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$Z(\lambda)$ model and flows: subgraph break–collapse method

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Abstract. A recursive algorithm previously developed for carrying out renormalisation calculations for the λ-state Potts model is generalised to the $Z(\lambda)$ model. The relations used are based on an expression we derive for the pair correlation function in terms of mod-λ flows, which represents an extension of a similar result for the partition function previously obtained by Biggs. The use of flows enables us to prove and extend the formulae which appear in the break-collapse method of Mariz and co-workers. It is argued that the use of fixed-flow bonds rather than the precollapsed bonds used by the latter authors leads to a more efficient algorithm.

1. Introduction

The $Z(\lambda)$ model contains as particular cases many statistical models of well known theoretical and experimental relevance (e.g. bond percolation, random resistor networks, spin-$\frac{1}{2}$ Ising, λ-Potts, discrete spin cubic, clock and classical XY models). During recent years, this model has been the subject of a considerable number of studies in both lattice gauge theory and statistical mechanics (Wu and Wang 1976, Elitzur *et al* 1979, Savit 1980, Cardy 1980, Alcaraz and Köberle 1980, 1981, Alcaraz and Tsallis 1982, Mariz *et al* 1985, Tsallis and Souletie 1986).

The $Z(\lambda)$ model is identical to the Ising and three-state Potts model for $\lambda = 2$ and 3, respectively. For $\lambda \geq 4$, the $Z(\lambda)$ model has a richer critical behaviour involving two or more interaction parameters. Several methods have been used to calculate its phase diagram which has three or more phases. One of these techniques, the *break-collapse method* (BCM), was described for $\lambda = 4$ (the symmetric Ashkin-Teller model) by Mariz *et al* (1985, hereafter referred to as MTF), for $\lambda = 6$ by Mariz *et al* (1989), and for a general value of λ by Mariz and Tsallis (private communication). This method is an extension of the BCM for the Potts model (Tsallis and Levy 1981, Tsallis 1989) and it allows the exact calculation of the partition function and correlation functions of finite clusters which are used in renormalisation group procedures. The latter have been successfully used in the calculation of approximate critical frontiers and critical exponents of the $Z(\lambda)$ model (MTF, Tsallis and Souletie 1986, de Souza 1988, Mariz 1989, Mariz *et al* 1989). In the case of $Z(4)$, both pure (isotropic or anisotropic) and random ferromagnetic (or antiferromagnetic) models on the square and cubic lattices have been considered. For $Z(6)$, existing calculations are restricted to the ferromagnetic model on the isotropic square lattice.

In a previous paper on the Potts model (de Magalhães and Essam 1988, hereafter referred to as PF3) we presented a more efficient recursive algorithm than the BCM. This algorithm was based on combinatorial formulae conjectured by Tsallis (1988), the proofs of which were given in PF3 through the use of flow polynomials (Tutte 1954, 1984). The connections between these graph theoretic polynomials and the Potts model were presented in Essam and Tsallis (1986, hereafter referred to as PF1) and de Magalhães and Essam (1986, hereafter referred to as PF2) (see also Wu (1988) for a less formal derivation of some of these connections). The algorithm of PF3 is known as the 'subgraph break-collapse method' (SBCM) and therein the Potts cluster is represented by a graph G, the vertices of which are the atoms; the occurrence of an edge in G represents a bond, or interaction, between the corresponding atoms. A graph with many vertices requires a prohibitive amount of computer time to calculate its partition function and correlation functions directly as a sum over states. The BCM and SBCM provide alternative and more efficient ways of calculating these functions. In both of these recursive methods, the above-mentioned functions for a graph G are expressed in terms of those for the 'broken' (deleted) and 'collapsed' (contracted) graphs. These are obtained from G by deleting and contracting, respectively, a chosen edge e. The extension of the techniques from the Potts model to the $Z(\lambda)$ model involves other graphs besides the broken and collapsed graphs. In the BCM these extra graphs are the 'precollapsed' graphs (in which the edge e is precollapsed), while in the SBCM they are the graphs with fixed flows on the edge e. Such an edge will be referred to as a 'frozen edge'. Here we interpret the precollapsed bonds in terms of flows and derive all equations necessary to extend the algorithm of PF3 to the general $Z(\lambda)$ model. From these equations, which we call 'graph reduction equations', we derive an extension of the formulae which appear in the BCM of Mariz and co-workers. We argue that our algorithm is more efficient than the BCM. One of the reasons for this is the fact that the use of frozen edges, rather than precollapsed edges, reduces the depth of recursion since the number of frozen edges can never be more than the number of independent cycles in the graph.

In § 2 we first summarise a previous result (Biggs 1976, 1977) in which the partition function of the $Z(\lambda)$ model is expressed as a sum over mod-λ flows, and then extend it to the correlation function. In § 3 we derive the graph reduction equations of the SBCM. In § 4 we present the SBCM algorithm and an extension of the BCM formulae. We also illustrate the SBCM by an example for $Z(4)$ and compare it with the BCM. Finally, the conclusions are presented in § 5.

2. The flow vector and correlation function

2.1. *The model*

We consider a $Z(\lambda)$ cluster represented by graph G with vertex set V and edge set E. With each vertex i of V is associated a state variable n_i which takes on the λ integer values, $0, \ldots, \lambda-1$. The Hamiltonian is given (Alcaraz and Köberle 1980, 1981) by

$$H(G) = k_B T \sum_{e \in E} h_e(n_i - n_j) \tag{2.1}$$

where the edge e has vertices i and j and $n_i - n_j$ is calculated mod λ. The sum over edges in (2.1) includes all interacting pairs of atoms and the interaction may depend on e so that, for example, lattice models with anisotropic couplings are included. The pair interaction energy is independent of the ordering of i and j so that

$$h_e(\lambda-\alpha)=h_e(\alpha). \tag{2.2}$$

It follows from (2.2) that for $\lambda \geqslant 2$ there are only $(\bar{\lambda}+1)$ distinct values of the energy of interaction between a given pair of atoms, where $\bar{\lambda}=[\lambda/2]$ is the integer part of $\lambda/2$.

The following are important special cases of (2.1), the Potts model:

$$h_e(n_i-n_j)=\begin{cases} h_e(0) & \text{for } n_i=n_j \\ \lambda K_e+h_e(0) & \text{for } n_i\neq n_j \end{cases} \tag{2.3a}$$

and the clock model:

$$h_e(n_i-n_j)=-K_e\cos[2\pi(n_i-n_j)/\lambda] \tag{2.3b}$$

where $K_e=\beta J_e$, with J_e being the coupling constant between the spins on vertices i and j, is positive for ferromagnets.

