II ELECTROSTATICS I

2.1 Coulomb's law

In 1785 Coulomb completed a series of experiments which encapsulated the laws governing the forces between stationary point electric charges. He concluded that for two charges Q_1 and Q_2:

i) the force between them is proportional to the product of the magnitude of the charges,

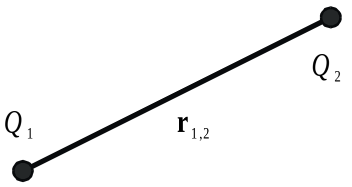
ii) the force is inversely proportional to the square of the separation of the charges and directed along the line joining them,

iii) the force is repulsive for like charges and attractive for unlike charges.

These statements may be summarised by:

$$\mathbf{F}_{1,2} \propto \frac{Q_1 Q_2}{r_{1,2}^2}\hat{\mathbf{r}}_{1,2}$$

Coulomb's law



In the SI (Système Internationale) system of measurement, which uses the *metre*, *kilogram* and *second* as its fundamental units (with the *ampère* as its basic electrical unit), the electrostatic force is written as

$$\mathbf{F}_{1,2} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 r_{1,2}^2}\hat{\mathbf{r}}_{1,2} \qquad (2.1)$$

– the constant of proportionality is $1/4\pi\varepsilon_0$. The $1/4\pi$ will be seen to make sense later. $\mathbf{r}_{1,2}$ is a unit vector pointing from Q_1 to Q_2.

F is measured in newtons
r is measured in metres } mechanical units

Q is measured in coulombs – electrical unit.

The constant ε_0 is known as the *permitivity of free space,* and it has the units of coulombs^2 metres^{-2} newtons^{-2} or farads metres^{-1}.

Once the coulomb is defined (it is specified as a current of one ampère flowing for one second) then the value of ε_0 may be determined experimentally. It is found to be

$$\varepsilon_0 = 8.85\times10^{-12}\ \text{f m}^{-1}. \qquad (2.2)$$

But from Maxwell's equations / relativity, one finds the direct result that *numerically*:

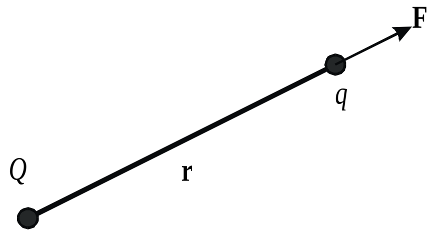
$$\varepsilon_0 = \frac{1}{4\pi \times 10^{-7} c^2} \tag{2.3}$$

where c is the velocity of light: 3.0×10^8 ms^{-1}.

2.2 The electric field

From Coulomb's law, we know that a *test* charge q in the neighbourhood of a charge Q, will experience a force **F**:

$$\mathbf{F} = q\frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$$



force on a test charge q

The force on the test object is proportional to its charge q; if we were to sample the field with a larger charge q' then there would be a correspondingly larger force $\mathbf{F}'$ produced. Similarly, a vanishingly small test charge would experience a vanishingly small force. And even in the absence of the test charge there is "something" in the space produced by the *source* charge – we define the *electric field intensity* **E** as the force at a point in space which would be experienced by a unit test charge q.

$$\mathbf{E} = \mathbf{F}/q \tag{2.4}$$

Mathematically, nothing new has been introduced in using the idea of the electric field. But *physically* this represents a change in one's view of the Coulomb interaction. Coulomb's law treats the two charges, Q_1 and Q_2 or Q and q in a completely symmetric way; the force is proportional to their product. But with the introduction of the electric field, two different aspects of charge are seen. The force is seen as the result of a two-step process. The source charge Q acts to *produce* an electric field, while the test charge q *responds* to the field by experiencing a force. There is thus an *active* and a *passive* aspect to electric charge.

