathbf{E}.\mathrm{d}\mathbf{r} = -\left(E_x^0 + \frac{\partial E_x}{\partial y}\frac{\mathrm{d}y}{2}\right)\mathrm{d}x$$

$$\text{side 3:}\quad \mathbf{E}.\mathrm{d}\mathbf{r} = -\left(E_y^0 - \frac{\partial E_y}{\partial x}\frac{\mathrm{d}x}{2}\right)\mathrm{d}y \qquad \text{side 4:}\quad \mathbf{E}.\mathrm{d}\mathbf{r} = \left(E_x^0 - \frac{\partial E_x}{\partial y}\frac{\mathrm{d}y}{2}\right)\mathrm{d}x$$

The line integral around the loop is found by adding the contribution from the four sides:

$$\oint_{\text{closed loop}} \mathbf{E}.\mathrm{d}\mathbf{r} = \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\mathrm{d}x\mathrm{d}y \; . \qquad (3.19)$$

This line integral was evaluated for a surface in the x–y plane. The normal to this surface points along the z axis (and it is actually the +z direction when the *direction* of the line integral is taken into account). Thus, recalling the definition of the curl, in Equation (3.18), we see that what we are in the process of evaluating is the z component of the vector curl**E**. For our surface dxdy:

$$\frac{1}{\text{area}}\oint_{\text{closed loop}} \mathbf{E}.\mathrm{d}\mathbf{r} = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = \operatorname{curl}\mathbf{E}\big|_z \qquad (3.20)$$

In a similar manner we may evaluate the other two components of the curl. In practice these are most simply obtained by cyclically permuting the x, y, z in the above expression. And assembling the three components into the vector

$$\operatorname{curl}\mathbf{E} = \operatorname{curl}\mathbf{E}\big|_x\,\hat{\mathbf{x}} + \operatorname{curl}\mathbf{E}\big|_y\,\hat{\mathbf{y}} + \operatorname{curl}\mathbf{E}\big|_z\,\hat{\mathbf{z}}\,,$$

the curl of **E** may be expressed in the convenient form

$$\operatorname{curl}\mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} \tag{3.21}$$

in terms of a determinant. One should, however, appreciate that this expression is essentially an *aide-mémoire*, strictly speaking the elements of a determinant should all be of a similar type.

3.2.4 The curl in electrostatics

Introduction of the curl was motivated by the conservative property of the **E** field which was expressed in Equation (2.13),

$$\oint_{\text{closed loop}} \mathbf{E}.\mathrm{d}\mathbf{r} = 0\ .$$

If we divide both sides of this equation by the area enclosed by the loop, then using the definition of the curl we may write

$$\operatorname{curl}\mathbf{E} \ = \ 0 \tag{3.22}$$

as the area shrinks to zero.

From this we see that Equation (3.22) is an equivalent statement of the conservative character of the **E** field.

3.2.5 The gradient

We have introduced the gradient of a scalar field already when we explored how to obtain the value of the electric field **E** from a knowledge of the electric potential V, the result of which was expressed in Equation (2.19)

$$\mathbf{E} = -\operatorname{grad} V\ .$$

Here we shall examine the gradient function in a slightly more abstract (and therefore more general) manner. In particular we shall, as we did with the divergence and the curl functions, define the gradient in a way that is independent of any particular co-ordinate system.

The gradient turns a scalar into a vector. To be more precise, it gives a vector from a scalar *field*. Given an arbitrary scalar field $V(\mathbf{r})$, the gradient of V, denoted by grad V is a vector that points in the direction of the most rapid variation of V with position. And the magnitude of grad V is the value of the derivative evaluated in this direction.

