Potts models, percolation, and duality

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A λ-state Potts model with multispin interactions is considered which includes models recently introduced by other workers in order to provide a Hamiltonian for site percolation. A duality transformation is obtained for this model which becomes the matching relation for site percolation when $\lambda = 1$.

1. INTRODUCTION

It has been known for some time[1] that the thermodynamic properties of the λ-state Potts model yield corresponding properties of bond percolation in the limite $\lambda \to 1$. Recently Kunz and Wu[2] have extended this result to site percolation by considering a generalization of the Potts model to include multispin interactions. Ashley and Temperley[3] have further developed this idea and shown that different Potts models can yield the same percolation problem.

The duality transformation for the Potts model with two-spin interactions was obtained by Potts[4] using a transfer matrix method. It was later obtained by topological arguments (see for example Refs. 5 and 6) and may be written

$$(z^{*-1} - 1)(z^{-1} - 1) = \lambda, \tag{1.1}$$

where z is defined in terms of the interaction parameter K by $z = e^{-\lambda K}$ and z^* denotes the corresponding variable for the dual problem. In Sec. 3 we generalize Eq. (1.1) to Potts models with multispin interactions. The duality relation for the zero field partition function is seen as an immediate extension of the matching relation for the mean number of clusters in a site percolation problem.[7] The latter is rederived by a simpler method which avoids the use of nonplanar graphs.

In the case $\lambda = 2$, the Potts model with two-spin interactions reduces to an Ising problem with two-spin interactions for which the duality relation was given much earlier by Kramers and Wannier.[8] Duality for Ising models with multispin intractions has recently received considerable attention.[9-13] The $\lambda = 2$ Potts model with three-spin interactions in zero field is also isomorphic with an Ising model with only two-spin interactions. Our result in this case enables the self-duality relation for the Ising model on the triangular lattice to be obtained without the usual reference to the honeycomb lattice (see also Wegner[11]).

In general the $\lambda = 2$ Potts model with l-spin interactions is isomorphic to an Ising model with even interactions of order 2 $[l/2]$ and less, and seems not to have been discussed previously when $l \geqslant 4$ except perhaps as a special case in the articles sited above.[9-13]

The model is introduced in Sec. 2 in a form which includes the conventional Potts model[4] as well as the models of Kunz and Wu[2] and Ashley and Temperley.[3] The correspondence with bond and site percolation is discussed in a unified scheme and the connecting formulas, some of which are required for the main discussion of Sec. 3, are derived.

2. THE MODEL AND ITS RELATION TO PERCOLATION

Consider a system of spins S. Each spin has λ states and selected subsets interact via a Potts interaction. Let the set of all such interacting subsets be denoted by I and define a bipartite interaction graph G which has vertex set $V = (S,I)$ and edge set

$$E = \{(s,i) : s \in S,\ i \in I \text{ and } s \in i.\}.$$

Suppose that each interaction involves at least two spins, so that the vertices of I have at least degree two, and also that spins involved in each interaction $i \in I$ are distinct so that there are no multiedges. Let the states of each spin be indexed by a variable $\alpha = 1, \cdots, \lambda$. In any state of the system let:

$$\xi = \{i \in I : \text{ not all spins of } i \text{ are in the same state}\},$$

$$\eta = \{s \in S : \ s \text{ not in state } \alpha = 1\},$$

$$\zeta = \{i \in I : \text{ not all spins of } i \text{ are in state } \alpha = 1\}.$$

Assign zero energy to the reference state of the system in which all spins are in state $\alpha = 1$ and an additional energy $-kT \log z_i$ for each $i \in \xi$, $-kT \log \mu_s$ for each $s \in \eta$ and $-kT \log \mu_i$ for each $i \in \zeta$. If the variables $\{\mu_s, s \in S\}$, $\{\mu_i, z_i, i \in I\}$ all lie in the interval [0,1], then the reference state is a ground state. The partition function is

$$\Lambda(z,\mu) = \sum_{\text{spin states}} z^{\xi} \mu^{\eta \cup \zeta}, \tag{2.1}$$

where

$$z^{\xi} = \prod_{i \in \xi} z_i \text{ and } \mu^{\eta \cup \zeta} = \prod_{s \in \eta} \mu_s \prod_{i \in \zeta} \mu_i. \tag{2.2}$$

(This convention for a variable raised to the power of a set enables all interaction variables to be considered as distinct without making the notation unduly heavy.) The model so defined will be called a generalized Potts model and denoted by (S,I).