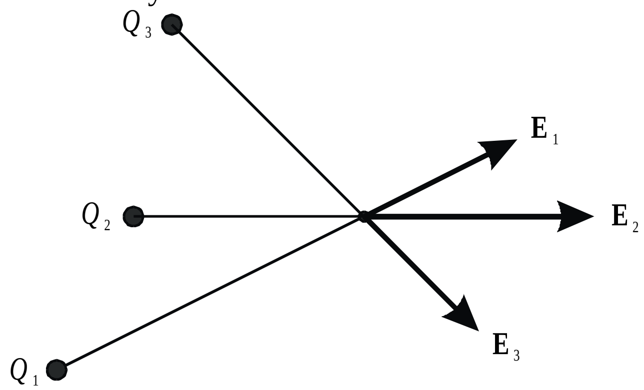
Active aspect of charge	**Passive aspect of charge**
A point charge Q produces an electric field **E**.	A point charge q responds to an electric field **E**.
$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$	$\mathbf{F} = q\mathbf{E}$
Q is the **source** of **E**.	q **responds** to **E**.

Is the electric field anything more than a mathematical convenience? Does it have a real existence? We shall see (from the energy stored in a capacitor) that one can regard energy as being contained in an electric field. Thus corresponding to a field distribution in space there should be an energy distribution. Is there really this energy in otherwise empty space? That would give definite support for the physical (as opposed to the mathematical) reality of the electric field. Feynman (in Lectures on Physics, vol. 2) states that the only way to demonstrate this would be by observing the gravitational effect of the energy – which, as yet, remains beyond the bounds of experimental possibility. However when charges accelerate, some electric field detaches itself from the charge and travels away (at the speed of light). This can be detected at a distance from the source and it can, for instance, do work on charges far from the source. Thus when the electric field travels, its associated energy travels with it. In this sense the electric field certainly is real.

2.3 Linearity and superposition

The force, on a charge, produced by a number of other charges is the sum of the forces that would be produced by the individual charges separately. Clearly the electric fields are additive in the same way.

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3$$



Principle of Superposition

The fact that the forces, or equivalently the electric fields, may be added in this way is an *experimental* observation. It tells us:

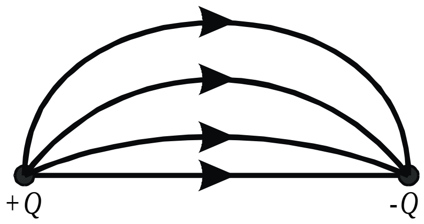
i) that electric fields may be added according to the usual rules of vector addition, and

ii) that the field distribution produced by one charge is not changed by the presence of other charges.

The rule that the fields may be added as independent vectors is known as the *Principle of Superposition*. It follows from the linearity of the equations of electromagnetism. Obviously it makes life very much easier; complex problems may be split up into smaller, simpler parts.

2.4 Lines of force – Electric flux

This is a pictorial representation of **E**. Lines of force start on positive charges and terminate on negative charges. They point in a direction parallel to **E** and the density of the lines is proportional to the magnitude of **E**. They are analogous to the *stream lines* of fluid flow.



lines of force

Lines of force were first used seriously by Faraday, and they helped him visualise electric (and magnetic) fields; he was a very non-mathematical person. Note that there is an undetermined constant in the specification of the density of lines of force.

The concept of lines of force is useful because (as we shall see) the lines are *conserved*. This is a direct consequence of Coulomb's inverse square law, and it is formalised in Gauss's law. Here let us take a sneak preview.

We first define the concept of *electric flux* Φ_{E}. This is a *scalar* quantity, essentially the number of lines of electric force passing perpendicularly through an area, given by

$$\begin{aligned}\Phi_{\mathrm{E}} &= \mathbf{E}\cdot\text{normal surface area}\\ &= \mathbf{E}\cdot\hat{\mathbf{n}}\ \text{times area}\end{aligned}$$

where $\hat{\mathbf{n}}$ is the unit vector pointing normal to the surface. This involves the vector dot product. A slight simplification follows if we introduce the idea of *vector area*. We define the vector area $\mathbf{a}$ as the (scalar) area a multiplied by the unit vector pointing normal to the surface:

$$\mathbf{a} = a\,\hat{\mathbf{n}}\,.$$

Then the electric flux is simply

$$\Phi_{\mathrm{E}} = \mathbf{E}\cdot\mathbf{a}\,.$$

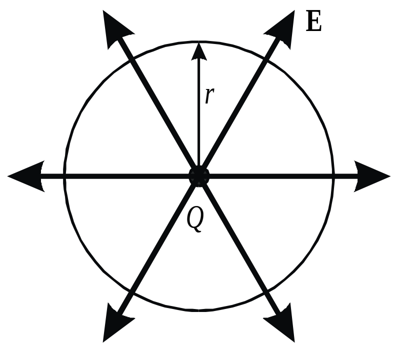
The above expression applies for a constant electric field **E**. In general, where **E** varies from place to place, for an infinitesimal vector area d**a**, the element of electric flux through this surface is given by

$$\mathrm{d}\Phi_{\mathrm{E}} = \mathbf{E}\cdot\mathrm{d}\mathbf{a}\,. \qquad (2.6)$$