In rectangular Cartesian co-ordinates we may evaluate the three components of the derivative of V with respect to position and then combine them together to form a vector. Now

$$\frac{\partial V}{\partial x} \quad \text{is the component in the } x \text{ direction}$$

$$\frac{\partial V}{\partial y} \quad \text{is the component in the } y \text{ direction}$$

$$\frac{\partial V}{\partial z} \quad \text{is the component in the } z \text{ direction}$$

so that we may assemble these into the vector:

$$\operatorname{grad} V = \frac{\partial V}{\partial x}\hat{\mathbf{x}} + \frac{\partial V}{\partial y}\hat{\mathbf{y}} + \frac{\partial V}{\partial z}\hat{\mathbf{z}} \tag{3.23}$$

And we recognise this to be the Cartesian form of the gradient as used in Equation (2.18).

3.2.6 An application of the gradient

An important example of the use of the gradient, which transcends electromagnetism, is in the evaluation of the (infinitesimal) change in a scalar function of position, say $V(\mathbf{r})$, as one moves a displacement d$\mathbf{r}$. Essentially this is simply an application of the multidimensional form of Taylor's expansion, expressed succinctly in vector form.

V is a function of position; in other words, it varies as a function of the co-ordinates:

$$V(\mathbf{r}) = V(x, y, z)$$

if we adopt rectangular co-ordinates. The value of V a small distance along from x, y, z, at position x + dx, y + dy, z + dz, is given by the Taylor expression

$$V(x + \mathrm{d}x, y + \mathrm{d}y, z + \mathrm{d}z) = V(x, y, z) + \frac{\partial V}{\partial x}\mathrm{d}x + \frac{\partial V}{\partial y}\mathrm{d}y + \frac{\partial V}{\partial z}\mathrm{d}z$$

taken to first order. But observe that the second part of this expression, representing the *change* in V, may be expressed as the dot product of d$\mathbf{r}$ with grad(V):

$$V(\mathbf{r} + \mathrm{d}\mathbf{r}) = V(\mathbf{r}) + \mathrm{d}\mathbf{r}.\operatorname{grad}(V)$$

or simply

$$\mathrm{d}V = \mathrm{d}\mathbf{r}.\operatorname{grad}(V) \tag{3.24}$$

3.2.7 Summary of vector calculus identities

We shall first state three important theorems of vector calculus. Although commonly regarded as theorems that must be derived through mathematical manipulations, we shall see from our *physical* definitions of the gradient, curl and divergence, that these results follow directly and quite trivially as identities.

Gauss's (divergence) theorem

This follows from the definition of div, Equation (3.13)

$$\operatorname{div}\mathbf{E} = \frac{1}{\text{volume}} \oiint\limits_{\substack{\text{closed}\\ \text{surface}}} \mathbf{E}.\mathrm{d}\mathbf{a}$$

as the volume shrinks to zero. If we multiply by the volume enclosed and integrate over this we obtain

$$\iiint\limits_{\text{volume}} \operatorname{div}\mathbf{E}\,\mathrm{d}v = \oiint\limits_{\substack{\text{bounding}\\ \text{surface}}} \mathbf{E}.\mathrm{d}\mathbf{a} \tag{3.25}$$

and this is referred to Gauss's *theorem* in vector calculus.

Stokes's (curl) theorem

This follows directly from our definition of the curl, Equation (3.18)

$$\operatorname{curl}\mathbf{E} = \frac{1}{\text{area}} \oint\limits_{\text{closed loop}} \mathbf{E}.\mathrm{d}\mathbf{r}$$

where the limit is taken as the area shrinks to zero. If we multiply by the area (strictly, we take the dot product with the area, since the above definition gives the component of the curl in the direction of the surface normal), and integrate over this we obtain

$$\iint\limits_{\text{area}} \operatorname{curl}\mathbf{E}.\mathrm{d}\mathbf{a} = \oint\limits_{\text{perimeter}} \mathbf{E}.\mathrm{d}\mathbf{r} \tag{3.26}$$

and this is referred to as Stokes's theorem in vector calculus.