Now consider the percolation model P on G in which each vertex of S is "occupied" with probability 1 and each vertex i of I is occupied independently with probability p_i. A percolation state may be specified by giving the vertices $I' \subseteq I$ which are occupied, and such a state occurs with probability $p^{I'}(1-p)^{I \setminus I'}$. A path is a sequence of vertices, successive members of which constitute an edge of E. An occupied path is one in which all vertices are occupied. In any percolation state two vertices are connected if there is an occupied path between them, and a maximal set of connected vertices is called a cluster. A single spin will be thought of

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as connected to itself, and so a cluster may be single isolated spin. The "percolation average" of a random variable $X(I')$ is defined by

$$\langle X\rangle_p = \sum_{I'\subseteq I} p^{I'}(1-p)^{I\setminus I'} X(I'). \tag{2.3}$$

For example $X(I')$ may be the number of clusters or the number of spins connected to a given spin when the vertices I' are occupied.

We now demonstrate that the partition function of (S,I) may be written as a percolation average for P. First note the identities

$$z^{\xi} = \prod_{i\in I}[z_i + (1-z_i)\delta(i)]$$

$$= \sum_{I'\subseteq I} z^{I\setminus I'}(1-z)^{I'}\prod_{i\in I'}\delta(i), \tag{2.4}$$

where the indicater variable $\delta(i)$ is defined by

$$\delta(i) = \begin{cases} 1, & \text{if } i\notin\xi \\ 0, & \text{if } i\in\xi. \end{cases} \tag{2.5}$$

With $p_i = 1 - z_i$ and taking I' to be vertices of I which are occupied in P we may write

$$z^{\xi} = \langle\prod_{i\in I'}\delta(i)\rangle_p. \tag{2.6}$$

By definition of ξ we see that

$$\prod_{i\in I'}\delta(i) = \begin{cases} 1, & \text{if all spins in each cluster are} \\ & \text{in the same state,} \\ 0, & \text{otherwise.} \end{cases} \tag{2.7}$$

Substituting (2.6) into (2.1) and interchanging the sum and average we obtain,

$$\Lambda(1-p,\mu) = \left\langle \prod_{c\in\mathscr{C}'}[1+(\lambda-1)\mu^{V(c)}] \right\rangle_p, \tag{2.8}$$

where $\mathscr{C}'$ is the set of all finite clusters corresponding to I' and $V(c)$ is the subset of vertices V which belong to the cluster c. In restricting the product to finite clusters we are assuming that sufficient μ variables are less than one, so that $\mu^{V(c)} = 0$ for any infinite cluster.

The result (2.8) will now be used to relate the thermodynamic properties of (S,I) to the percolation functions of P. We shall take $\mu = 1$ to mean $\mu_v \to 1^-, \forall\, v\in V$.

The free energy $\ln\Lambda$ yields a generating function $G(p,\mu)$ for percolation theory, since

$$\mathscr{G}(p,\mu) \equiv \frac{\partial}{\partial\lambda}\ln\Lambda\Big|_{\lambda=1} = \left\langle \sum_{c\in\mathscr{C}'}\mu^{V(c)} \right\rangle_p. \tag{2.9}$$

When $\mu = 1$, $\mathscr{G}(p,u) = \langle n\rangle_p$, the mean number of finite spin clusters, which for an infinite G is normally calculated per spin or per interaction, since $\mathscr{G}$ itself would be infinite.

Differentiating (2.9) w.r.t. μ_v,

$$\frac{\partial\mathscr{G}}{\partial\ln\mu_v} = \langle\delta_v\mu^{V(c_v)}\rangle_p, \tag{2.10}$$

where $\delta_v = 1$ if there is a finite cluster c_v containing vertex v but zero otherwise. Setting $\mu = 1$ gives the probability that v is part of a finite cluster, so that if G is finite,

$$\frac{\partial G}{\partial\mu_v}\Big|_{\mu=1} = \begin{cases} 1, & v\in S, \\ p_v, & v\in I. \end{cases} \tag{2.11}$$

For G infinite and $v\in V$ the percolation probability P_v may be defined as the conditional probability that, given v belongs to a cluster, it belongs to an infinite cluster, and hence,

$$P_v = 1 - p_v^{-1}\frac{\partial\mathscr{G}}{\partial\mu_v}\Big|_{\mu=1}, \tag{2.12}$$

where $p_v = 1$ if $s\in S$.