Vector area is an important concept in electromagnetism. You should appreciate that there is an ambiguity in the specification of the direction of the surface. For a closed surface we adopt the convention that the normal vector points *outward*.

Let us take a point charge Q. At a distance r away the magnitude of the electric field intensity is

$$E = \frac{Q}{4\pi\varepsilon_0 r^2}$$



flux through a sphere

and the lines of force are radial. In other words, they cut the surface of the sphere perpendicularly.

The electric flux through the sphere is given by

$$\begin{aligned}\Phi_{\mathrm{E}} &= \mathbf{E}\boldsymbol{.}\text{area} \\ &= \frac{Q}{4\pi\varepsilon_0 r^2}\times 4\pi r^2\end{aligned}$$

and note that the $4\pi r^2$ cancels out, leaving:

$$\Phi_{\mathrm{E}} = Q/\varepsilon_0 \, . \tag{2.7}$$

We see that the total number of lines of force is independent of *r*, and proportional to the charge *Q*. Thus we can interpret Φ_{E} as the number of lines of force emanating from the charge *Q*. And therefore it is a useful concept: all as a consequence of the inverse square law.

Another way of regarding this result is to take the conservation of the number of lines of force as fundamental. Then the inverse square law follows as a consequence of the geometry of three-dimensional space. From this point of view the electric field **E** is interpreted as the *density of electric flux*.

We also observe that the 4π has vanished in the expression for Φ_{E}. If there had been no 4π in the original Coulomb's law expression then these factors would have cropped up in all sorts of places where they had no business.

2.5 Gauss's law

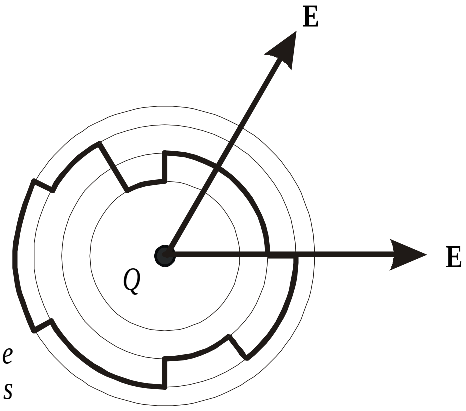
Gauss's law is essentially the extension of the result of the previous section to a more general situation. There are two aspects to the generalisation:

i) the volume surrounding the charge is not restricted to being spherical, and

ii) an arbitrary distribution of charges must be allowed within the volume.

We shall see that once i) is accomplished, then ii) follows from the linearity condition.

The result of the previous section was that the electric flux through a spherical surface surrounding a charge *Q* was given by Q/ε_0, independent of the radius of the sphere. Any sphere centred on *Q* will do. Furthermore, one could imagine an assembly of spherical elements connected by planes in the radial direction.



If a general volume is constructed in this way then no lines of **E** penetrate the flat areas; there is flux only through the spherical surfaces. And thus the flux through the constructed surface is the same Q/ε_0 . As in the usual procedures of calculus, we may build the surface of a larger number of surfaces with smaller steps, and in this way approach the limit of an arbitrary given shape. So we conclude that the electric flux through an arbitrary closed surface is given by Q/ε_0 .

Since Coulomb's law is linear, it follows that we can add the effects of a number of charges, giving the final result

$$\Phi_{\mathrm{E}} = \sum_i Q_i/\varepsilon_0 \tag{2.8}$$

where the sum is over all charges enclosed by the volume. Alternatively, recalling the definition of Φ_{E}, we may write this result directly as an integral over the closed surface:

$$\oiint\limits_{\substack{\text{closed}\\\text{surface}}} \mathbf{E}.\mathrm{d}\mathbf{a} = \sum_i Q_i/\varepsilon_0 \tag{2.9}$$

We see that we have succeeded in making the two generalisations required. The result is Gauss's law, expressed here in one of its forms. In words: the electric flux through a closed surface is given by the total charge enclosed divided by ε_0 .

