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Duality relation for Potts multispin correlation functions

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Abstract. The duality relation for multispin correlation functions of the Potts model on a plane graph is derived using the formulation of a percolation average. It is shown that in simple cases the correlation functions are related to those of the dual graph. But more generally the correlation functions are expressible as linear combinations of ratios of certain percolation averages, the dual partitioned equivalent transmissivities, and dual correlation functions. An overall multiplication factor arises in these considerations, and we elucidate the graph-theoretic meaning of this function.

1. Introduction

Duality constructions and relations have played an important role in field theory and statistical mechanics (for a review of duality see Savit [1]). In the usual formulation of duality for statistical systems, one studies how the partition function of a spin system on a lattice is related to that of the dual lattice. In such considerations all lattice sites enter the picture on an equal footing, and it is therefore natural to use *unrooted* graphs, i.e., graphs which do not have specially distinguished vertices, for its interpretation (for definitions of graph-theoretical terms see Essam and Fisher [2]). In considerations of quantities such as correlation functions, the pair connectedness, equivalent transmissivities [3, 4], and the conductance of resistor networks [5], special lattice sites appear. In the interpretation of these cases, it is necessary to use *rooted* graphs, i.e., graphs possessing specially distinguished vertices. In this way, *m-rooted* graphs, or graphs with m roots, have appeared in real-space renormalisation group considerations for spin systems on regular [3, 6-12] fractal, and hierarchical lattices [13-16].

Construction of the geometric dual for unrooted and one-rooted (root residing in the infinite face) graphs is well established [2]. For two-rooted graphs, Tsallis [17], in a consideration of bond percolation, introduced a duality construction which is identical to the one given by Melrose [13] for plane fractal and hierarchical lattices. This $m=2$ duality construction has proven to be very useful in real-space renormalisation group considerations [3, 7, 10, 11, 18-22]. For $m=3$, specific examples of three-rooted dual graphs have been previously given [3, 8, 9, 13], although, to our knowledge, explicit rules for the geometric duality construction have not been presented. Here, we extend duality constructions to m-rooted plane graphs for general m, and obtain the duality relation for multispin correlation functions of the Potts model (for a review on the Potts model see Wu [23]). Our results are presented in terms of edge variables (*thermal transmissivities* [3]), and of partitioned equivalent transmissivities [4, 24] which are ratios of certain percolation averages.

The organisation of this paper is as follows. In section 2 we introduce the Potts model and its multispin correlation functions as well as the dual of a rooted plane graph. In section 3 we review the derivation of the duality relation for the Potts partition function using the formulation of a percolation average. It is subsequently shown in section 4 that this formulation can be straightforwardly extended to obtaining the duality for the partitioned equivalent transmissivity. The duality relation for multispin correlation functions is discussed, and explicitly derived, in some special cases in section 5. It is shown that, in the most general case, the duality relates a correlation function to the partitioned equivalent transmissivities in the dual space, and it is not always possible to express the final result solely in terms of the dual correlation functions. Finally, in section 6, some graph-theoretic meaning is given to an overall multiplication factor arising in our considerations.

2. Definitions

Consider a standard Potts model on a plane connected graph G with vertex set V of N sites, and edge set $\mathscr{E}$ of E edges. The spin at the ith site can take on q distinct values $\sigma_i = 1, 2, \ldots, q$, and the Hamiltonian is

$$-\beta\mathscr{H} = K \sum_{e\in\mathscr{E}} \delta_{Kr}(\sigma_i, \sigma_j) \tag{1}$$

where the summation extends to all edges in $\mathscr{E}$. Here, we have explicitly assumed the same interaction K along all edges, but our results can be extended to include edge-dependent interactions.

It is convenient to regard the spin σ_i as being represented by a unit vector $\hat{s}_i$ pointing in one of the q-symmetric directions of a hypertetrahedron in $q-1$ dimensions. Then the m-spin correlation function for spins at sites $1, 2, \ldots, m$ is defined to be†

$$\Gamma_{12\ldots m}(G) \equiv (q-1)^{m/2} \frac{1}{Z(G)} \operatorname{Tr}[s_{1\alpha}s_{2\alpha}\ldots s_{m\alpha} \exp(-\beta\mathscr{H})] \tag{2}$$

where

$$Z(G) = \operatorname{Tr} \exp(-\beta\mathscr{H}) \tag{3}$$

is the partition function of the Potts model (1), $s_{i\alpha} = \hat{s}_i \cdot \hat{e}_\alpha$, and $\hat{e}_\alpha$ is a unit vector pointing in the symmetric direction α of the hypertetrahedron.

