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Directed compact percolation near a damp wall: mean length and mean number of wall contacts

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Abstract

Key aspects of the cluster distribution in the case of directed, compact percolation near a damp wall are derived as functions of the bulk occupation probability p and the wall occupation probability p_w. The mean length of finite clusters and mean number of contacts with the wall are derived exactly, and we find that both results involve elliptic integrals and further multiple sum functions of two variables. Despite the added complication of these multiple sum functions, our analysis shows that the critical behaviour is similar to the dry wall case where these functions do not appear. We derive the critical amplitudes as a function of p_w.

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(Some figures may appear in colour only in the online journal)

1. Introduction

1.1. Background

Percolation processes were first examined thoroughly by Broadbent and Hammersley [6], who coined the term *percolation* from the physical situation of a fluid percolating through a medium. The general percolation model can be modified in many ways: one of the most natural of these is to add a direction or directions in which the fluid may flow through the medium. The case of directed percolation [7, 8] arises in many different physical situations [17–19, 22] but this modification alone does not lead to an exactly solvable model. However, if we also add the condition of compactness, meaning that clusters cannot branch off or contain holes, then this *directed, compact* percolation model, introduced by Domany and Kinzel [9], is an exactly solvable model, and it is this model which we will consider in this paper.

Directed compact percolation in the bulk was considered by Essam [10], who calculated the percolation probability, mean size and mean length of clusters in this case, as well as their associated critical exponents of $\beta = 1$, $\gamma = 2$ and $\tau = 1$, respectively. An important modification to the directed compact percolation model, as it is to many statistical mechanical models, is to add a wall to restrict the growth of the cluster. This was first considered by Essam and TanlaKishani [14], who considered a *wet* wall, where the sites on the wall are always occupied, and calculated the percolation probability, mean cluster size and mean cluster length. The critical exponents for these properties were found to be $\beta = 1$, $\gamma = 2$ and $\tau = 1$, respectively—that is, identical to the exponents for the bulk case.

A *dry* wall, where the wall sites are never occupied, was first considered by Bidaux and Privman [2], and the percolation probability was derived exactly by Lin [15]. It was found that the critical behaviour differed from that of the bulk and wet wall cases, with a critical exponent of $\beta = 2$. The mean cluster size for this case was calculated by Essam and Guttmann [11]—solving exactly in the low-density region and using differential approximants for the high-density region—and again it was found that the associated critical exponent, $\gamma = 1$, differed from the bulk and wet wall. Brak and Essam [4] derived the mean cluster length exactly for the dry wall case, expressing it in terms of elliptic integrals. Of particular importance was the presence of $K(m)$, the complete elliptic integral of the first kind, as defined in 17.3.1 of [1],

$$K(m) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - m\sin^2\theta}}\, d\theta, \tag{1.1}$$

which can be expressed as the power series,

$$K(m) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[\frac{(2n)!}{2^{2n}(n!)^2} \right]^2 m^{2n}. \tag{1.2}$$

The behaviour of the mean size near the critical point was found in [4] to be dominated by the logarithmic singularity of $K(m)$ near $m = 1$. As such the critical exponent τ for a dry wall was effectively zero, which differed from the bulk and wet results of $\tau = 1$. The mean number of contacts with the wall was defined in the dry wall case [4] to be the mean number of times a cluster included a site adjacent to the wall. As with the mean length, the mean number of contacts [4] was able to be expressed in terms of elliptic integrals with a logarithmic singularity near the critical point.

It was natural to introduce a *damp* wall model, which interpolates between the wet and dry walls, with a variable probability p_w of occupation of wall sites. This was first considered in [16]. The percolation probability as a function of both p and p_w was calculated for the damp wall, and its critical behaviour was found to follow that of the dry wall, with critical exponent $\beta = 2$, for all $p_w < 1$. The crossover to the wet wall limit, where $\beta = 1$, was demonstrated explicitly. The percolation probability was calculated using the generating function for a particular lattice path problem involving two walks. The generating function involves a set of infinite sums over single walk partition functions and is not directly expressible in terms of known functions. Intriguingly, however, the percolation probability for the damp wall model is a rational function of p and p_w.