2.2. *The partition function*

A theorem of Biggs (1976, 1977) concerning the partition function $Z(G)$, which he refers to as algebraic duality, may be written in the form

$$Z(G)=\lambda^{\nu-\varepsilon}\left(\prod_{e\in E} z_e\right)D(G) \tag{2.4}$$

where ν is the number of vertices and ε is the number of edges in G. λz_e is the partition function of the edge e in isolation, where z_e is given by

$$z_e=\sum_{\alpha=0}^{\lambda-1}\exp(-h_e(\alpha)) \tag{2.5}$$

and $D(G)$ is the following generating function for flows:

$$D(G)=\sum_{\varphi\in F(G)}\prod_{e\in E} t_e(\varphi(e)). \tag{2.6}$$

Here the function $t_e(\alpha)$ is the component α of the λ-dimensional vector transmissivity $\boldsymbol{t}_e$ (Alcaraz and Tsallis 1982) for edge e defined by

$$t_e(\alpha)=\frac{1}{z_e}\sum_{\beta=0}^{\lambda-1}\exp(2\pi\mathrm{i}\alpha\beta/\lambda)\exp(-h_e(\beta)) \qquad (\alpha=0,1,\ldots,\lambda-1) \tag{2.7}$$

and $\varphi(e)$ is the value of the flow φ on the edge e. A flow is a function defined on the edge set E which assigns an integer value, in the range $0,\ldots,\lambda-1$, to each edge, subject to a conservation condition at each vertex (see, for example, PF1). The conservation condition may be expressed as follows. Each edge is given an arbitrary directing and an incidence matrix S is defined for $j\in V$ and $e\in E$ by

$$S_{je}=\begin{cases} +1 & \text{if } e \text{ is directed into } j \\ -1 & \text{if } e \text{ is directed out of } j \\ 0 & \text{if } j \text{ is not a vertex of } e. \end{cases} \tag{2.8}$$

We say that φ is a flow on G (i.e. $\varphi\in F(G)$) if for each $j\in V$,

$$\partial\varphi(j)\equiv\sum_{e\in E} S_{je}\varphi(e)=0 \qquad \operatorname{mod}\lambda \tag{2.9}$$

i.e. the signed sum of the flows at each vertex is zero mod λ.

We note that it follows from (2.2) and (2.7) that

$$t_e(\lambda-\alpha)=t_e(\alpha) \tag{2.10}$$

and that $t_e(0)=1$. For the case of the Potts model $t_e(\alpha)$, for $\alpha\neq 0$, is independent of α and is given by (2.2) of PF3. Also in this case (2.6) reduces to (2.7) of PF3.

2.3. *The correlation function*

Now let us extend Bigg's result to pair correlation functions. Such a function will normally be the thermal average of some function $f(n_1-n_2)$ where, as usual, the difference of the state variables n_1, n_2, for arbitrarily chosen vertices 1 and 2, is calculated mod λ. The special vertices 1 and 2 are known as roots of the graph. Making a Fourier decomposition of f gives

$$\langle f(n_1-n_2)\rangle_{\text{thermal}}=\frac{1}{\lambda}\sum_{\alpha=0}^{\lambda-1} f_{\lambda-\alpha}T_\alpha(1,2;\,G) \tag{2.11}$$

where

$$\begin{aligned}T_\alpha(1,2;\,G)&=\langle\exp[-2\pi\mathrm{i}\alpha(n_1-n_2)/\lambda]\rangle_{\text{thermal}}\\&=\frac{1}{Z(G)}\sum_{n_1=0}^{\lambda-1}\cdots\sum_{n_\nu=0}^{\lambda-1}\exp[-2\pi\mathrm{i}\alpha(n_1-n_2)/\lambda]\prod_{e\in E}\exp[-h_e(n_i-n_j)].\end{aligned} \tag{2.12}$$

This definition, together with (2.2), implies that

$$T_{\lambda-\alpha}(1,2;\,G)=T_\alpha(1,2;\,G). \tag{2.13}$$

In PF1 the pair correlation function of the Potts model (given by the thermal average of $s_1\cdot s_2$) is related to the equivalent transmissivity $t_{12}^{\text{eq}}(G)$. This relation may be recovered as an example of (2.11) and (2.12) by letting

$$f(n_1-n_2)=s_1\cdot s_2=\lambda\delta(n_1-n_2)-1 \tag{2.14}$$

in which case $T_1(1,2,G)=T_2(1,2,G)=\ldots=T_{\lambda-1}(1,2,G)=t_{12}^{\text{eq}}(G)$ and $T_0(1,2,G)=1$ (see (2.22) below).

We now extend (2.6) in order to express $T_\alpha(1,2;\,G)$ in terms of flow generating functions. Inverting (2.7) gives

$$\exp[-h_e(n_i-n_j)]=\frac{z_e}{\lambda}\sum_{\beta=0}^{\lambda-1}\exp[-2\pi\mathrm{i}\beta(n_i-n_j)/\lambda]t_e(\beta) \tag{2.15}$$

and hence

$$\prod_{e\in E}\exp[-h_e(n_i-n_j)]=\lambda^{-\varepsilon}\left(\prod_{e\in E}z_e\right)\sum_{\varphi\in\Phi}\left[\exp\left(-\frac{2\pi i}{\lambda}\sum_{e\in E}\varphi(e)\delta n(e)\right)\prod_{e\in E}t_e(\varphi(e))\right] \tag{2.16}$$

where $\varphi(e)$ is the value of β on edge e, Φ is the set of all functions on E with values in the range $0,\ldots,\lambda-1$, and $\delta n(e)=n_i-n_j$. Now using lemma 1 of Biggs (1976), namely

$$\sum_{e\in E}\varphi(e)\delta n(e)=\sum_{j\in V}n_j\partial\varphi(j) \tag{2.17}$$

where $\partial\varphi(j)$ is defined in (2.9) and combining it with (2.4), (2.12) and (2.16) we obtain

$$T_\alpha(1,2;\,G)=N_\alpha(1,2;\,G)/D(G) \tag{2.18a}$$

where

$$N_\alpha(1,2;G)=\lambda^{-\nu}\sum_{\varphi\in\Phi}\sum_{n_1=0}^{\lambda-1}\cdots\sum_{n_\nu=0}^{\lambda-1}\exp\left[\frac{2\pi\mathrm{i}}{\lambda}\left((n_2-n_1)\alpha-\sum_{j\in V}n_j\partial\varphi(j)\right)\right]\prod_{e\in E}t_e(\varphi(e)). \tag{2.18b}$$

Using a well known property of the λ roots of unity, the sum over n_i yields a factor zero unless

$$\partial\varphi(i)=\begin{cases}-\alpha & \text{if } i=1\\ +\alpha & \text{if } i=2\\ 0 & \text{otherwise.}\end{cases} \tag{2.19}$$

This may be expressed by saying that the flow is conserved at every non-rooted vertex and that there is a net external flow α entering at root 1 and leaving at root 2. A flow which satisfies (2.19) will be called a *rooted α-flow*. The set of such flows will be denoted by $F_\alpha(G)$ and hence

$$N_\alpha(1,2;G)=\sum_{\varphi\in F_\alpha(G)}\prod_{e\in E}t_e(\varphi(e)). \tag{2.20}$$

We will call $\boldsymbol{N}(1,2;G)=\{N_\alpha(1,2;G),\ \alpha=0,1,\ldots,\lambda-1\}$ the *flow vector*, although strictly speaking each of its components is a flow generating function for rooted α-flows. Comparison between (2.20) and (2.6) shows that $D(G)=N_0(1,2;G)$ and we note that (2.18a) and (2.13) imply that

$$N_{\lambda-\alpha}(1,2;G)=N_\alpha(1,2;G). \tag{2.21}$$

For the Potts model (2.20) reduces, for $\alpha\neq 0$, to (see proof in the appendix)

$$\begin{aligned}N_1(1,2;G)=N_2(1,2;G)=\ldots=N_{\lambda-1}(1,2;G)&\equiv N_{12}(G)\\ &=\sum_{G'\subseteq G}F_{12}(\lambda,G')\prod_{e\in E'}t_e\end{aligned} \tag{2.22}$$

which is (4.3) of PF1. Here $F_{12}(\lambda,G')$ is the two-rooted flow polynomial defined in PF1.