Equation (2) may be generalised to define a *partitioned* m-spin correlation function [4]

$$\Gamma_{P(m)}(G)$$

for which the m spins are partitioned into b blocks according to the partition $P(m)$ of the m integers $1, 2, \ldots, m$, such that the α indices in (2), instead of being all the same, are the same for all spins within a block and *distinct* for spins in different blocks. The duality relation for the partition function $Z(G)$ is well known (see, e.g., Wu [23]); here we consider duality relations for the correlation function $\Gamma_{P(m)}(G)$.

† Note that a factor $(q-1)^{m/2}$ has been introduced in (2) so as to make the definition of the correlation functions to agree with that of de Magalhães and Essam [4].

The special vertices $1, 2, \ldots, m$, which are distinct from other vertices, constitute the roots of the graph G. In order to properly describe the duality relation for the correlation functions we consider the case that G is connected, and has no single-edge loops or unrooted vertices of degree one. (These are not stringent restrictions, since loops and unrooted vertices of degree one have no effect on the correlation functions). We shall also suppose that the articulation points of G are unrooted, since the correlation function for a graph with a rooted articulation point can be factored. We also suppose that, as shown in figure 1, all m roots can be placed on the boundary of G bordering the *infinite* (exterior) face. We shall picture that the rooted graph has m (imaginary) infinite faces defined by introducing m (imaginary) non-intersecting boundary lines running from each root out to infinity. We define the graph D dual to G to be another m-rooted graph with

(i) the m roots of D each residing in one of the m infinite faces of G,

(ii) one unrooted vertex of D residing in each of the $c(G) = E - N + 1$ finite (interior) faces of G, and

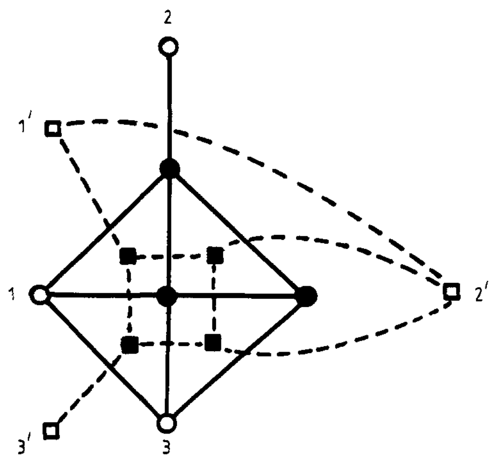

Figure 1. A 3-rooted graph G (full lines) and its dual D (broken lines). The roots are denoted by open circles and open squares.

(iii) E edges each crossing one edge of G and joining the vertices which lie in the faces adjoining the edge.

An example of G and its dual D is shown in figure 1. Clearly, the dual graph D generally has

$$N^* = c(G) + m = E - N + m + 1 \tag{4}$$

vertices, where $c(G)$ is the number of independent circuits in G. The infinite face of D can also be pictured as being partitioned into m infinite faces by m (imaginary) non-intersecting boundaries running from each root to infinity. The dual of D is clearly G.

3. Duality for the partition function

It is known that the thermodynamic properties of the Potts model may be expressed in terms of percolation averages [24, 25].

Consider the percolation average defined by

$$\langle X\rangle_{G,t} \equiv \sum_{G'\subseteq G} X(G')t^{E(G')}(1-t)^{E-E(G')} \tag{5}$$

where the summation is extended over all partial graphs, $G'\subseteq G$, $X(G')$ is any property of the partial graph G', and $E(G')$ is the number of edges in G'. It is readily verified (Essam and Tsallis [24]) that the Potts partition function can be written as

$$Z(q,G) = q^N(1-t)^{-E}\langle q^c\rangle_{G,t} \tag{6}$$

where t is the thermal transmissivity defined by

$$t = \frac{1-e^{-K}}{1+(q-1)\,e^{-K}}. \tag{7}$$

In our discussion of duality for the m-spin correlation functions in the next section it will be necessary to consider a percolation average which reduces to $\langle q^c\rangle_{G,t}$ when $m=1$. It is therefore instructive to rederive the usual duality relation for the partition function by considering this quantity first. When $m=1$ our definition of the dual graph D coincides with the standard definition [2] of the dual for an unrooted graph in which only one vertex is placed in the infinite face. Introducing the Euler relation

$$E(G') = N + c(G') - n(G') \tag{8}$$

where $n(G')$ is the number of components, including isolated sites, in G', we can rewrite the percolation average appearing in (6) as

$$\begin{aligned}\langle q^c\rangle_{G,t} &\equiv \sum_{G'\subseteq G} q^{c(G')}t^{E(G')}(1-t)^{E-E(G')}\\ &= (1-t)^E\left(\frac{t}{1-t}\right)^N \sum_{G'\subseteq G}\left[\frac{qt}{1-t}\right]^{c(G')}\left[\frac{1-t}{t}\right]^{n(G')}\end{aligned} \tag{9}$$

The derivation of the duality relation for $\langle q^c\rangle_{G,t}$, or the partition function, is a consequence of the existence of a bijection (one-one correspondence) between partial graphs $G'\subseteq G$ and partial graphs $D'\subseteq D$: for each edge of D present in D', the crossing edge of G is absent in G' and vice versa. This leads to the following proposition.