Gradient theorem

This follows directly from the 'chain rule' expression involving the gradient, Equation (3.24):

$$\mathrm{d}V = \operatorname{grad}(V)\,.\,\mathrm{d}\mathbf{r}\,.$$

This gives the small increment in the scalar field quantity V in moving a small displacement d$\mathbf{r}$. For a general displacement from $\mathbf{r}_1$ to $\mathbf{r}_2$ the change in V will then be found by integrating, so that

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \operatorname{grad}(V)\,.\,\mathrm{d}\mathbf{r} = V(\mathbf{r}_2) - V(\mathbf{r}_1) \tag{3.27}$$

and this is referred to as the gradient theorem of vector calculus.

Collection of other identities

There follows a collection of identities relating to the operators div, grad and curl, most of which we will not prove. Students should, however, be familiar with them from their courses in vectors. This summary is compiled from the book "Advanced Electricity and Magnetism" by W. Duffin and elsewhere. First we consider operations on the products of two vectors.

1	$\operatorname{grad}(UV)$	$=$	$U\operatorname{grad}(V)+V\operatorname{grad}(U)$	(3.28)
2	$\operatorname{grad}(\mathbf{A}\cdot\mathbf{B})$	$=$	$(\mathbf{A}\cdot\operatorname{grad})\mathbf{B}+(\mathbf{B}\cdot\operatorname{grad})\mathbf{A}+\mathbf{A}\times\operatorname{curl}\mathbf{B}+\mathbf{B}\times\operatorname{curl}\mathbf{A}$	
3	$\operatorname{div}(V\mathbf{A})$	$=$	$V\operatorname{div}(\mathbf{A})+\mathbf{A}\cdot\operatorname{grad}(V)$	(3.29)
4	$\operatorname{div}(\mathbf{A}\times\mathbf{B})$	$=$	$\mathbf{B}\cdot\operatorname{curl}(\mathbf{A})-\mathbf{A}\cdot\operatorname{curl}(\mathbf{B})$	(3.30)
5	$\operatorname{curl}(V\mathbf{A})$	$=$	$V\operatorname{curl}(\mathbf{A})-\mathbf{A}\times\operatorname{grad}(V)$	(3.31)
6	$\operatorname{curl}(\mathbf{A}\times\mathbf{B})$	$=$	$(\mathbf{B}\cdot\operatorname{grad})\mathbf{A}-(\mathbf{A}\cdot\operatorname{grad})\mathbf{B}-\mathbf{B}\operatorname{div}\mathbf{A}+\mathbf{A}\operatorname{div}\mathbf{B}$	

The operator **A•**grad in the second and fifth are given, in Cartesian co-ordinates by

$$(\mathbf{A}\cdot\operatorname{grad})B_x = A_x\frac{\partial B_x}{\partial x}+A_y\frac{\partial B_x}{\partial y}+A_z\frac{\partial B_x}{\partial z}$$

Next we consider the successive application of two vector differential operations.

7	$\operatorname{curl}\operatorname{grad}(V)$	$=$	0	(3.32)
8	$\operatorname{div}\operatorname{curl}(\mathbf{A})$	$=$	0	(3.33)
9	$\operatorname{div}\operatorname{grad}(V)$	$=$	$\nabla^2 V$	(3.34)
10	$\operatorname{curl}\operatorname{curl}(\mathbf{A})$	$=$	$\operatorname{grad}\operatorname{div}(\mathbf{A})-\nabla^2\mathbf{A}$	(3.35)

Of these, the seventh and ninth are particularly important; they will be proved below.

The tenth result is slightly odd since it involves ∇^2 of a *vector* quantity. In rectangular Cartesian co-ordinates this is simply the vector whose x component is ∇^2 of the x, and similarly for the y and the z components. In other, general, co-ordinate systems the curl curl expression may be taken as the *definition* of ∇^2 of the vector.

The following two proofs are established using rectangular Cartesian co-ordinates. Since the div, grad and curl have been defined *independently* of any co-ordinate system, it follows that any convenient co-ordinate system can be used and the result will be true in the general case.