There are two main uses for Gauss's law. It may be used in the calculation of electric fields, but only for systems which have a high degree of symmetry. Secondly, Gauss's law is one of the direct ingredients of Maxwell's equations. The differential form of the above Equation (2.9) is actually one of the four Maxwell equations.

2.6 Work in an electric field

The work done by a constant force **F** when moving a distance **l** in the direction of the force is given by

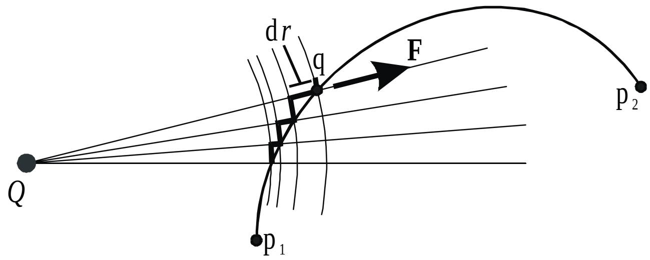
$$W = \mathbf{F}.\mathbf{l}$$

If the applied force varies with position, and if the direction of the motion is not parallel to the force then the work done in moving an infinitesimal displacement d**l** is given by the vector dot product

$$\begin{aligned} dW &= \mathbf{F}.d\mathbf{l} \\ &= F\, dl \cos\theta \end{aligned} \qquad (2.10)$$

where θ is the angle between the direction of the force and the displacement.

To calculate the work done in moving a test charge q in the electric field of a source charge Q we may use a procedure similar to that in the previous section (but applied to a line rather than to a surface).



calculation of work in an electric field

The path is decomposed into elements radial and tangential with respect to the source charge Q. As the test charge q moves between the two points indicated, only the radial element of the motion works against the force: the (scalar) distance dr.

The force *applied* must be such as to counterbalance the electrostatic force: it is therefore in the *opposite* direction to the electric force, thus the work done in this infinitesimal displacement is

$$dW = -\frac{qQ}{4\pi\varepsilon_0 r^2}\, dr \qquad (2.11)$$

that may immediately be integrated to give the work done in traversing a finite path:

$$\begin{aligned} W &= -\int_{r_1}^{r_2} \frac{qQ}{4\pi\varepsilon_0 r^2}\, dr \\ &= \frac{qQ}{4\pi\varepsilon_0}\left(\frac{1}{r_2} - \frac{1}{r_1}\right). \end{aligned} \qquad (2.12)$$

We shall investigate the numerical aspects of this result shortly. However it should be recognised, from the above derivation, that only the initial and the final position are of consequence, and the actual path the test charge moves through is irrelevant. In particular, this means that if a charge is taken through an arbitrary path and then returned to its initial position the work done will be zero. The work done may be expressed in terms of the line integral of **E**:

$$W = -q \oint_{\text{closed loop}} \mathbf{E}.d\mathbf{r} = 0\,.$$

Thus we conclude that the closed loop line integral of **E** is necessarily zero

$$\oint_{\text{closed loop}} \mathbf{E}.\mathrm{d}\mathbf{r} = 0\,. \tag{2.13}$$

Since the physical content of this equation is that energy is conserved when a charge traverses a closed loop in an electric field, the **E** field is referred to as a *conservative* field.

Here, then, we have another property of the electric field, also expressed in integral form. Gauss's law is valid very generally for electromagnetic systems. By contrast the above property of the **E** field, while it is an important element of another of Maxwell's equations, will need to be modified when moving charges and electric currents are considered.

2.7 Electric potential

If we take a test charge q starting from some fixed position $\mathbf{r}_0$ then it requires a certain amount of energy to move it to another point **r** in an electric field **E**. From the previous section we know that the amount of energy is independent of the actual path taken. This means that every point in space may be characterised by a scalar quantity: the energy it took to get the charge there. The *electric potential V* at a point p is defined as the energy required to bring a unit charge to that point. Clearly there is an arbitrary additive constant which depends on the initial position.