Proposition 1. (General m). If D' is the partial graph corresponding to G', then

(a) vertices of G' in different faces of D' are not connected;

(b) vertices of G' in the same face of D' are connected;

(c) the results (a) and (b) are true with G and D interchanged.

A proof for $m=1$ may be found in Essam [26, 27] and the extension to general m is discussed in section 4. Proposition 2 follows immediately.

Proposition 2. Each face of D' (finite or infinite) contains exactly one component of G' and vice versa. Consequently, there exists the identity (for $m=1$)

$$n(G') = c(D') + 1 \tag{10a}$$

and on interchanging G and D,

$$c(G') = n(D') - 1. \tag{10b}$$

It follows from the existence of the bijection that we may regard the summation in (9) as being taken over $D' \subseteq D$ and from (10) that the roles of the two quantities inside the square brackets in (9) are interchanged in the dual graph.

Introducing the dual variable

$$t^* \equiv \mathrm{e}^{-K} = \frac{1-t}{1+(q-1)t} \tag{11}$$

or, equivalently,

$$\frac{qt^*}{1-t^*} = \frac{1-t}{t} \qquad \frac{1-t^*}{t^*} = \frac{qt}{1-t} \tag{12}$$

we obtain after substituting (10*a*, *b*) and (12) into (9) as well as introducing (9) for D,

$$\begin{aligned}\langle q^{\mathrm{c}}\rangle_{G,t} &= \left(\frac{1-t}{1-t^*}\right)^E \left(\frac{t}{1-t}\right)^N \left(\frac{1-t^*}{t^*}\right)^{N^*} \frac{(1-t)^2}{qt^2}\langle q^{\mathrm{c}}\rangle_{D,t^*}\\ &= q^{1-N}[1+(q-1)t]^E\langle q^{\mathrm{c}}\rangle_{D,t^*} \qquad m=1.\end{aligned} \tag{13}$$

Here, we have used the identity $E = N + N^* - 2$, namely (4) with $m = 1$, in writing down the second line in (13). The expression (13) gives the duality relation for the percolation average (9). For the convenience of later use, we note the further identities between t and t^*:

$$\left(\frac{1-t}{t}\right)\left(\frac{1-t^*}{t^*}\right) = [1+(q-1)t][1+(q-1)t^*] = q. \tag{14}$$

The duality relation (13), which is valid for $m = 1$, can also be interpreted as being valid for general m in terms of a *collapsed* graph. Starting from D, we construct a

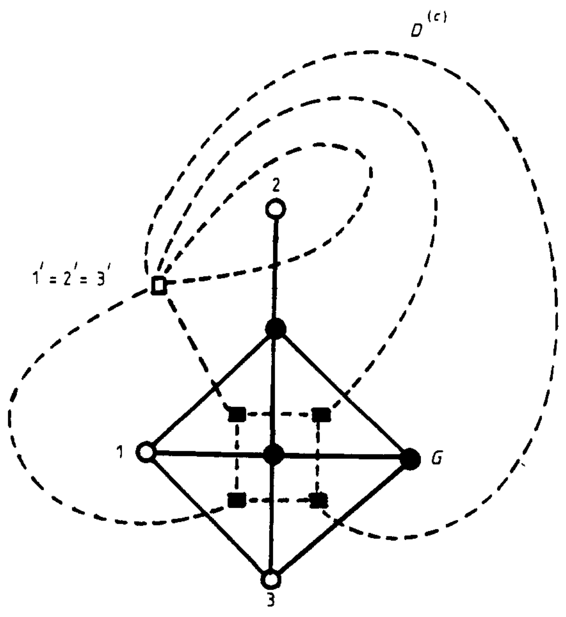

Figure 2. The 3-collapsed graph $D^{(c)}$ constructed from the 3-rooted graph D in figure 1, by moving two of the three roots (open squares) of D and superimposing them onto the third root so that $D^{(c)}$ is the conventional dual of G.