In PF1 it was shown that the correlation function for a two-rooted Potts cluster is proportional to the transmissivity of a single effective edge with Hamiltonian defined in terms of a partial trace over the internal spins (see (3.15) of PF1). We now extend this result to the $Z(\lambda)$ model. Following the derivation of (2.18b) we can show that the sum of the left-hand side of equation (2.16) over all n except n_1 and n_2 (denoted below by Tr′) depends only on the difference n_1-n_2. It is therefore possible to define the equivalent Hamiltonian $h_{\mathrm{eq}}(n_1-n_2)$ by

$$\mathrm{Tr}'\left(\exp\left(-\sum_{e\in E}h_e(n_i-n_j)\right)\right)=C\exp[-h_{\mathrm{eq}}(n_1-n_2)] \tag{2.23}$$

where C is a constant. Carrying out the further sum over n_1 and n_2 with and without the factor $\exp[-2\pi\mathrm{i}(n_1-n_2)\alpha/\lambda]$ and taking the ratio of the results gives, using (2.7) and (2.12),

$$T_\alpha(1,2;G)=t_{\mathrm{eff}}(\alpha) \tag{2.24}$$

where $t_{\mathrm{eff}}(\alpha)$ is the component α of the vector transmissivity of a single pair of atoms 1 and 2 interacting with Hamiltonian $h_{\mathrm{eq}}(n_1-n_2)$. We therefore call $\boldsymbol{T}(1,2;G)=\{N_\alpha(1,2;G)/N_0(1,2;G),\ \alpha=0,1,\ldots,\lambda-1\}$ the equivalent vector transmissivity between the roots 1 and 2 of G. For $Z(4)$, $\boldsymbol{T}(1,2;G)$ is the equivalent vector transmissivity of Mariz *et al* (1985) which they denote by $\boldsymbol{G}$.

3. Graph reduction equations of the SBCM

In this section we extend the equations of the SBCM algorithm from the Potts model to the $Z(\lambda)$ model. The major step is to replace the denominator and numerator of the equivalent transmissivity of an effective edge used in the Potts model by a flow vector.

In PF3, three ways were used to reduce the size of the graph under consideration: (a) splitting into pieces at articulation vertices; (b) replacement of subgraphs attached at only two vertices by *effective edges*; (c) removal of (effective) edges using an effective break-collapse equation. The first of these was made possible by the fact that the correlation function for an articulated graph might be factorised. Secondly, three types of subgraph were considered for replacement by an effective edge, namely (a) edges in series, (b) edges in parallel, and (c) subgraphs which were not combinations of series and/or parallel edges. The latter was referred to briefly as non-reducible subgraph replacement. Finally edge removal might only be carried out at the expense of replacing the graph by two further graphs; one in which the edge was deleted and the other in which the edge was contracted. It was therefore only used as a last resort when the replacement by effective edges was not possible. All three ways were used recursively and applied to effective edges and subgraphs containing effective edges as well as ordinary edges. The formulae which enabled the reduction processes to be carried out were derived for subgraphs, with the understanding that they could be used for effective edges since the latter can always be expanded into subgraphs.

3.1. Splitting of articulated graphs

Suppose that G is separated into two subgraphs G_1 and G_2 by an articulation vertex i (see figure 1). There are two cases to consider depending on whether both roots 1 and 2 are in the same subgraph (figure 1(a)) or whether there is one root in G_1 and one root in G_2 (figure 1(b)). In the latter case we suppose that $i \neq 1$ or 2 and the graphs are said to be in series.

3.1.1. Both roots in G_1. If $i \neq 1$ or 2, by the conservation condition (2.19), any flow in $F_\alpha(G)$ is such that the signed sum of $\varphi(e)$ over the edges of G_1 incident with i is zero, and hence we say that there is no flow between G_1 and G_2. It follows that any flow in $F_\alpha(G_1)$ combined with any flow in $F(G_2)$ gives rise to a flow in $F_\alpha(G)$ and

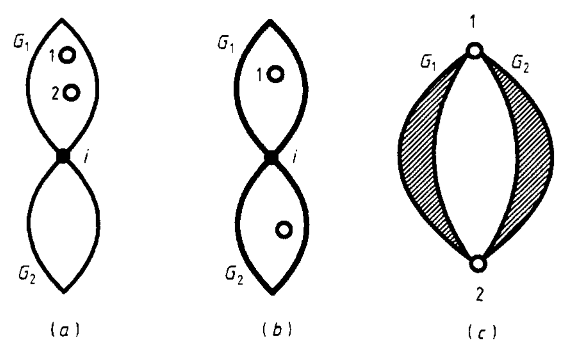

Figure 1. Pictorial representations of two graphs G_1 and G_2 which share an articulation vertex i ((a) and (b)) or which are in parallel (c). The roots 1 and 2 are represented by small circles and unrooted vertices by full dots. In (b) the graphs G_1 and G_2 are in series.

that all such flows may be obtained in this way. The sum in (2.20) may therefore be reorganised as follows:

$$N_\alpha(1,2;\,G)=\sum_{\varphi_1\in F_\alpha(G_1)}\ \sum_{\varphi_2\in F(G_2)}\ \prod_{e\in E_1} t_e(\varphi_1(e))\prod_{e\in E_2} t_e(\varphi_2(e)) \tag{3.1}$$

and hence, using (2.6) and (2.20),

$$N_\alpha(1,2,\,G)=N_\alpha(1,2;\,G_1)D(G_2). \tag{3.2}$$

If $i=2$, there is a net flow of α into i from G_1 and again using (2.19) there is no flow into G_2 since $\partial\varphi(2)=\alpha$. Similarly there is no flow into G_2 when $i=1$ and (3.1) holds in all cases.

3.1.2. G_1 and G_2 are in series. For graphs in series (figure 1(b)) (2.19) implies that there is a flow of α from G_1 to G_2 and hence any flow in $F_\alpha(G)$ may be composed from a flow in $F_\alpha(G_1)$ and a flow in $F_\alpha(G_2)$. As in § 3.1.1 the sum in (2.20) may be factorised:

$$\begin{aligned}N_\alpha(1,2;\,G)&=\sum_{\varphi_1\in F_\alpha(G_1)}\ \sum_{\varphi_2\in F_\alpha(G_2)}\ \prod_{e\in E_1} t_e(\varphi_1(e))\prod_{e\in E_2} t_e(\varphi_2(e))\\&=N_\alpha(1,i;\,G_1)N_\alpha(i,2;\,G_2).\end{aligned} \tag{3.3}$$

For two edges in series (3.3) reduces to (9) of Alcaraz and Tsallis (1982).