Proof of $\operatorname{curl}\operatorname{grad}(V)=0$

Let us evaluate the x co-ordinate of curl grad(V). Denoting the vector grad(V) by **A**, the x component of the curl is given by

$$\operatorname{curl}(\mathbf{A})\Big|_x = \frac{\partial A_y}{\partial z}-\frac{\partial A_z}{\partial y}.$$

But since **A** is given by grad(V), the components A_y and A_z are given by

$$A_y=\frac{\partial V}{\partial y}, \qquad A_z=\frac{\partial V}{\partial z}$$

from which we see that the x component of the curl is

$$\operatorname{curl}(\mathbf{A})\Big|_x = \frac{\partial^2 V}{\partial z \partial y} - \frac{\partial^2 V}{\partial y \partial z}.$$

The two second derivatives in this equation differ solely in the order of differentiation. However we know that it is a general rule of calculus that for differentiable functions the order of differentiation is unimportant. Thus the two terms in the above equation cancel, so that the x component of curl grad vanishes. And so in general any component will vanish by a similar argument, leading us to the general conclusion that for an arbitrary differentiable scalar function of position $V(\mathbf{r})$,

$$\operatorname{curl}\operatorname{grad}(V) = 0$$

the result we required to prove.

Proof of $\operatorname{div}\operatorname{grad}(V) = \nabla^2 V$

The derivation of this is quite straightforward. Writing, as above, the vector grad(V) as $\mathbf{A}$, the components of $\mathbf{A}$ are

$$A_x = \frac{\partial V}{\partial x}, \qquad A_y = \frac{\partial V}{\partial y}, \qquad A_z = \frac{\partial V}{\partial z}.$$

Then taking the divergence,

$$\begin{aligned} \operatorname{div}\mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\ &= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \end{aligned}$$

and we recognise the second line as the Laplacian, ∇^2, so that

$$\operatorname{div}\operatorname{grad}(V) = \nabla^2 V$$

which is the result we required to prove.

3.2.8 Nabla – or Del notation

The operations of grad, div and curl have been related to different ways of differentiating vectors. We now introduce a general vector differentiation operator ∇, called del (or nabla in old English), in terms of which these three vector calculus functions can be expressed in a unified manner. The del *operator* is defined by

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \tag{3.36}$$

Observe that $\boldsymbol{\nabla}$ is a vector; it has components, here, in the x, y and the z direction. Also, note that it is an *operator*. The derivatives are 'open'; they are waiting for something to *operate* on.

Del applied to a scalar function, say $V(\mathbf{r})$, will give a vector:

$$\nabla V(\mathbf{r}) = \hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

which we immediately recognise as the gradient of *V*:

$$\nabla V(\mathbf{r}) \equiv \mathrm{grad}\, V(\mathbf{r}) . \qquad (3.37)$$

If we apply del to a vector function then we must ask how to do the vector multiplication. Since del is a vector we can *dot* it onto a vector to give a scalar, or we can *cross* it onto a vector to give another vector. Let us examine the dot product first.

Let us apply del to the vector **E** as a dot product. Thus we want to evaluate

$$\nabla.\mathbf{E} = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \right) . \left(E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}} \right)$$

In accordance with the rules for the dot product, the *x* component of the first term goes with the *x* component of the second term, and similarly for *y* and *z*. This gives

$$\nabla.\mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

which we recognise as the divergence of the vector **E**:

$$\nabla.\mathbf{E}(\mathbf{r}) \equiv \mathrm{div}\, \mathbf{E}(\mathbf{r}) \qquad (3.38)$$

If we apply del to the vector **E** as a cross product,

$$\nabla \times \mathbf{E} = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \right) \times \left(E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}} \right)$$

we may evaluate this using the determinant mnemonic for the vector cross product:

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

This, we immediately recognise, is the expression for the curl of the vector **E**, expressed in rectangular co-ordinates:

$$\nabla \times \mathbf{E}(\mathbf{r}) \equiv \mathrm{curl}\, \mathbf{E}(\mathbf{r}) . \qquad (3.39)$$

We now consider a double application of the del operator. Del operating on a scalar function $V(\mathbf{r})$ gives the gradient of *V*, a vector. We shall convert this back to a scalar by *dotting* a further del onto this, that is, taking the divergence:

$$\mathrm{div}\,\mathrm{grad}\, V(\mathbf{r}) = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \right) \times \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \right) V(\mathbf{r}) .$$

The two brackets may be regarded as a *scalar* operator, produced by dotting del onto itself. Thus we may write

$$\operatorname{div}\operatorname{grad} = \nabla\boldsymbol{.}\nabla = \nabla^2 , \tag{3.40}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} . \tag{3.41}$$

This is known as the Laplacian operator, or simply just “del squared”. In some books it is denoted by the symbol Δ.