Considering the electric potential in the electric field of a point charge Q, we may take over the result of the previous section. If the initial position $\mathbf{r}_0$ is taken to be at infinity (so that $1/r_0 = 0$) then the electric potential at a displacement **r** from Q is given by

$$V(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0 r} \tag{2.14}$$

The electric potential has two main uses. It corresponds to the voltage of an electric circuit: the work done is equal to the charge times the potential difference through which it has been moved. However V is also particularly useful, in electrostatics, as an auxiliary quantity in the calculation of electric fields. From a given charge distribution it is easier to calculate the *scalar V* than to calculate the *vector* **E**. When V is found, it then remains to evaluate **E**, which is a relatively easy procedure.

2.8 Calculation of E from V

Recall that V is defined in terms of the work done in moving a unit charge. If the charge is moved through an infinitesimal displacement d**r** then the change in potential dV is given by

$$\mathrm{d}V = -\mathbf{E}.\mathrm{d}\mathbf{r}\,. \tag{2.15}$$

This can be expressed in terms of rectangular Cartesian coordinates as

$$\mathrm{d}V(x,y,z) = -\left(E_x\mathrm{d}x + E_y\mathrm{d}y + E_z\mathrm{d}z\right),$$

where the position dependence of V is explicitly indicated. From this expression the components of **E** are immediately recognised as the various partial derivatives of *V*:

$$E_x = -\frac{\partial V}{\partial x},\ E_y = -\frac{\partial V}{\partial y},\ E_z = -\frac{\partial V}{\partial z}\,. \tag{2.16}$$

These components may be assembled together to recover the electric field vector:

$$\begin{aligned}\mathbf{E} &= E_x\mathbf{i} + E_y\mathbf{j} + E_z\mathbf{k} \\ &= -\left(\frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}\right).\end{aligned} \tag{2.17}$$

The expression for **E** in terms of *V* involves the gradient function of vector calculus:

$$\mathrm{grad}(V) = \frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k} \tag{2.18}$$

(in rectangular cartesian coordinates), so that the electric field may be expressed as

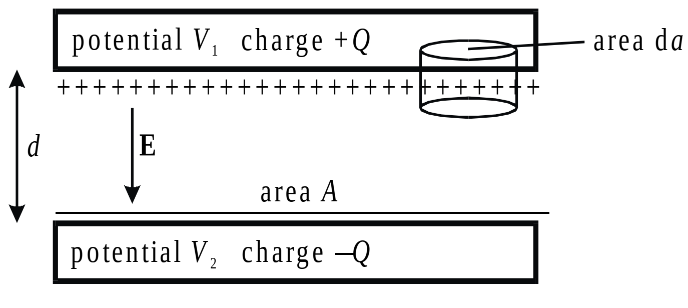
$$\mathbf{E} = -\mathrm{grad}(V). \tag{2.19}$$

This will be investigated further when we have the full machinery of vector calculus at our disposal.

2.9 The capacitor

We now consider extended charge distributions on conductors rather than the point charges of the previous sections. Note that in a future section we shall examine the properties of conductors in some detail; for the present we will not be too pedantic except to note that the electric field is zero inside a conductor (Section 4.4.1).

Two conductors containing charge will produce an electric field in the space between. We consider the practically important case where the charge has been transferred from one conductor to the other: one conductor has a charge $+Q$ and the other $-Q$. The difference in electric potential between the conductors is proportional to the transferred charge, the constant of proportionality depending on geometric factors. (The constant also depends on the material in the space between the conductors, but recall our statement of philosophy on material media; at this stage we are only interested in the vacuum and not in material media.)



a prototypical capacitor

We shall start by calculating the electric field intensity between the plates. To do this we utilise the device of the “Gaussian pill box” drawn on the top plate. Neglecting edge effects the lines of **E** will be normal to the plates, piercing only the bottom of the pill box. The area of this is da, so using Gauss’s law we have

$$E\,\mathrm{d}a = \mathrm{d}Q/\varepsilon_0\,, \tag{2.20}$$

which may be written as

$$E\,\mathrm{d}a = \sigma\,\mathrm{d}a/\varepsilon_0$$

where σ is the surface density of charge, Q/A.