3.2. Parallel combination of graphs

Suppose now that G is composed of two subgraphs G_1 and G_2 having only the root vertices 1 and 2 in common (see figure 1(c)). Suppose that the flow into the edges of G_1 which are incident with root 1 is β, then the flow into G_2 is $\alpha-\beta$. Subdividing the flows on G according to the value of β gives

$$\begin{aligned}N_\alpha(1,2;\,G)&=\sum_{\beta=0}^{\lambda-1}\ \sum_{\varphi_1\in F_\beta(G_1)}\ \sum_{\varphi_2\in F_{\alpha-\beta}(G_2)}\ \prod_{e\in E_1} t_e(\varphi_1(e))\prod_{e\in E_2} t_e(\varphi_2(e))\\&=\sum_{\beta=0}^{\lambda-1} N_\beta(1,2;\,G_1)\sum_{\varphi_2\in F_{\alpha-\beta}(G_2)}\ \prod_{e\in E_2} t_e(\varphi_2(e))\\&=\sum_{\beta=0}^{\lambda-1} N_\beta(1,2;\,G_1)N_{\alpha-\beta}(1,2;\,G_2)\end{aligned} \tag{3.4}$$

which, for the Potts model, reduce to (4.14a) and (4.14b) of PF3 in the respective cases of $\alpha\neq 0$ and $\alpha=0$.

Equation (3.4) has the form of a convolution and hence the discrete Fourier transform:

$$\tilde{N}_\beta(1,2;\,G)=\sum_{\alpha=0}^{\lambda-1}\exp(2\pi \mathrm{i}\alpha\beta/\lambda)N_\alpha(1,2;\,G) \tag{3.5}$$

may be factorised as

$$\tilde{N}_\beta(1,2;\,G)=\tilde{N}_\beta(1,2;\,G_1)\tilde{N}_\beta(1,2;\,G_2). \tag{3.6}$$

The N_α may be polynomials in several variables and the product of two of these polynomials is usually the most time-consuming operation in the determination of $N_\beta(1,2;\,G)$. If so, then taking the Fourier transform, using the product rule (3.6) and then inverting is more efficient than the direct convolution. The advantage of the transform method increases with the number of graphs in parallel; if there are n such graphs then

$$\tilde{N}_\beta(1,2;\,G)=\prod_{k=1}^{n}\tilde{N}_\beta(1,2;\,G_k). \tag{3.7}$$

For the Potts model, $\tilde{N}_0$ is the X of PF3 (4.15c), $\tilde{N}_1$ is Y (4.15d) and the inversion of (3.7) leads for $\beta=0$ and $\beta\neq0$ to the respective equations (4.15a) and (4.15b) of PF3. In the case that G is the single edge $[1,2]$ we note that $\tilde{N}_\beta/\lambda$ is equal to the probability $p^{(\beta)}$ defined by Alcaraz and Tsallis (1982) and that $\tilde{N}_\beta/\tilde{N}_0$ is their $t(\beta)^D$, i.e. the dual variable of $t(\beta)$. For a pair of edges in parallel (3.6) leads to (11) of Alcaraz and Tsallis (1982).

3.3. *Replacement of a subgraph by an effective edge*

We now consider a generalisation of the parallel combination formula which allows the size of a graph to be reduced by replacing a subgraph by a single edge. In order for this to be possible G must be the union of two subgraphs H and L which have only two vertices i and j in common. Furthermore, both of the root points must be in H (see figure 2) with the possibility that i and/or j are rooted. The case when both i and j are rooted is the parallel combination above. In general the flows in $F_\alpha(G)$ may again be subdivided, but this time, according to the flow β into L at i (and out at j), by which we mean the signed sum of $\varphi(e)$ over the edges of L incident with j. By the conservation condition this implies an additional flow into H at j (and out at i). The generalisation of (3.4) is therefore

$$N_\alpha(1,2;\,G)=\sum_{\beta=0}^{\lambda-1}N_\beta(i,j;\,L)N_{\alpha\beta}(1,2;j,i;\,H) \tag{3.8a}$$

with

$$N_{\alpha\beta}(1,2;j,i;\,H)=\sum_{\varphi\in F_{\alpha\beta}(H)}\ \prod_{e\in E_H}t_e(\varphi(e)) \tag{3.8b}$$

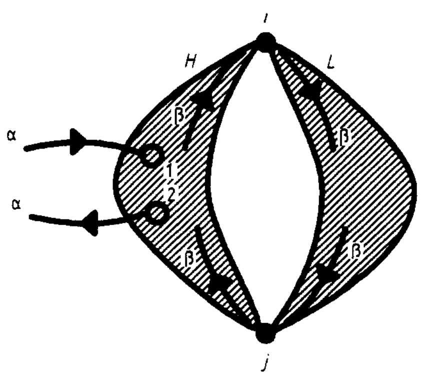

Figure 2. Pictorial representation of a two-reducible graph $G=L\cup H$ with the roots 1 and 2 in H and subjected to an external flow α in at 1 and out at 2. A net flow β goes from H to L at i and from L to H at j. Each graph is represented by a half-moon shape.

where $F_{\alpha\beta}(H)$ is the set of mod-λ flows on H with an external flow α in at 1 and out at 2 and β in at j and out at i. If $i=1$, the net flow in H at the common vertex is $\alpha-\beta$ and if, in addition $j=2$, then there is net flow in H of $\alpha-\beta$ out at 2, in agreement with (3.4).

The sum over β in (3.8*a*) followed by the sum over φ in (3.8*b*) may be replaced by a single sum over flows in $F_\alpha(H\cup e_L)$, where e_L is an effective edge replacing the subgraph L and having flow vector equal to the flow vector of L. This result may be summarised by

$$N_\alpha(1,2;\,G)=N_\alpha(1,2;\,H\cup e_L). \tag{3.9}$$

This replacement may be repeated as long as there are further subgraphs which satisfy the above conditions on L so that the flow vector of G may be equal to that of a graph with several effective edges. The subgraph selected for substitution may itself contain effective edges. Figure 3 shows an example of successive replacements.

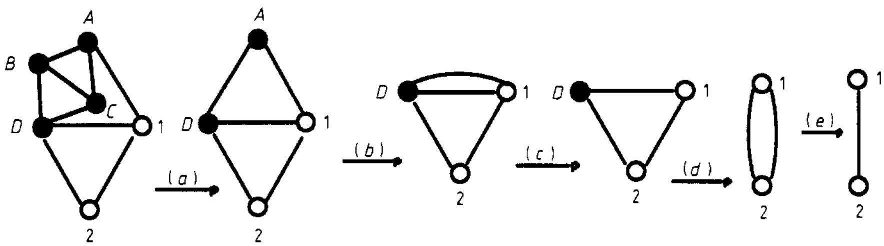

Figure 3. An example of successive replacements of a subgraph by an effective edge carried out in the application of the SBCM algorithm. Step (*a*) shows a non-reducible subgraph replacement. In (*b*) and (*d*) ((*c*) and (*e*)) two effective edges in series (in parallel) are replaced by a new effective edge.