Finally we examine another double application of the del operator. This time we will take the curl of the gradient of a scalar:

$$\operatorname{curl}\operatorname{grad} V(\mathbf{r}) = \nabla \times \nabla V(\mathbf{r})$$

which may be written as

$$\nabla \times \left(\frac{\partial V}{\partial x}\hat{\mathbf{x}} + \frac{\partial V}{\partial y}\hat{\mathbf{y}} + \frac{\partial V}{\partial z}\hat{\mathbf{z}} \right)$$

or, in determinant form:

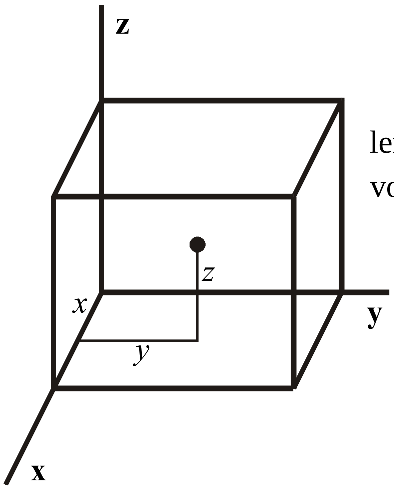
$$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \end{vmatrix} V(\mathbf{r}) = 0$$

This expression is zero because we observe that two rows of the determinant are identical. Recall that particular property of determinants. Actually, we are being a little sloppy here. Think this one through.

Thus we conclude that the curl of the gradient of *any* scalar function is zero.

3.3 Vector Differential Operators in Various Co-ordinate Systems

3.3.1 Cartesian co-ordinates



length element $\mathrm{d}\mathbf{l} = \mathrm{d}x\,\hat{\mathbf{x}} + \mathrm{d}y\,\hat{\mathbf{y}} + \mathrm{d}z\,\hat{\mathbf{z}}$
volume element $\mathrm{d}v = \mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z$

$$\operatorname{grad}\psi(x,y,z) = \hat{\mathbf{x}}\frac{\partial\psi}{\partial x} + \hat{\mathbf{y}}\frac{\partial\psi}{\partial y} + \hat{\mathbf{z}}\frac{\partial\psi}{\partial z}.$$

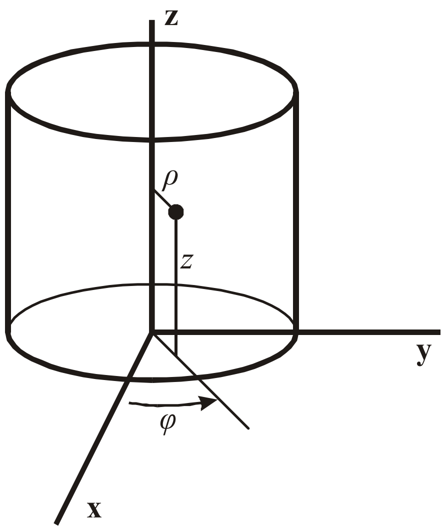
$$\operatorname{div}\mathbf{A}(x,y,z) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}.$$

$$\begin{aligned}
\operatorname{curl}\mathbf{A}(x,y,z) &= \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right), \\
&= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}.
\end{aligned}$$

$$\nabla^2\psi(x,y,z) = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}.$$

$$\begin{aligned}
\nabla^2\mathbf{A}(x,y,z) &= \hat{\mathbf{x}}\nabla^2 A_x + \hat{\mathbf{y}}\nabla^2 A_y + \hat{\mathbf{z}}\nabla^2 A_z, \\
&= \hat{\mathbf{x}}\left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right) + \hat{\mathbf{y}}\left(\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}\right) \\
&\quad + \hat{\mathbf{z}}\left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right).
\end{aligned}$$