'm-collapsed' graph $D^{(c)}$ by moving and superimposing all m roots of D into a single root (which resides in the infinite face of G) such that G has a single root in the infinite face of $D^{(c)}$. An example of the construction of the 3-collapsed graph $D^{(c)}$ starting from the graph D in figure 1 is shown in figure 2. Since in the conventional dual graph there is always one site of D residing in the infinite face of G, clearly $D^{(c)}$ is precisely the conventional dual. We can therefore rewrite (13) as

$$\langle q^c\rangle_{G,t} = q^{1-N}[1+(q-1)t]^E \langle q^c\rangle_{D^{(c)},t^*} \tag{15a}$$

and, similarly if instead we collapse G to give $G^{(c)}$,

$$\langle q^c\rangle_{D,t^*} = q^{1-N^*}[1+(q-1)t^*]^E \langle q^c\rangle_{G^{(c)},t}. \tag{15b}$$

Here, it is to be noted that the right-hand side of (15a, b) pertain to general m. In addition, as a consequence of (15a, b) and the use of (14) and (4), these percolation averages satisfy the identity

$$\langle q^c\rangle_{G,t}\langle q^c\rangle_{D,t^*} = q^{1-m}\langle q^c\rangle_{G^{(c)},t}\langle q^c\rangle_{D^{(c)},t^*}. \tag{16}$$

4. Duality relation for the partitioned equivalent transmissivity

It is known [4] that the m-site correlation function of the Potts model is expressible as a linear combination of the partitioned equivalent transmissivity

$$t_{\boldsymbol{P}}^{\mathrm{eq}}(t, G) = \frac{\langle q^c \gamma_{\boldsymbol{P}}\rangle_{G,t}}{\langle q^c\rangle_{G,t}} \tag{17}$$

where $\boldsymbol{P} = P(m)$ is a partition of the m roots of G into b blocks. For example, one possible partition for $m = 6$ is $\boldsymbol{P} = \{1, 3\}, \{2, 5, 6\}, \{4\}$ for $b = 3$. In (17),

$$\gamma_{\boldsymbol{P}}(G') = \begin{cases} 1 & \text{if roots in the same block of the partition } \boldsymbol{P} \text{ are} \\ & \text{connected among themselves in } G' \text{ and if} \\ & \text{roots of different blocks are not connected} \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

If $\gamma_{\boldsymbol{P}}(G') = 1$, we say that $\boldsymbol{P}$ is the partition of the roots induced by G'. For example, in figure 3, the partial graph G' of a rectangular section G of the square lattice induces the partition $\boldsymbol{P} = \{1, 11\}, \{2, 9, 10\}, \{3, 5, 8\}, \{4\}, \{6\}, \{7\}$. As a first step of deriving the duality relation for the correlation function, we obtain in this section the duality for the equivalent transmissivity (17).

We have already seen that the denominator of (17) satisfies the duality relation (15a). It therefore remains only to consider the duality for the numerator in (17). To accomplish this, we proceed as in (9) by writing

$$\begin{aligned} \langle q^c \gamma_{\boldsymbol{P}}\rangle_{G,t} &\equiv \sum_{G'\subseteq G} \gamma_{\boldsymbol{P}}(G') q^{c(G')} t^{E(G')} (1-t)^{E-E(G')} \\ &= (1-t)^E \left(\frac{t}{1-t}\right)^N \sum_{G'\subseteq G} \gamma_{\boldsymbol{P}}(G') \left(\frac{qt}{1-t}\right)^{c(G')} \left(\frac{1-t}{t}\right)^{n(G')}. \end{aligned} \tag{19}$$

Further progress requires that we relate $\gamma_{\boldsymbol{P}}(G')$, $c(G')$, and $n(G')$ to corresponding quantities for D'. This may be achieved by noticing, first, that proposition 1 continues to be valid for any m-rooted graph $G' \subseteq G$. The dividing boundaries introduced in

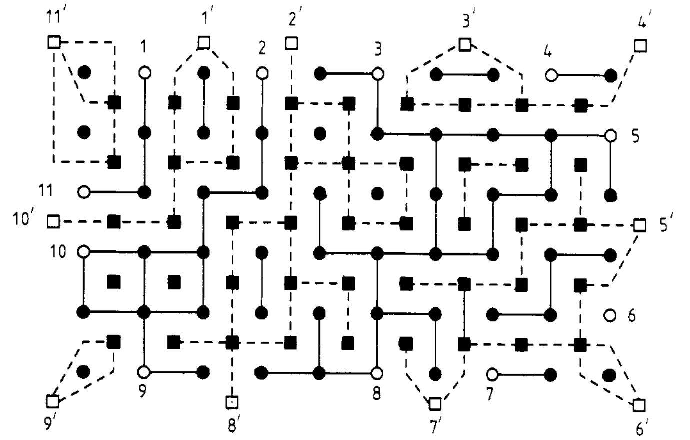

Figure 3. A $m=11$ rooted partial graph G' (full lines) of a rectangular lattice G and the corresponding partial graph D' (broken lines) of D. Vertices of G are denoted by circles and vertices of D by squares, with open circles and squares denoting rooted sites.

defining the infinite faces of the rooted graph G are maintained but different infinite faces can become connected in G' via union with finite faces of G. One may convince oneself of the truth of proposition 1 by inspection of figure 3 which shows a partial graph G' of G together with the corresponding partial graph D' of D for $m=11$.