Setting $\mu_s = 1$, $\forall\, s$ and $\mu_i = \mu, \forall\, i$ in (2.10) gives

$$\frac{\partial\mathscr{G}}{\partial\ln\mu_v}\Big|_{\substack{\mu_s=1\\ \mu_v=\mu}} = \langle\delta_v\mu^{|I(c_v)|}\rangle, \tag{2.13}$$

where $I(c)$ is the set of interaction vertices in c. $|I(c_v)|$ is a measure of the cluster size, and if $v\in I$, it is the standard measure of cluster size in the site percolation problem $\bar{P}$, defined later. Equation (2.13) therefore gives the moment generating function for the cluster size distribution. The moment generating function for $|S(c_v)|$ may be obtained by setting $\mu_s = \mu$, $\forall s$ and $\mu_i = 1$, $\forall\, i$. This is an alternative measure of the cluster size and is, in fact, the number of sites in a bond problem (see later), whereas $|I(c_v)|$ is the number of bonds (see also Stephen[14]).

For $v, v'\in V$, the pair connectedness $P_{vv'}$ is the probability that v and v' belong to the same finite cluster. This may be obtained from (2.10) by differentiating first with respect to $\mu_{v'}$ and then with respect to μ_v, since the only terms which survive the second differentiation are the ones where v and v' belong to the same member of $\mathscr{C}'$, thus

$$P_{vv'} = \frac{\partial^2\mathscr{G}}{\partial\ln\mu_v\,\partial\ln\mu_{v'}}\Big|_{\mu=1},$$

$$= \frac{\partial^2\mathscr{G}}{\partial\mu_v\,\partial\mu_{v'}}\Big|_{\mu=1} \quad \text{if } v\neq v'. \tag{2.14}$$

The corresponding derivative of $\ln\Lambda$ is a spin correlation function.

The above relationship between percolation and the Potts model has been derived by other authors for less general models.

Kastelyn and Fortuin[1] demonstrated a correspondence between Potts models with only two-spin interactions and bond percolation. This was recently developed by Stephen[14] and Wu.[15] For such a model (S,I) the vertices I of the graph G have degree two, and for each $i\in I$ the pair of edges $\{(s_1,i),(i,s_2)\}$ may be replaced by a single edge (s_1,s_2) to give a graph H. This operation is called suppression of the vertices I. The edges so formed are deemed to be occupied when the corresponding vertex of i is occupied and an occupied edge provides a connection between its terminal vertices. The percolation model P corresponding to (S,I) is clearly the bond problem on H. Figure 1(a) illustrates the Potts model for the bond problem on the square lattice.

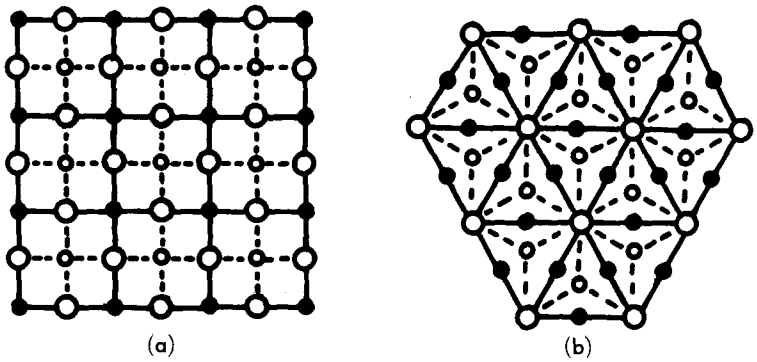

FIG. 1. (a) Kasteleyn–Fortuin correspondence between a Potts model with two-spin interactions and the bond problem on the square lattice. The self duality of this model is illustrated (b) Kunz–Wu correspondence between a Potts model with six-spin interactions and the site problem on the triangular lattice. The dual model is also shown.

Key : 0 = interaction vertex, • = primary spin, ° = dual spin,— = primary edge, ---- = dual edge.