Equation (3.9) could also be obtained by performing a partial trace over the internal vertices of L as in the derivation of (2.24). Our use of effective edges here is consistent with that in § 2 since (2.24) may be rederived by replacing H by a pair of isolated root points.

The simplest case of subgraph replacement is when L consists of a pair of edges in series. These edges may or may not be effective but in any case it follows from (3.3) that $N_\beta(i,j;\,L)$ may be calculated by multiplying the β components of the flow vectors of the two edges. Similarly, when L is the parallel combination of two (effective) edges the equivalent flow vector is obtained using the Fourier transform technique of § 3.2. A replacement which is made when no series or parallel combination of edges exists will be called, as in PF3, a non-reducible subgraph replacement. In this case the calculation of $N_\beta(i,j;\,L)$ is a subproblem of the same type as the calculation of $N_\alpha(1,2;\,G)$, which is one reason why the SBCM is recursive.

3.4. *Effective break-collapse equation*

When G is such that no further subgraph replacements may be made then further reduction methods must be considered. In the case of the Potts model a formula known as the effective break-collapse equation was used (see PF3). This will now be rederived and extended to the $Z(\lambda)$ model. Let f be an edge of G, possibly effective, and subdivide the flows in $F_\alpha(G)$ according to the flow $\beta=\varphi(f)$ in f. Definition (2.20) gives

$$N_\alpha(1,2;G)=\sum_{\beta=0}^{\lambda-1} t_f(\beta)N_{\alpha\beta}(1,2;f;G) \tag{3.10}$$

where $N_{\alpha\beta}(1,2;f;G)$ is the generating function of the flows on G with external flow α in at 1 and out at 2 and a fixed flow β in the edge f. We call such an edge f a *frozen edge.* Observe that a fixed flow β from i to j in the edge $f=[i,j]$ is equivalent to an external flow β in at j and out at i with f deleted. Therefore $N_{\alpha\beta}(1,2;f;G)$ is equal to $N_{\alpha\beta}(1,2;j,i;H)$ in (3.8), where H is the 'broken graph' G_f^δ obtained from G by deleting the edge f. If $\beta=0$ then

$$N_{\alpha 0}(1,2;f;G)=N_\alpha(1,2;G_f^\delta) \tag{3.11a}$$

and in particular

$$N_{00}(1,2;G)=D(G_f^\delta). \tag{3.11b}$$

If f is the edge $[i,j]$ of G we denote by G_f^γ the 'collapsed graph' obtained from G_f^δ by identifying the vertices i and j. The flows on G_f^γ may be obtained from those on G by restriction to the edge set $E\backslash f$ and, hence, the generating function for these flows may be found by setting $t_f(\beta)=1$ in (3.10), i.e.

$$N_\alpha(1,2;G_f^\gamma)=\sum_{\beta=0}^{\lambda-1} N_{\alpha\beta}(1,2;f;G). \tag{3.12}$$

Therefore to contract an edge is equivalent to summing over all possible flows for this frozen edge.

The effective break-collapse equations for the models considered below follow from (3.10) using (3.11) and (3.12).

3.4.1. The λ-state Potts model. For the Potts model, if f is an effective edge then $t_f(0)=D_{\text{eff}}$ and for $\beta>0$, $t_f(\beta)=N_{\text{eff}}$ (the same for all $0<\beta<\lambda$). Thus in this case all components of the flow vector are determined by the flow vectors for the broken and collapsed graphs. From (3.10) and (3.11a) we obtain

$$N_\alpha(1,2;G)=D_{\text{eff}}N_\alpha(1,2;G_f^\delta)+N_{\text{eff}}\sum_{\beta=1}^{\lambda-1} N_{\alpha\beta}(1,2;f;G) \tag{3.13}$$

and using (3.12) and (3.11a)

$$\begin{aligned} N_\alpha(1,2;G)&=D_{\text{eff}}N_\alpha(1,2;G_f^\delta)+N_{\text{eff}}[N_\alpha(1,2;G_f^\gamma)-N_\alpha(1,2;G_f^\delta)]\\ &=(D_{\text{eff}}-N_{\text{eff}})N_\alpha(1,2;G_f^\delta)+N_{\text{eff}}N_\alpha(1,2;G_f^\gamma) \end{aligned} \tag{3.14}$$

which is the 'effective break-collapse' rule (4.13) of PF3.

3.4.2. The Z(4) model. For the $Z(4)$ model, because of the symmetry condition (2.21), the flow vector of an effective edge can have at most three different components:

$$\boldsymbol{N}_{\text{eff}}=(D_{\text{eff}},N_{1\text{eff}},N_{2\text{eff}},N_{1\text{eff}}).$$

Equation (3.10) and (3.11a) now yield

$$\begin{aligned} N_\alpha(1,2;G)=D_{\text{eff}}N_\alpha(1,2;G_f^\delta)&+N_{1\text{eff}}[N_{\alpha 1}(1,2;f;G)+N_{\alpha 3}(1,2;f;G)]\\ &+N_{2\text{eff}}N_{\alpha 2}(1,2;f;G) \end{aligned}$$

which, combined with (3.12), leads to

$$N_\alpha(1,2;\,G)=(D_{\text{eff}}-N_{1\text{eff}})N_\alpha(1,2;\,G_f^\delta)+N_{1\text{eff}}N_\alpha(1,2;\,G_f^\gamma)$$
$$+(N_{2\text{eff}}-N_{1\text{eff}})N_{\alpha 2}(1,2;f;\,G) \tag{3.15}$$

where the third term did not exist for the Potts model. Thus, in addition to the broken and collapsed graphs required for the Potts model we must also consider the graph G with the chosen edge frozen with a fixed flow of 2.

3.4.3. The $Z(\lambda)$ model. The extension of (3.15) to general λ may be obtained by subtracting $t_f(1)$ times (3.12) from (3.10). Writing $t_f(\beta)=N_{\beta\text{eff}}$ we have, for $\lambda\geqslant 4$,

$$N_\alpha(1,2;\,G)=(D_{\text{eff}}-N_{1\text{eff}})N_\alpha(1,2;\,G_f^\delta)+N_{1\text{eff}}N_\alpha(1,2;\,G_f^\gamma)$$
$$+\sum_{\beta=2}^{\lambda-2}(N_{\beta\text{eff}}-N_{1\text{eff}})N_{\alpha\beta}(1,2;f;\,G) \tag{3.16}$$

and, in addition to the flow vectors for the broken and collapsed graphs, there are now a further $\lambda-3$ vectors corresponding to the edge f being frozen with fixed flows from 2 to $\lambda-2$.

4. SBCM and BCM for the $Z(\lambda)$ model

In this section we generalise the SBCM algorithm for the Potts model described in PF3 to the $Z(\lambda)$ model. Furthermore, we extend to effective edges the formulae which appear in the BCM of Mariz and co-workers, and interpret their precollapsed bonds in terms of frozen edges. Finally, we illustrate the SBCM using the Wheatstone bridge $Z(4)$ cluster and compare it with the results of MTF obtained through their break-collapse algorithm.