Proposition 3. Let G_1' and G_2' be partial graphs of G, and let D_1' and D_2' be the corresponding partial graphs of D. If G_1' and G_2' induce the same partition of the roots of G, then D_1' and D_2' induce the same partition of the roots of D.

Proof. Number the m roots $1, 2, \ldots, m$ of G in ascending order around the boundary except that m is followed by 1, and number the m roots $1', 2', \ldots, m'$ of D in relation to those of G such that i' resides between roots i and $i+1$, except that m' is between m and 1. If i' and j' are connected in D_1', then from proposition 1 there is no path on G_1' between the root sets $R_1=\{i+1, \ldots, j\}$ and $R_2=\{j+1, \ldots, i\}$ of G, since these sets lie in different faces of D_1'. Since G_1' and G_2' induce the same partition of the roots of G, there is no path on G_2' between R_1 and R_2, and by proposition 1 there must therefore be a path between i' and j' on D_2'. Conversely, if i' and j' are not connected in D_1', then there is at least one path between R_1 and R_2 on G_1' and hence on G_2'. Consequently, there is no path between i' and j' on D_2'.

It follows from proposition 3 that we may define a bijection between the root partitions of G and D by saying that if G' induces the partition $\boldsymbol{P}$ and D' corresponds to G', then the corresponding partition $\boldsymbol{Q}$ is the one induced by D'. Clearly, any G' which induces $\boldsymbol{P}$ will suffice to find $\boldsymbol{Q}$, and the simplest one is to take a forest. That is, for any partition $\boldsymbol{P}$ of the m sites of G, we connect those rooted vertices belonging to a given block in $\boldsymbol{P}$ by lines. As an example, we show in figure 3 an $m=11$ partial graph, and in figure 4 the associated forest construction. The solid lines in the forest construction now divide the lattice into regions. Then, the partition $\boldsymbol{Q}$ is one in which all rooted vertices i' residing in one region of the lattice are associated to one block of $\boldsymbol{Q}$. The forest construction of $\boldsymbol{Q}$ is shown in figure 4 by broken lines.

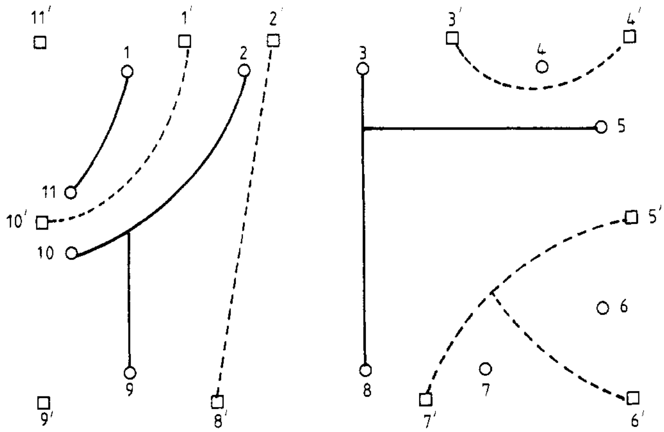

Figure 4. The partitions $P = \{1, 11\}, \{2, 9, 10\}, \{3, 5, 8\}, \{4\}, \{6\}, \{7\}$, $Q = \{1', 10'\}, \{2', 8'\}, \{3', 4'\}, \{5', 6', 7'\}, \{9'\}, \{11'\}$, and the associated forests corresponding to the partial graphs G' and D' in figure 3. Rooted vertices belonging to one block in P and Q are connected by, respectively, full and broken lines.

It is clear that the correspondence between P and Q is a bijection. If P and Q are corresponding partitions, then clearly

$$\gamma_Q(D') = \gamma_P(G'). \tag{20}$$

Further, we note that if P has b blocks and Q has b^* blocks, then

$$b + b^* = m + 1 \tag{21}$$

which follows from the facts that $b + b^*$ is invariant under addition or deletion of an edge from G', and, when $b = 1$, the roots of G' are all connected and by proposition 1 there is no connection between the roots of D' so that $b^* = m$. The above invariance is also a consequence of proposition 1, since if deleting an edge from G' increases the number of blocks in P by one, then addition of the corresponding edge to D' must introduce a path between different blocks of Q, and hence b^* is reduced by 1.