The percolation model P corresponding to any Potts model (S,I) may be converted into a more conventional percolation model $\bar{P}$ in which all sites are randomly occupied. The sites S of G which are occupied with probability one may be removed by the following "star" transformation. For each vertex $s \in S$ let $I(s)$ be the vertices of I which are adjacent to s. Then replace the edges $\{(s,i), i \in I(s)\}$ by the complete graph on $I(s)$ and remove s from the vertex set of G. If s has degree two this corresponds to suppression. If s has degree three it is the star–triangle transformation. Multiedges formed during the transformation may be replaced by single edges without changing the connectedness of the model. The resulting graph will be called $\bar{G}$. The site clusters of $\bar{P}$ correspond to the clusters of P, but a cluster of P which is just a single spin has no counterpart in $\bar{P}$. However, the following simple relation holds between the mean number of clusters for the two problems

$$\langle n \rangle_p = \langle \bar{n} \rangle_p + \sum_{s \in S} (1-p)^{I(s)}. \tag{2.15}$$

The pair connectedness for two vertices $i_1, i_2 \in I$ is the same in both problems.

A given site problem may correspond to several Potts models. Kunz and Wu[2] showed that the site problem on a graph H corresponds to the Potts model (S,I) obtained by taking I to be the sites of H and associating a vertex of S with each edge of H. Clearly H is the graph $\bar{G}$ defined above. The Potts model for the site problem on the triangular lattice is shown in Fig. 1(b). This model involves six-spin interactions. An alternative Potts model for which $\bar{P}$ is the site problem on the triangular lattice was given by Ashley and Temperley.[3] This model is obtained by taking S and I to be the two triangular sublattices of a honeycomb lattice as shown in Fig. 2. In contrast to the Kunz-Wu model only three-spin interactions are involved.

It is well known that any bond problem may be transformed into a site problem on a different graph. In the present context, if the bond problem on a graph H corresponds by the Kastelyn-Fortuin transformation[1] to a Potts model (S,I), then the equivalent site problem is $\bar{P}$ described above.

3. DUALITY FOR PERCOLATION AND THE ZERO FIELD PARTITION FUNCTION

Consider a Potts model (S,I) and for simplicity assume that all vertices have at least degree two. Assume also that the interaction graph G is connected and planar, and that it has been drawn in the plane to form a plane graph (also denoted by G). The graph $\bar{G}$ obtained from G by the star transformation is not necessarily planar, and is in general a decorated mosaic.[7] The dual model (S^*,I) is constructed by placing a vertex of S^* in each face of G and connecting it by an edge to each vertex of I which lies in the boundary of the face in such a way that no two edges intersect. The plane graph so formed will be denoted by G^* and $G^{**} = G$. Figure

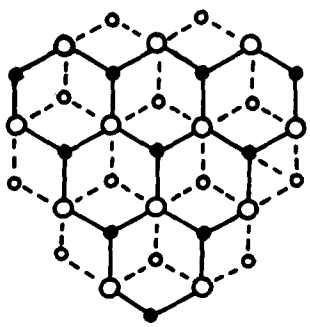

FIG. 2. Ashley–Temperley correspondence between a Potts model with three-spin interactions and the site problem on the triangular lattice. The key is as in Fig. 1.

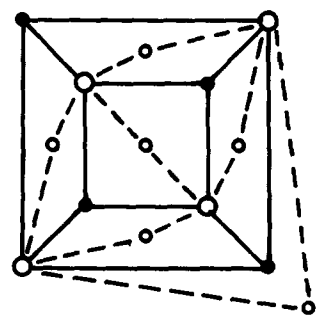

FIG. 3. Duality on a finite graph. Both Potts models correspond to the site problem on the tetrahedron. The key is as in Fig. 1.

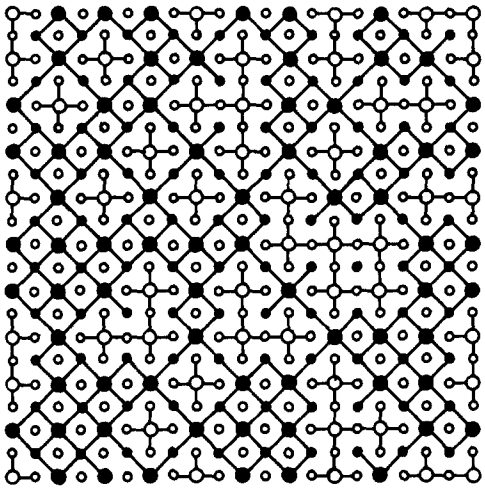

FIG. 4. Corresponding percolation states for the site problem on the square lattice and its matching lattice. Open circles are occupied on the square lattice, whereas solid circles are occupied on the matching lattice.