Thus we have found that the electric field between the plates of the capacitor can be written as

$$E = \sigma/\varepsilon_0 = Q/\varepsilon_0 A\,. \tag{2.21}$$

This expression is independent of position within the volume of the capacitor, indicating that the electric field within a parallel plate capacitor is uniform (neglecting edge effects).

Now let us consider the potential difference between the two conductors. The separation between the plates is d. Moving a unit charge a distance d parallel to a uniform electric field **E** will take an energy Ed. This is the difference in potential between the conductors:

$$V = Ed\,, \tag{2.22}$$

from which we see that the potential difference is proportional to the charge

$$V = Q\frac{d}{\varepsilon_0 A}\,. \tag{2.23}$$

The constant of proportionality between Q and V is known as the *capacitance*; its symbol is C and it is measured in Farads.

In general we have

$$V = Q/C \tag{2.24}$$

where, for the parallel plate capacitor

$$C = \varepsilon_0 A/d\,. \tag{2.25}$$

2.10 Electric field energy

It is necessary to perform work in the establishment of an electric field. To calculate the work done in creating a field **E** we shall imagine charging a capacitor by the successive transfer of charge from one plate to the other. If the present charge on the capacitor is q then the work done in moving an infinitesimal charge dq from one plate to the other, a distance d apart, is given by

$$\mathrm{d}W = \mathrm{d}q\,E\,d\,. \tag{2.26}$$

But since the electric field E is given by $q/\varepsilon_0 A$, we can write

$$\mathrm{d}W = \frac{d}{\varepsilon_0 A} q\,\mathrm{d}q \tag{2.27}$$

so that in building up the charge from nothing to Q the work done is

$$\begin{aligned} W &= \frac{1}{C}\int_0^Q q\,\mathrm{d}q \\ &= Q^2/2C \end{aligned} \tag{2.28}$$

which, incidentally, can also be written as $CV^2/2$. These are all expressions for the energy stored in a capacitor. But our particular interest is in an expression involving the electric field.

Using the result from Equation (2.28), $W = Q^2/2C$, and the expression of Equation (2.21) for the charge, $Q = \varepsilon_0 AE$, we obtain

$$\begin{aligned} W &= \varepsilon_0^2 A^2 E^2 \frac{d}{2\varepsilon_0 A} \\ &= \frac{\varepsilon_0 E^2}{2} \times \text{volume.} \end{aligned} \tag{2.29}$$

The underlying idea here is that the work done in charging the capacitor is the energy contained in the capacitor. And then *if* that energy is considered to reside in the electric field, then the magnitude of that energy is $\varepsilon_0 E^2/2$ per unit volume. In other words, $\varepsilon_0 E^2/2$ is the electric field energy *density*, which we denote by U_E:

$$U_\mathrm{E} = \frac{\varepsilon_0 E^2}{2} \tag{2.30}$$

When you have completed this chapter you should:

- know the physical phenomena contained in Coulomb’s law;
- know the units of the physical quantities in the Coulomb’s law formula;
- be able to calculate the force on a stationary charge due to other static charges;
- be familiar with idea of the electric field;
- appreciate the active and passive aspects of electric charge;

- be able to calculate electric field of an arrangement of charges;
- be able to calculate force on a charge in a given electric field;
- understand that the linearity of the Coulomb force permits electric forces and electric fields to be combined according to the usual rules for vectors;
- be familiar with concept of electric flux and its connection with the charge contained in a volume;
- know the meaning of vector area and be able to perform related calculations;
- be happy with the formalisation of the ideas of flux and enclosed charge in terms of Gauss's law
- understand that the inverse square law leads to conservation of lines of flux;
- be able to calculate total flux emerging from arbitrary charge distributions;
- be able to calculate the work done when a charge is moved in an electric field (produced by other charges)
- understand that the electric field is conservative and how this leads to the idea of the electric or electrostatic potential;
- be able to calculate the potential for given charge distributions;
- be able to calculate the electric field from a given potential;
- be familiar with the capacitor and its electrical properties;
- be able to utilise the artifice of the Gaussian 'pill box';
- understand that **E** is uniform in a parallel plate capacitor;
- be able to calculate the work done in charging a capacitor;
- interpret the work done in charging a capacitor in terms of electric field energy.