4.1. The SBCM algorithm

The SBCM algorithm of PF3, for the Potts model, uses a recursive procedure T which executes the operations of splitting into pieces and series, parallel and non-reducible subgraph replacement as long as possible and then uses the effective break-collapse equation. Non-reducible subgraph replacement and use of the effective break-collapse equation both require calls to T and hence the need for recursion. The procedure terminates when a graph with only two vertices is obtained, at which point the equivalent transmissivity is calculated by the parallel rule. Three main changes need to be made in order to extend this algorithm to the $Z(\lambda)$ model.

(i) Firstly the effective break-collapse equation must be replaced by (3.16), which entails calculating $N_{\alpha\beta}(1,2;f;\,G)$ for $\beta=2$ to $\lambda-2$. This may be achieved by replacing step $(d4)$ of the algorithm by a loop containing a further call to T for the graph G with the flow vector for edge f replaced by a constant vector representing the fixed flow β. For $Z(4)$ the only fixed flow required is 2 which is represented by the vector $(0,0,1,0)$. With this replacement the series and parallel equations work without modification.

(ii) Step $(d1)$ of the algorithm selects an (effective) edge for application of the effective break-collapse equation. The edge selected must now not be a frozen edge, nor must it be an (effective) edge, the flow in which is already determined by the flow in the frozen edges together with the conservation condition.

(iii) A further terminal step must be added before the terminal condition mentioned in (IIe) of PF3. This is used when the current graph has more than two vertices and yet no further applications of the effective break-collapse rule are necessary since the flow in all (effective) edges is already determined by the flows on the frozen edges, the external flow and the conservation condition. The component α of the flow vector for the graph is now determined by calculating the implied flow in each (effective) edge and taking the product of the appropriate components of the flow vectors for these edges. This terminating condition will arise when the number of frozen edges is equal to the number of independent cycles in the graph. In this case we call it a *frozen graph.*

4.2. *The* BCM *for effective edges*

4.2.1. The Z(4) model. Mariz *et al* (1985) presented for the $Z(4)$ model a break-collapse equation similar to (3.15), but in terms of ordinary edges rather than effective edges. In addition to the flow vectors for the broken and collapsed graphs they used a third flow vector $N_\alpha^{\mathrm{bc}}(1, 2;\, G)$ defined for the graph G with the chosen edge, f, 'precollapsed'. This was defined to be $N_\alpha(1, 2;\, G)$ with $t_f(0)=1$, $t_f(1)=t_f(3)=0$ and $t_f(2)=1$. Interpretation in terms of flows was not mentioned in the MTF paper but from (3.10) we obtain

$$N_\alpha^{\mathrm{bc}}(1, 2;\, G) = N_{\alpha 0}(1, 2;\, G) + N_{\alpha 2}(1, 2;\, f;\, G) \tag{4.1}$$

from which it follows that $N_\alpha^{\mathrm{bc}}(1, 2;\, G)$ is the generating function for internal flows having value 0 or 2 on the chosen edge f and subject to an external flow α entering at 1 and leaving at 2. Equation (4.1) combined with (3.15) and restricted to ordinary edges (i.e. non-effective edges) yields (8) of MTF. Mariz *et al* (1985) applied the latter equation recursively until graphs with *all* edges precollapsed are arrived at. For such a graph (which we will denote by G_{pr}) $N_\alpha(1, 2;\, G_{\mathrm{pr}})$ is the number of rooted mod-4 α flows with the constraint that the flow on any edge must be 0 or 2. Such flows will be called even flows. Tsallis (1988) has stated without proof that

$$N_1(1, 2;\, G_{\mathrm{pr}}) = N_3(1, 2;\, G_{\mathrm{pr}}) = 0 \tag{4.2a}$$

$$N_2(1, 2;\, G_{\mathrm{pr}}) = 2^{c(G_{\mathrm{pr}})}\gamma_{12}(G_{\mathrm{pr}}) \tag{4.2b}$$

$$D(G_{\mathrm{pr}}) = 2^{c(G_{\mathrm{pr}})} \tag{4.2c}$$

where $c(G_{\mathrm{pr}})$ is the number of independent cycles in G_{pr}; $\gamma_{12}(G_{\mathrm{pr}})$ is 1 if the roots are connected and zero otherwise. We now argue that these results follow directly from our interpretation in terms of flows. We first note that, for even flows, $\partial\varphi(i)$ is even for all i, and hence (2.19) can only be satisfied when α is even. Equation (4.2a) therefore follows immediately. Further we note that there is a correspondence (bijection) between the even rooted mod-4 2-flows and the unrestricted mod-2 1-flows obtained by replacing edges with flow 2 by edges with flow 1. Equation (4.2b) follows from the fact that the number of unrestricted rooted mod-2 1-flows is $2^{c(G_{\mathrm{pr}})}$, when the roots are connected on G_{pr}, and 0 otherwise (see PF1). Equation (4.2c) results from a similar correspondence between even mod-4 flows and unrestricted mod-2 flows.

4.2.2. The Z(λ) model. Using (2.21) we can rearrange (3.16) for $\alpha = 0, 1, \ldots, \bar{\lambda}$ in the following form:

$$N_\alpha(1,2;G)=\left(D_{\text{eff}}+(\bar\lambda-2)N_{1\text{eff}}-\sum_{\beta=2}^{\bar\lambda}N_{\beta\text{eff}}\right)N_\alpha(1,2;G_f^\delta)+N_{1\text{eff}}N_\alpha(1,2;G_f^\gamma)$$

$$+\sum_{\beta=2}^{\bar\lambda}(N_{\beta\text{eff}}-N_{1\text{eff}})N_\alpha^{bb\ldots c\ldots b}(1,2;G) \tag{4.3}$$

with the superscript c occupying the βth position. $N_\alpha^{bb\ldots c\ldots b}(1,2;G)$ is defined (Tsallis private communication) as $N_\alpha(1,2;G)$ with the chosen edge f being a precollapsed edge of type β, i.e.

$$t_f(\gamma)=\begin{cases}1 & \text{if } \gamma=0,\beta \text{ or } \lambda-\beta\\ 0 & \text{otherwise.}\end{cases} \tag{4.4}$$

Using (3.10) it follows that for $\beta=2,3,\ldots,\bar\lambda$

$$N_\alpha^{b\ldots c\ldots b}(1,2;G)=N_{\alpha 0}(1,2;f;G)+N_{\alpha\beta}(1,2;f;G)+N_{\alpha,\lambda-\beta}(1,2;f;G) \tag{4.5}$$

where for λ even and $\beta=\lambda/2$ the last two terms become equal and should be included only once.

Equation (4.3) reduces, when f is an ordinary edge, to the break-collapse equation conjectured by Mariz *et al* (1985, 1988) for $\lambda=4$ and 6, and by Mariz and Tsallis (Tsallis private communication) for a general value of λ. In the break-collapse algorithm of Mariz and co-workers, (4.3) is applied as many times as needed to arrive at graphs G_{pr} with all edges precollapsed. But, since G_{pr} can contain, for $\lambda>4$, different types of precollapsed edges, there are no simple formulae for the components of their flow vectors such as the ones for $\lambda=4$ (see 4.2). Their calculation involves the explicit enumeration of all mod-λ flows which can take the values 0, β or $\lambda-\beta$ on each precollapsed edge of type β.