3 shows the dual transformation between the cube G and the decorated tetrahedron G^*. Notice that the Ashley–Temperley model of Fig. 2 is self-dual, as is the square lattice with two-spin interactions [Fig. 1(a)]. The dual of the Kunz–Wu model of Fig. 1(b) is the diced lattice in which the sixth order vertices are interaction vertices. The percolation model $\bar{P}$ corresponding to the latter model is also the triangular site problem.

The percolation model $\bar{P}^*$ corresponding to (S^*,I) together with $\bar{P}$ constitute a pair of matching site problems.[7] These have the property that the mean number of clusters at probability p for $\bar{P}$ is related to the mean number of clusters at probability $1-p$ for $\bar{P}^*$. (For the models in Fig. 3, $\bar{P}$ and $\bar{P}^*$ are both the site problem on the tetrahedron which is self-matching.) We now obtain a similar relation for P and P^* which by (2.17) implies the matching relation between $\bar{P}$ and $\bar{P}^*$. A relation between the partition functions of dual models is also obtained by the same technique.

Suppose that the occupied vertices of G in a percolation state of P are colored black. This means that all the S vertices are black, together with the subset I' of I. The black vertices together with the edges of G connecting black vertices form a subgraph which we shall call G_B, the components of which are the clusters. A corresponding state of P^* may be obtained by supposing that the vertices $I \setminus I'$ are occupied on G^*. If these vertices together with the vertices S^* are colored white, the white vertices define a subgraph G_W the components of which are the clusters for this state of P^*. Figure 4 shows a pair of dual models and the corresponding states. The model (S^*,I) is the Kunz–Wu model[2] for which $\bar{P}^*$ is the site problem on the square lattice, and its dual (S,I) is such that $\bar{P}$ is the site problem on the square lattice with first and second-neighbor connections. The latter is the matching lattice of the square lattice. The figure illustrates the following theorem.

Theorem: With G_B and G_W defined as above:

(a) every component of G_W is contained within a single face of G_B, and

(b) each face of G_B contains exactly one component of G_W. The same statement is true with G_B and G_W interchanged by symmetry of the dual construction.

Proof: G_B and G_W never intersect, since the only possible intersection points are vertices of I, and in any percolation state each vertex of I either belongs to G_B or to G_W. It follows that any two vertices which are connected in G_W must lie in the same face of G_B, which proves part (a).

To prove part(b) we first notice that the faces of G_B are made up of faces of G each of which contains a vertex of S^*. It follows immediately that every face of G_B contains at least one component of G_W. To show that there cannot be more than one, we must prove that any two vertices of G_W which lie in the same face of G_B are connected on G_W. It is sufficient to prove this for two vertices of S^*, since any vertex of $I \setminus I'$ is on the boundary of some face of G and is therefore connected to a vertex of S^*. Suppose that the vertices s_1 and s_2 of S^* lie in the faces F_1 and F_2 of G which are within the face F_B of G_B. Since F_B is a connected region of the plane, the faces F_1 and F_2 must be connected by a chain of faces of G which are within F_B, successive members of which have edge to edge contact along at least one edge of G which is internal to F_B. An edge of G which is internal to F_B must have a white vertex which is in $I \setminus I'$, and hence vertices of S^* which lie in successive faces of the chain must be connected on G_W via this white vertex. Hence the vertices s_1 and s_2 are connected by a chain of white vertices.

It follows from the theorem that the number of faces f_B of G_B, including the infinite face, is equal to the number of components n_W of G_W, which is the number of clusters for the problem P^*. Also the number of faces of G_B is determined in terms of n_B by Euler's law applied to G_B, thus,

$$n_W = f_B = e_B - v_B + n_B + 1. \tag{3.1}$$

On taking averages over all percolation states,

$$\langle n_B \rangle_p = \Phi(p) + \langle n_W \rangle^*_{1-p}, \tag{3.2}$$

where the average on the right is calculated for P^*, for which $1-p_i$ is the probability of occupation of i. The function $\Phi(p)$ is linear in the variables p_i, and is given by

$$\Phi(p) = |S| + \sum_{i\in I} p_i - \sum_{i\in I} \gamma_i p_i - 1, \tag{3.3}$$

where γ_i is the number of vertices of S which are adjacent to i. Notice that

$$\begin{aligned}\Phi(1-p) &= |S| + |I| - |E| - \sum_{i\in I} p_i + \sum_{i\in I} \gamma_i p_i - 1,\\ &= -|S^*| - \sum_{i\in I} p_i + \sum_{i\in I} \gamma_i p_i + 1,\\ &= -\Phi^*(p),\end{aligned} \tag{3.4}$$

where we have used (3.3) with $p_i = 1$. Notice also that by construction γ_i is also the number of vertices of S^* adjacent to i.