3.3.2 Cylindrical polar co-ordinates



$$x = \rho\cos\varphi$$
$$y = \rho\sin\varphi$$
$$z = z$$

unit vectors:

$$\hat{\boldsymbol{\rho}} = \cos\varphi\,\hat{\mathbf{x}} + \sin\varphi\,\hat{\mathbf{y}}$$
$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}}$$
$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

length element $\mathrm{d}\mathbf{l} = \mathrm{d}\rho\,\hat{\boldsymbol{\rho}} + \rho\,\mathrm{d}\varphi\,\hat{\boldsymbol{\varphi}} + \mathrm{d}z\,\hat{\mathbf{z}}$

volume element $\mathrm{d}v = \rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}z$

$$\operatorname{grad}\psi(\rho,\varphi,z) = \hat{\boldsymbol{\rho}}\frac{\partial\psi}{\partial\rho} + \hat{\boldsymbol{\varphi}}\frac{1}{\rho}\frac{\partial\psi}{\partial\varphi} + \hat{\mathbf{z}}\frac{\partial\psi}{\partial z}.$$

$$\operatorname{div}\mathbf{A}(\rho,\varphi,z) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho A_\rho\right) + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}.$$

$$\operatorname{curl}\mathbf{A}(\rho,\varphi,z) = \hat{\boldsymbol{\rho}}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right) + \hat{\boldsymbol{\varphi}}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right) + \hat{\mathbf{z}}\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}\left(\rho A_\varphi\right) - \frac{\partial A_\rho}{\partial\varphi}\right),$$

$$= \frac{1}{\rho}\begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho\hat{\boldsymbol{\varphi}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\varphi} & \dfrac{\partial}{\partial z} \\ A_\rho & \rho A_\varphi & A_z \end{vmatrix}.$$

$$\nabla^2\psi(\rho,\varphi,z) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^2\psi}{\partial z^2}.$$

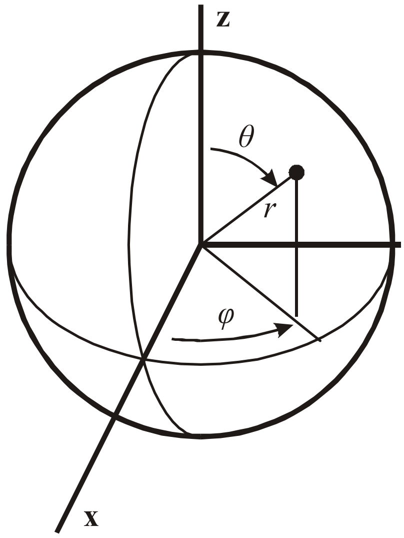
$$\nabla^2\mathbf{A}(\rho,\varphi,z) = \hat{\boldsymbol{\rho}}\left(\nabla^2 A_\rho - \frac{2}{\rho^2}\frac{\partial A_\varphi}{\partial\varphi} - \frac{A_\rho}{\rho^2}\right) + \hat{\boldsymbol{\varphi}}\left(\nabla^2 A_\varphi + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial\varphi} - \frac{A_\varphi}{\rho^2}\right) + \hat{\mathbf{z}}\nabla^2 A_z,$$

$$= \hat{\boldsymbol{\rho}}\left(\frac{\partial^2 A_\rho}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 A_\rho}{\partial\varphi^2} + \frac{\partial^2 A_\rho}{\partial z^2} + \frac{1}{\rho}\frac{\partial A_\rho}{\partial\rho} - \frac{2}{\rho^2}\frac{\partial A_\varphi}{\partial\varphi} - \frac{A_\rho}{\rho^2}\right)$$

$$+\hat{\boldsymbol{\varphi}}\left(\frac{\partial^2 A_\varphi}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 A_\varphi}{\partial\varphi^2} + \frac{\partial^2 A_\varphi}{\partial z^2} + \frac{1}{\rho}\frac{\partial A_\varphi}{\partial\rho} + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial\varphi} - \frac{A_\varphi}{\rho^2}\right)$$

$$+\hat{\mathbf{z}}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A_z}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 A_z}{\partial\varphi^2} + \frac{\partial^2 A_z}{\partial z^2}\right).$$