4.3. An illustration of the SBCM for the Z(4) model and comparison with the BCM

Now let us illustrate the SBCM for the $Z(4)$ model by calculating the equivalent transmissivity of the Wheatstone bridge graph G of figure 4. The same calculation was carried out in MTF using their BCM algorithm and we are therefore able to compare the number of steps required by the two algorithms. We note that with the choice $h(0)=-K_1-2K_2$, $h(1)=h(3)=K_1+2K_2$ and $h(2)=3K_1-2K_2$, the Hamiltonian (2.1) can be written in terms of two coupled Ising variables as in (1) of MTF. The transmissivity components $t(1)$ and $t(2)$ defined by (2.7), when expressed in terms of K_1 and K_2, are seen to be the parameters t_1 and t_2 defined in (2a) and (2b) of MTF. We shall use their notation in the rest of this section and assume that the vector transmissivity $\boldsymbol{t}_e=\boldsymbol{t}$, the same for all edges.

Applying (3.15), to graph G in figure 4, with f being the edge e_3, we get for $\alpha=0,1,2$:

$$N_\alpha(1,2;G)=(1-t_1)N_\alpha(1,2;G_b)+t_1N_\alpha(1,2,G_c)+(t_2-t_1)N_{\alpha 2}(1,2;e_3,G_d) \tag{4.6}$$

where the graphs G_b, G_c and G_d are shown in figure 4. The terms corresponding to the deleted graph G_b and contracted graph G_c can be easily calculated by using the series and parallel equations (3.3) and (3.4). The expressions for the flow vectors for these graphs agree with (9a)–(9f) of MTF, where their superscripts bb and cc refer to our graphs G_b and G_c, respectively.

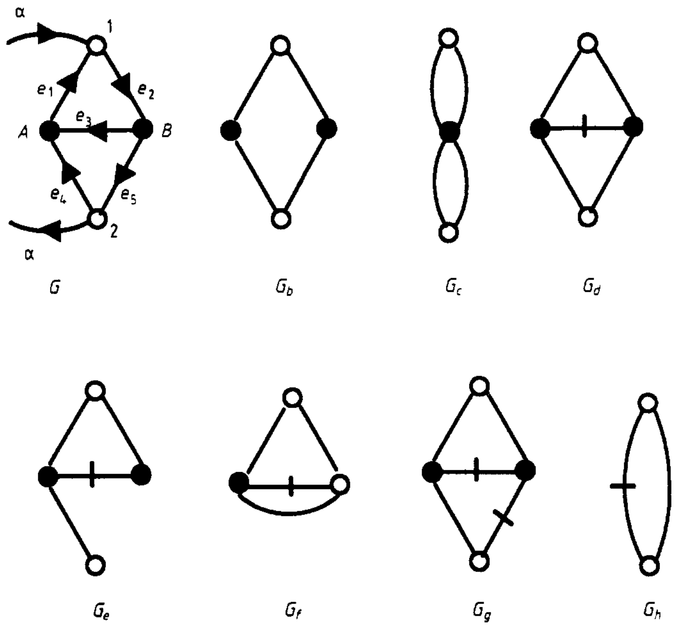

Figure 4. Graphs generated during the application of the SBCM to the $Z(4)$ two-rooted graph G. Each edge e_i is given an arbitrary directing indicated by the arrow. α represents the external flow in at the root 1 and out at the root 2. The barred line indicates a frozen edge with flow 2. To each non-frozen edge is associated a vector transmissivity $\boldsymbol{t} = (1, t_1, t_2, t_1)$.

In order to calculate the last term of (4.6), we apply (3.15) to the graph G_d with the edge f chosen to be e_5:

$$\begin{aligned} N_{\alpha 2}(1, 2; e_3; G_d) \\ &= (1-t_1)N_{\alpha 2}(1, 2; e_3; G_e) + t_1 N_{\alpha 2}(1, 2; e_3; G_f) \\ &\quad + (t_2 - t_1)N_{\alpha 22}(1, 2; e_3, e_5; G_g) \qquad (\alpha = 0, 1, 2) \end{aligned} \tag{4.7}$$

where G_e, G_f and G_g are shown in figure 4.

G_e and G_g are frozen graphs so that the flows in each edge are determined by the flow on the frozen edges together with the external flow α. This yields the results in table 1. The flow vector of G_f is obtained by replacing the subgraph G_h by an effective edge with flow vector equal to that of an ordinary edge with flow vector $(1, t_1, t_2, t_1)$ in parallel with a frozen edge with flow 2 whose flow vector is $(0, 0, 1, 0)$. Using (3.4)

Table 1. Intermediate flow vectors used in the calculation of the flow vector of the graph G in figure 4.

	$\alpha = 0, 2$	$\alpha = 1, 3$
$N_{\alpha 2}(1, 2; e_3; G_e)$	t_2^2	$t_1^2 t_2$
$N_{\alpha 2}(1, 2; e_3; G_f)$	$t_2 + t_2^2 + 2t_1^3$	$t_1(t_1 + 2t_2 + t_1 t_2)$
$N_{\alpha 22}(1, 2; e_3, e_5; G_g)$	t_2	t_1^2
$N_{\alpha 2}(1, 2; e_3; G_d)$	$2(t_2^2 + t_1^4)$	$4t_1^2 t_2$

the flow vector of G_h is $(t_2, t_1, 1, t_1)$ and that of G_f is now easily obtained using the series-parallel rules (3.3) and (3.4). The result is given in table 1 along with that for G_d obtained by substitution in (4.7).

Finally using (9*a*)-(9*f*) of MTF with the result for G_d in (4.6) leads to

$$D(G)=1+4t_1^3+2t_2^3+2t_1^4+t_2^4+4t_1^3t_2^2+2t_1^4t_2 \tag{4.8a}$$

$$N_1(1,2;\,G)=2t_1^2(1+t_1+3t_2^2+2t_1t_2+t_1t_2^2) \tag{4.8b}$$

and

$$N_2(1,2;\,G)=2(t_2^2+t_2^3+4t_1^3t_2+t_1^4+t_1^4t_2) \tag{4.8c}$$

which agree with (6) and (7) of MTF when the latter are specialised to the isotropic case.

Notice that combining (4.1), the result for G_d and (9*a*)-(9*c*) of MTF we recover their expressions (9*g*)-(9*i*) for $N_\alpha^{\mathrm{bc}}(1,2;\,G)$ $(\alpha=0,1,2)$ as expected. In the case of the Potts model $(t_1=t_2)$ (4.8) reproduce (5) of Tsallis and Levy (1981). It is worthwhile stressing that the application of the SBCM to the graph G (figure 4) involved the use of the effective break-collapse equation (3.15) twice, which generated five graphs, the flow vectors of which were easily computed by the series and parallel equations. On the other hand, Mariz *et al* (1985) applied their break-collapse equation five times generating eleven graphs which are combinations of series and/or parallel edges. Therefore, for the $Z(4)$ model on the graph G, our method was more efficient than the BCM of MTF since it required a smaller number of iterations.