The matching relation obtained by Sykes and Essam[7] was

$$\langle \bar{n}_B \rangle_p = N\phi(p) + \langle \bar{n}_W \rangle^*_{1-p}, \tag{3.5}$$

where $N = |I|$. Using (2.7) we find

$$N\phi(p) = \Phi(p) + \sum_{s \in S^*} p^{I(s)} - \sum_{s \in S} (1-p)^{I(s)}, \tag{3.6}$$

which should be compared with their equation (6.14).

The above relations are exact when G is a finite graph but should also apply to an infinite lattice graph in the thermodynamic limit, in which case one works with the mean number of clusters per vertex of I. A proper treatment of this limit will not be attempted here. However, we note that although our main theorem is true when the lattice is made finite by embedding it in a torus in the usual way (see Fig. 4), not all subgraphs G_B are properly embedded in the torus. This creates difficulties when applying Euler's formula for the number of regions.

As an illustration of the above formulas if P is the site problem on the square lattices with first and second neighbor connections,

$$\Phi(p) = N(1-3p)$$

and

$$\phi(p) = 1 - 3p + 2p^2 - (1-p)^4,$$

$$= p - 4p^2 + 4p^3 - p^4,$$

in agreement with Table II of Sykes and Essam.[7]

A duality relation for the zero field partition function will now be obtained. Setting $\mu = 1$ in (2.8),

$$\Lambda(1-p,1) = \langle \lambda^n \rangle_p, \tag{3.7}$$

where $n = |\mathscr{C}'|$, the number of finite clusters in the percolation model P. The duality for Λ may therefore be considered as a percolation problem, and the notation of the first part of the section will be maintained. By definition,

$$\langle \lambda^n \rangle_p \equiv \langle \lambda^{n_B} \rangle_p = \sum_{I' \subseteq I} p^{I'} (1-p)^{I \setminus I'} \lambda^{n_B}. \tag{3.8}$$

From (3.1)

$$n_B = n_W + |S| + |I'| - \sum_{i \in I'} \gamma_i - 1, \tag{3.9}$$

which is Eq. (3.3) before averaging. Substituting in (3.8) gives

$$\langle \lambda^{n_B} \rangle_p = \lambda^{|S|-1} \sum_{I' \subseteq I} (p\lambda^{1-\gamma})^{I'} (1-p)^{I \setminus I'} \lambda^{n_W}, \tag{3.10}$$

and renormalizing the distribution gives

$$\langle \lambda^{n_B} \rangle_p$$

$$= \lambda^{|S|-1}(1-p+p\lambda^{1-\gamma})^I \sum_{I' \subseteq I} p^{*I'} (1-p^*)^{I \setminus I'} \lambda^{n_W},$$

$$= \lambda^{|S|-1}(1-p+p\lambda^{1-\gamma})^I \langle \lambda^{n_W} \rangle^*_{p^*}, \tag{3.11}$$

where

$$p^* = \frac{1-p}{1-p+p\lambda^{1-\gamma}}. \tag{3.12}$$

When $\lambda = 1, p^* = 1-p$, which is the usual percolation relation. In terms of the Potts model variables,

$$\Lambda(z,1) = \lambda^{|S|-1}(z + (1-z)\lambda^{1-\gamma})^I \Lambda^*(z^*,1), \tag{3.13}$$

where

$$z^* = \frac{(1-z)}{z\lambda^{\gamma-1} + (1-z)}, \tag{3.14}$$

which may be written in the symmetric form

$$(z^{*-1} - 1)(z^{-1} - 1) = \lambda^{\gamma - 1}. \tag{3.15}$$

Equations (3.12), (3.14), and (3.15) are local relations and the subscript i on the variables z, p, and γ has been suppressed for convenience. For model with only two-spin interactions $\gamma_i = 2$, $\forall\ i$ and Eq. (1.1) is obtained.

It is worth noting that when $\lambda = 2$ the Ashley–Temperley model[3] is isomorphic to a triangular lattice Ising model with a two-spin interaction parameter $y = z^{1/2}$ for which the duality relation reads (with $\gamma = 3$),

$$(y^{*-2} - 1)(y^{-2} - 1) = 4,$$

and since the model is self-dual, the critical point (assumed to be unique) is located at $y_c^2 = \frac{1}{3}$ as found by Onsager.[16]

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