To relate $n(G')$ to $c(D')$, (10a) must be extended using a generalisation of proposition 2, which again follows from proposition 1, namely:

Proposition 4. Each face (finite or infinite) of D' contains exactly one component of G', and there is an infinite face of D' corresponding to each block of P.

Instead of (10a), we now have, for general m,

$$n(G') = c(D') + b \tag{22a}$$

and, interchanging G and D,

$$c(G') = n(D') - b^*. \tag{22b}$$

Proceeding exactly as in the derivation of (13), we substitute (22a, b) and (20) into (19), write the summation in (19) as being taken over $D' \subseteq D$, and use (21) to eliminate b^*. This leads to

$$\begin{aligned}\langle q^c \gamma_P \rangle_{G,t} &= \left(\frac{1-t}{1-t^*}\right)^E \left(\frac{t}{1-t}\right)^N \left(\frac{1-t^*}{t^*}\right)^{N^*} \left(\frac{1-t}{t}\right)^{m+1} \left(\frac{1}{q}\right)^{m+1-b} \langle q^c \gamma_Q \rangle_{D,t^*} \\ &= q^{b-N}[1 + (q-1)t]^E \langle q^c \gamma_Q \rangle_{D,t^*} \end{aligned} \tag{23}$$

which is the generalisation of (13) to arbitrary m. Here, use has been made of (4) and (14) in writing down the last step in (23). Note that (23) reduces to (13) for $m=1$ after setting $b=1$.

The duality relation for the partitioned equivalent transmissivity $t_P^{\mathrm{eq}}(t, G)$ can now be written down by substituting (15a) and (23) for, respectively, the denominator and the numerator of (17). This leads to the relation†

$$t_P^{\mathrm{eq}}(t, G) = q^{b-1}\frac{\langle q^c \gamma_Q\rangle_{D,t^*}}{\langle q^c\rangle_{D^{(c)},t^*}}. \tag{24}$$

If we define

$$A(t, G) \equiv \frac{\langle q^c\rangle_{G^{(c)},t}}{\langle q^c\rangle_{G,t}} \tag{25}$$

which, according to (16), satisfies the duality relation

$$A(t, G)A(t^*, D) = q^{m-1} \tag{26}$$

we can rewrite (24) as

$$t_P^{\mathrm{eq}}(t, G) = q^{b-m}A(t, G)t_Q^{\mathrm{eq}}(t^*, D). \tag{27}$$

This is the desired duality relation for the partitioned equivalent transmissivity. Here, m is the number of roots, b is the number of blocks in $\boldsymbol{P}$, and t and $t^* \equiv \mathrm{e}^{-K}$ are related by (7), (11), or (14). Note that a common multiplication factor $A(t, G)$, which is independent of the partition $P(m)$, arises in the above formulation. We shall in section 6 elucidate the graphical meaning associated to this multiplication factor.

5. Duality for the correlation function

In this section we consider the duality relation for the correlation functions $\Gamma_{P(m)}(G)$. To be more precise, we shall derive duality relations which relate the spin correlation functions to the partitioned equivalent transmissivities in the dual space. We shall see that, in some cases, these equivalent transmissivities can be expressed in terms of spin correlation functions of the dual space.

Consider first the case of $m=2$, for which there exist two partitioned correlation functions corresponding to the partitions $\{12\}$ and $\{1, 2\}$ of the integers 1, 2. It has been established [4] that these correlation functions are related to the equivalent transmissivity t_{12}^{eq} as follows:

$$\begin{aligned}\Gamma_{12}(G) &= t_{12}^{\mathrm{eq}}(t, G)\\ \Gamma_{1,2}(G) &= -(q-1)^{-1}t_{12}^{\mathrm{eq}}(t, G).\end{aligned} \tag{28}$$

Now using (27) and the following identity which holds for either G or D,

$$t_{12}^{\mathrm{eq}} + t_{1,2}^{\mathrm{eq}} = 1 \tag{29}$$

we obtain, after using (28) for D,

$$\begin{aligned}\Gamma_{12}(G) &= q^{-1}A(t, G)[1-\Gamma_{12}(D)]\\ \Gamma_{1,2}(G) &= -q^{-1}A(t, G)[(q-1)^{-1}+\Gamma_{1,2}(D)].\end{aligned} \tag{30}$$

† The relation (24) can also be derived by using (17) and (3.20) of de Magalhães and Essam [4].

Here, for simplicity, $\Gamma_{12}(D)$ for D denotes $\Gamma_{1'2'}(D)$, etc. These are the desired duality relations relating correlation functions to their duals.