5. Conclusions

We have generalised to the $Z(\lambda)$ model the subgraph break-collapse method (SBCM) of the Potts model which we presented in a previous paper (de Magalhães and Essam 1988). The essential change is to replace the denominator and numerator (D, N) of the equivalent transmissivity of an effective edge used in the Potts model by a flow vector $(N_0, N_1, \ldots, N_{\lambda-1})$. The effective break-collapse equation contains graphs with frozen edges having fixed flows in addition to the broken and collapsed graphs which appear in the Potts model. Detailed modifications of the SBCM algorithm are given in § 4.1.

In an alternative algorithm of Mariz and co-workers known as the break-collapse method (BCM), graphs with precollapsed edges were considered rather than graphs with frozen edges. The effective break-collapse equation with frozen edges (3.16) generates $(\lambda-1)$ flow vectors, while the one with precollapsed edges (4.3) leads to $[\lambda/2]+1$ flow vectors. Therefore, for $\lambda>4$, the BCM generates, in each iteration, less flow vectors to be computed than the SBCM. On the other hand, in the BCM more iterations are needed, and the determination of the flow vectors for the terminal graphs (i.e. graphs with all edges precollapsed) requires the examination of all the mod-λ flows that can be formed such that the flow on each precollapsed edge of type β is 0, β or $\lambda-\beta$. This is an enumeration problem, the computing time for which grows exponentially with the number of cycles in the graphs, except in the case of $\lambda=4$ for which formulae are available (4.2). By comparison, our terminal graphs are frozen graphs having a fixed flow and the flow vector is therefore immediately determined (see § 4.1).

We have shown by example that, for $\lambda = 4$, our algorithm takes less steps than the BCM of Mariz and co-workers. Taking into account the considerations of the previous paragraph, we believe that on balance, even for values of λ greater than 4, our algorithm is still the most efficient.

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Appendix. Proof that $N_\alpha(1,2;G)$ for $\alpha \neq 0$ is independent of α in the case of the Potts model

This result is stated in (2.22).

In § 2 of PF1 it is shown how one can generate, in the Potts model, all the $\lambda^{c(G)}$ ($c(G)$ is the number of independent cycles in G) mod-λ flows in a graph G. For this, one chooses a spanning tree τ on G. Each edge not in τ defines an independent cycle formed by the chosen edge together with the unique path in τ which joins the endpoints of the edge. The primitive flows in the set of cycles so formed provides a basis in the cycle space of G. For example, for the graph G of figure 4, if we choose the spanning tree drawn in figure 5(a), then we obtain the independent cycles C_1 and C_2 shown in figure 5(b) and (c), respectively. All possible mod-λ flows can be generated by assigning the values $0, 1, \ldots, \lambda - 1$ to the strength f_i of the flow in each one of these cycles (the strength of the flow may be taken as the value of the flow on the edge of the cycle not in τ since this occurs in exactly one of the independent cycles).

Let us now suppose that the roots 1 and 2 of G are connected (i.e. $\gamma_{12}(G) = 1$), otherwise $N_\alpha(1,2;G)$ is zero. In order to generate the rooted α-flows ($\alpha = 1, 2, \ldots, \lambda - 1$) we now add to each of the above $\lambda^{c(G)}$ unrooted flows, a flow having value α on the unique path θ in τ from 1 to 2 and zero on all other edges. (Figure 5(d) shows

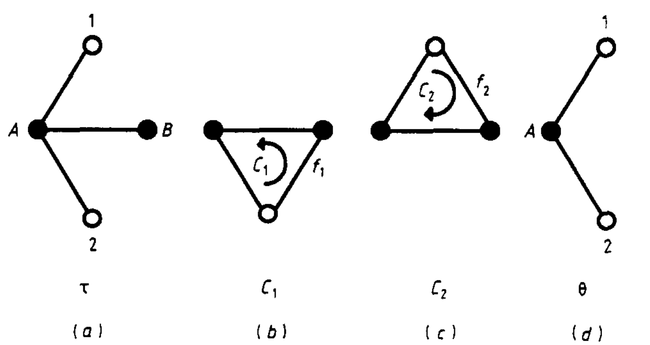

Figure 5. An arbitrary spanning tree τ (see (a)) of the graph G (figure 4) and its corresponding independent cycles C_1 (b) and C_2 (c). f_i indicates the strength of the flow in each cycle C_i ($i = 1, 2$). The path θ which connects the roots is shown in (d).

such a path for the forest drawn in figure 5(a).) This will generate, for any fixed α, all the rooted α-flows which occur in the sum defining $N_\alpha(1, 2; G)$ without duplication because of the independence of the cycles. Although, for any values of α_1 and $\alpha_2 \neq \alpha_1$ ($\alpha_i = 1, 2, \ldots, \lambda - 1$), the rooted α_1-flows are different from the rooted α_2-flows, the number of flows is equal to $\lambda^{c(G)}$ in both cases. The total number of rooted α-flows is therefore independent of α.

We now argue that the number $F_{12}^{(\alpha)}(\lambda, G')$ of proper rooted α-flows (flows which are non-zero on every edge) on any partial graph G' of G is also independent of α. This was implicitly assumed in PFI where the number of such flows was denoted by $F_{12}(\lambda, G)$. For $\gamma_{12}(G') \neq 0$ let us partition the total number $\lambda^{c(G')}$ of rooted α-flows (proper and improper) on G' according to the subset of edges on which they are proper (non-zero), which gives for any $G' \subseteq G$:

$$\gamma_{12}(G')\lambda^{c(G')} = \sum_{E'' \subseteq E'} F_{12}^{(\alpha)}(\lambda, G'') \tag{A1}$$

where the factor $\gamma_{12}(G')$ has been included on the left since the number of α-flows is zero when the roots are not connected on G'. In the latter circumstance the right-hand side is also zero since the number of proper flows is zero on any subgraph of G'. The above equation may be inverted (Rota 1964) to yield

$$F_{12}^{(\alpha)}(\lambda, G') = \sum_{E'' \subseteq E'} (-1)^{|E' \setminus E''|} \gamma_{12}(G'')\lambda^{c(G'')}. \tag{A2}$$

Since the right-hand side of (A2) is independent of α, it follows that

$$F_{12}^{(1)}(\lambda, G') = F_{12}^{(2)}(\lambda, G') = \ldots = F_{12}^{(\lambda-1)}(\lambda, G') \equiv F_{12}(\lambda, G'). \tag{A3}$$

On the other hand, (2.20) particularised for the Potts model gives, for $\alpha = 1, 2, \ldots, \lambda - 1$,

$$N_\alpha(1, 2; G) = \sum_{G' \subseteq G} F_{12}^{(\alpha)}(\lambda, G') \prod_{e \in E'} t_e. \tag{A4}$$

The combination of (A3) and (A4) shows that the generating function $N_\alpha(1, 2; G)$ for the mod-λ flows subjected to a fixed non-zero external flow α is independent of α in the case of the Potts model (2.22).

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