3.3.3 Spherical polar co-ordinates



$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

unit vectors:

$$\hat{\mathbf{r}} = \sin\theta\cos\varphi\,\hat{\mathbf{x}} + \sin\theta\sin\varphi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\varphi\,\hat{\mathbf{x}} + \cos\theta\cos\varphi\,\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}}$$

length element $\mathrm{d}\mathbf{l} = \mathrm{d}r\,\hat{\mathbf{r}} + r\mathrm{d}\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\mathrm{d}\varphi\,\hat{\boldsymbol{\varphi}}$

volume element $\mathrm{d}v = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi$

$$\operatorname{grad}\psi(r,\theta,\varphi) = \hat{\mathbf{r}}\frac{\partial\psi}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}.$$

$$\operatorname{div}\mathbf{A}(r,\theta,\varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}.$$

$$\begin{aligned}\operatorname{curl}\mathbf{A}(r,\theta,\varphi) &= \hat{\mathbf{r}}\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\,A_\varphi\right) - \frac{\partial A_\theta}{\partial\varphi}\right] + \\ &\quad + \hat{\boldsymbol{\theta}}\left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\varphi\right)\right] + \hat{\boldsymbol{\varphi}}\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right], \\ &= \frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\,\hat{\boldsymbol{\varphi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\varphi} \\ A_r & rA_\theta & r\sin\theta A_\varphi\end{vmatrix}.\end{aligned}$$

$$\nabla^2\psi(r,\theta,\varphi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}.$$

$$\left[\text{Note that }\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) \equiv \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi)\right]$$

$$\begin{aligned}\nabla^2\mathbf{A}(r,\theta,\varphi) &= \hat{\mathbf{r}}\left(\nabla^2 A_r - \frac{2}{r^2}A_r - \frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,A_\theta\right) - \frac{2}{r^2\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}\right) \\ &\quad + \hat{\boldsymbol{\theta}}\left(\nabla^2 A_\theta - \frac{A_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial A_r}{\partial\varphi} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\varphi}{\partial\varphi}\right) \\ &\quad + \hat{\boldsymbol{\varphi}}\left(\nabla^2 A_\varphi - \frac{A_\varphi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial\varphi}\right).\end{aligned}$$

When you have completed this chapter you should:

- understand the distinction between scalars and vectors;
- know the meaning of the vector dot product;
- be able to manipulate expressions involving dot products;
- know the meaning of the vector cross product and appreciate that it is an anticommutative operation;
- be able to manipulate expressions involving cross products;
- understand the scalar triple product of vectors and interpret as the volume of a parallelepiped;
- be aware of the vector triple product and its expansion;
- understand the divergence of a vector and its interpretation in terms of the flux through a closed surface;
- know the expression for the divergence in rectangular Cartesian co-ordinates;
- understand the curl of a vector and its interpretation in terms of a closed loop line integral;
- know the expression for the curl in rectangular Cartesian co-ordinates;
- understand the gradient of a scalar field function and its meaning as a vector;
- know the expression for the gradient in rectangular Cartesian co-ordinates;
- understand the use of the divergence in electrostatics and its connection with Gauss's law.
- understand the various possible double applications of the above vector calculus functions, including div grad = ∇^2 and curl grad = 0;
- be familiar with the ∇ operator, known as 'del' or 'nabla' and its connection with div, grad and curl;
- be able to manipulate expressions using ∇;
- relate the functions div, grad and curl to the properties of the electric field and electric potential already encountered.