Note that the numerator of $A(t, G)$ satisfies the following relation

$$\langle q^{c}\rangle_{G^{(c)},t}=\langle q^{c}\rangle_{G,t}+(q-1)\langle q^{c}\gamma_{12}\rangle_{G,t} \tag{31}$$

where we have applied the 'onion property' (see equation (4.17) of Essam and Tsallis [24]) to the original graph $G^{(c)}$ and the 'peeled' graph G.

Combining the above relation with (17) and (25), we get

$$A(t, G)=1+(q-1)t_{12}^{\mathrm{eq}}(t, G) \tag{32}$$

which, together with (28) and (30), leads to Tsallis and Levy's result [3], namely,

$$t_{12}^{\mathrm{eq}}(D)=[1-t_{12}^{\mathrm{eq}}(G)][1+(q-1)t_{12}^{\mathrm{eq}}(G)]^{-1}. \tag{33}$$

For $m=3$, there are five independent partitioned correlation functions corresponding to the partitions $\{123\}$, $\{1, 23\}$, $\{2, 31\}$, $\{3, 12\}$, $\{1, 2, 3\}$ of the integers 1, 2, 3. It has been shown [4] that these correlation functions are all related to the equivalent transmissivity t_{123}^{eq}:

$$\begin{aligned}
&\Gamma_{123}(G)=(q-2)(q-1)^{-1/2}t_{123}^{\mathrm{eq}}(t, G)\\
&\Gamma_{1,23}(G)=\Gamma_{2,31}(G)=\Gamma_{3,12}(G)=-(q-1)^{-3/2}(q-2)t_{123}^{\mathrm{eq}}(t, G)\\
&\Gamma_{1,2,3}(G)=2(q-1)^{-3/2}t_{123}^{\mathrm{eq}}(t, G).
\end{aligned} \tag{34}$$

Introducing (27) and the identity

$$t_{123}^{\mathrm{eq}}+t_{1,23}^{\mathrm{eq}}+t_{2,31}^{\mathrm{eq}}+t_{3,12}^{\mathrm{eq}}+t_{1,2,3}^{\mathrm{eq}}=1 \tag{35}$$

and using (34) for D, we obtain after some steps the expression

$$\begin{aligned}
\Gamma_{123}(G)=\frac{1}{q^{2}}A(t, G)\bigg(&-\Gamma_{123}(t^{*}, D)\\
&+\frac{q-2}{(q-1)^{1/2}}[1-t_{1,23}^{\mathrm{eq}}(t^{*}, D)-t_{2,31}^{\mathrm{eq}}(t^{*}, D)-t_{3,12}^{\mathrm{eq}}(t^{*}, D)]\bigg).
\end{aligned} \tag{36}$$

This is the duality relation. However, we note that the right-hand side of (36) contains terms of equivalent transmissivities whose partitions Q contain blocks of single sites. As we shall see below, these dual transmissivities in general cannot be expressed in terms of the dual correlation functions.

For general m the partitioned correlation function $\Gamma_{P(m)}(G)$ can be expressed† as a linear combination of the transmissivities $t_{P}^{\mathrm{eq}}(t, G)$ which, in conjunction with (27), expresses $\Gamma_{P(m)}(G)$ in terms of $t_{Q}^{\mathrm{eq}}(t^{*}, D)$. Similarly, we can express $\Gamma_{Q(m)}(D)$ as a linear combination of $t_{Q}^{\mathrm{eq}}(t^{*}, D)$ and use these expressions to solve for $t_{Q}^{\mathrm{eq}}(t^{*}, D)$ in terms of $\Gamma_{Q(m)}(D)$. However, it turns out that equivalent transmissivities $t_{Q}^{\mathrm{eq}}(t^{*}, D)$ whose partition $Q(m)$ contains one or more blocks consisting of a single site never appear in this formulation‡. As a result, we cannot obtain expressions for these t_{Q}^{eq}, and, consequently, they remain in the resulting duality relation.

† See equation (3.14*a*) and (3.44) of de Magalhães and Essam [4].

‡ This is a consequence of the fact that the coefficient of t_{Q}^{eq} occurring in (3.44) of de Magalhães and Essam [4] is the flow polynomial $F(q, I_{P,Q})$, which vanishes identically for Q containing blocks of single sites.

The duality relation for other partitioned correlation functions can be worked out on a case by case basis. For example, for $m=4$ we find

$$\Gamma_{1234}(G)=\frac{A(t,G)}{q^2}\left[\left(\frac{q^2-3q+3}{q(q-1)}\right)t^{\mathrm{eq}}_{1,2,3,4}(t^*,D)+t^{\mathrm{eq}}_{1,24,3}(t^*,D)+t^{\mathrm{eq}}_{2,13,4}(t^*,D)\right]. \quad (37)$$

Here, use has been made of the fact that, as a consequence of G being planar, some of the equivalent transmissivities such as $t^{\mathrm{eq}}_{13,24}$ vanishes identically.

6. Expression for $A(t, G)$ in terms of generating functions for flow polynomials

In the preceeding sections we have seen that a common factor $A(t, G)$, which is independent of the partition $P(m)$, arises in the duality relation for $\Gamma_{P(m)}(G)$. It is therefore useful to examine the graphical interpretation of this function.

The function $A(t, G)$ is defined by (25). Following Wu [28] we introduce

$$f_{ij}\equiv\delta_{ij}-\frac{1}{q}\equiv\frac{q-1}{q}\,\hat{s}_i\cdot\hat{s}_j \quad (38)$$

so that the Boltzmann factor occurring in (2) and (3) can be written as

$$\mathrm{e}^{-\beta\mathscr{H}}=(1-t)^{-E}\prod_{e\in G}(1+qtf_{ij}). \quad (39)$$

It then follows from (6) and (39) that the denominator of (25) is

$$\langle q^c\rangle_{G,t}=q^{-N}\,\mathrm{Tr}\prod_{e\in G}(1+qtf_{ij}). \quad (40)$$

Now, the flow polynomial [24] for any graph g having $N(g)$ sites can be written as [28, 29]

$$\begin{aligned}F(q,g)&\equiv\sum_{g'\subseteq g}(-1)^{E(g)-E(g')}q^{c(g')}\\&=q^{E(g)-N(g)}\,\mathrm{Tr}\prod_{e\in g}f_{ij}.\end{aligned} \quad (41)$$

Therefore, by affecting a partial graph expansion of the product in (40), we obtain

$$\langle q^c\rangle_{G,t}=\sum_{G'\subseteq G}F(q,G')t^{E(G')}. \quad (42)$$

This relation which expresses the percolation average in terms of the flow polynomial was first obtained by Essam and Tsallis [24].

We now proceed in exactly the same fashion to write the numerator of (25) as a percolation average. Now, the m-collapsed graph $G^{(c)}$ is obtained from G by moving and overlapping the m roots into a single point. Since this procedure reduces the number of sites by $m-1$ while leaving the number of edges unchanged, we have from (6)

$$\langle q^c\rangle_{G^{(c)},t}=q^{m-N-1}(1-t)^E Z(q,G^{(c)}). \quad (43)$$

We can achieve the same effect of collapsing G by not physically moving the roots, but instead requiring the m roots to be in the *same* spin state. This can be done formally by introducing a ghost spin σ_0 and rewriting in (43)

$$Z(q,G^{(c)})=\mathrm{Tr}^+[\delta_{01}\delta_{02}\ldots\delta_{0m}\,\mathrm{e}^{-\beta\mathscr{H}}] \quad (44)$$

where the trace Tr^+ is taken over σ_0 as well as the N spins in G and

$$\delta_{0i} = \delta_{Kr}(\sigma_0, \sigma_i).$$

Now, substituting (39), (44) into (43) and introducing $\delta_{0i} = q^{-1}(1+qf_{0i})$, we obtain

$$\langle q^c \rangle_{G^{(c)},t} = q^{-(N+1)} \mathrm{Tr}\left[\prod_{i=1}^{m} (1+qf_{0i}) \prod_{e\in G} (1+qtf_{ij})\right]. \tag{45}$$

It is then convenient to introduce the graphs

$$\begin{aligned} G^+ &= G \cup g \\ g &= g_1 \cup g_2 \cup \ldots \cup g_m \end{aligned} \tag{46}$$

where g_i is an edge connecting the ith root to the spin σ_0. Expanding the products in (45) and introducing (41) for the flow polynomial, we obtain from (45)

$$\langle q^c \rangle_{G^{(c)},t} = \sum_{G' \subseteq G+} F(q, G') t^{E(G') - E(g \cap G')}. \tag{47}$$

Combining (42) and (47), we finally obtain from (25) the following expression for $A(t, G)$:

$$A(t, G) = \frac{\sum_{G' \subseteq G+} F(q, G') t^{E(G') - E(g \cap G')}}{\sum_{G' \subseteq G} F(q, G') t^{E(G')}}. \tag{48}$$

7. Summary

We have considered the duality relation for the multispin correlation functions of the standard Potts model. Useful graph-theoretic terms are introduced in section 2, and in section 3 we derive the duality relation of the partition function, which is also the one-site correlation function, using the formulation of a percolation average. This derivation is generalised in section 4 to obtain (27), the duality relation for the partitioned equivalent transmissivity. In section 5, we establish that duality relations for correlation functions can be related to those of the partitioned equivalent transmissivity. Finally, in section 6, we elucidate the graph-theoretic meaning of a factor arising in these considerations.

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