It is of interest to calculate the cluster properties for this generalized damp wall model that have been calculated in the other cases, both to extend the conclusion that the damp and dry models share the same critical behaviour and to understand the functions involved. This paper proceeds to find expressions in the damp wall case for the mean length of finite clusters and mean number of contacts with the wall. We find that the exponents for each of these properties in the damp wall case follow the result of the dry wall case and show that in the wet wall limit

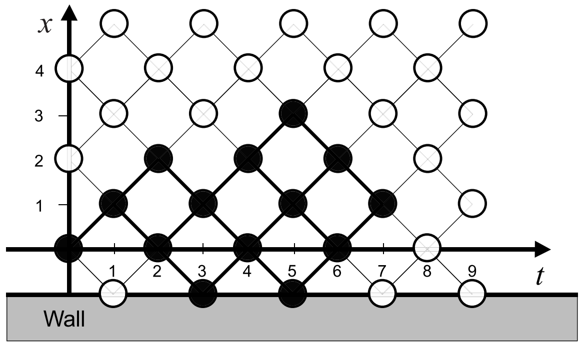

Figure 1. An example of a cluster near a damp wall, with a combination of wet and dry wall sites. The probability of this cluster growing from a seed of a single site at the origin is $p^5q^5p_w^2q_w^2$, as calculated in section 1.2.4. This particular cluster has length 8, with two wall contacts, and its cluster parameters are given in table 1.

they cross over to the bulk values. We have calculated the critical behaviour explicitly giving the critical amplitudes as a function of p_w.

As might be expected following the dry wall case, we find that both the mean length and the mean number of contacts in the damp wall case can be expressed in terms of elliptic integrals, plus additional terms involving double sums, which is in contrast to the rational form found for the percolation probability.

1.2. Model

The model is defined on a directed square lattice, the sites of which are the points in the t, x plane with integer coordinates such that $t \in \mathbb{N} \cup 0$, $x \in \mathbb{Z}$ and $t + x$ is even. The growth rule is that the site (t, x) becomes wet with certainty if both the sites $(t - 1, x \pm 1)$ are wet, and with probability p if only one of these sites is wet, defining $q = 1 - p$ to be the probability of remaining dry in this case. Where both of the sites $(t - 1, x \pm 1)$ are dry, the site (t, x) remains dry with certainty, ensuring that a single compact cluster is produced from a *seed* of m contiguous sites at $t = 0, x \geqslant 0$, where we often restrict ourselves to the case $m = 1$ in this work. The bulk case allows unrestricted growth in the x-direction, whereas in all other cases considered here we have the introduction of a wall which restricts the growth of the cluster. The wall is represented by the sites $x = -1$ and odd $t > 0$, where the wall sites are *wet*, or occupied, with probability p_w and *dry* (unoccupied) with probability $q_w = 1 - p_w$.

1.2.1. Cluster parameters. We define a *compact* cluster of occupied sites, as in figure 1, which we can measure by the following parameters:

$$a_t = \text{the } x\text{-coordinate of the highest occupied site in a column at } t, \tag{1.3}$$

$$b_t = \text{the } x\text{-coordinate of the lowest occupied site in a column at } t, \tag{1.4}$$

$$m_t = \text{the number of sites at a given value of } t: \tag{1.5}$$

$$= \frac{a_t - b_t}{2} + 1 \tag{1.6}$$

$$m := m_0 \tag{1.7}$$

$$= \text{the `seed' which begins the cluster,} \tag{1.8}$$

Table 1. Values of cluster parameters for the cluster shown in figure 1.

t	0	1	2	3	4	5	6	7	8
a_t	0	1	2	1	2	3	2	1	–
b_t	0	1	0	−1	0	−1	0	1	–
m_t	1	1	2	2	2	3	2	1	0

$$T = \text{ the } t\text{-value where the cluster terminates, for finite clusters:} \tag{1.9}$$

$$\ell := T + 1 \tag{1.10}$$

$$= \text{ the cluster } \textit{length}\text{, defined as the number of occupied columns.} \tag{1.11}$$

See table 1 for the cluster parameters corresponding to the example in figure 1.

For any finite cluster, we will always have $a_T = b_T$ and $m_T = 1$, as the cluster terminates with a total of $T + 1$ occupied columns for $t \in [0, T]$. By definition, $m_t = 0$ for $t > T$.

1.2.2. Wall location. Although the location of the wall is a relative measure (if we specify a seed to start 'adjacent to the wall', then changing the x-value of the wall location does not change the problem), for clarity we will explain the differing wall locations used in work on directed compact percolation.

Our earlier work on the damp model in [16] had a wall located at $x = 1$, simply for convenience so that the walks associated with the cluster remained in the region $x \geqslant 0$. However, in this paper we choose to situate the wall at $x = -1$, in line with the location of the wet wall considered in [14] and the dry wall considered in [15], [11] and [4]. This makes it easier to see that the extremes of the damp wall case map to the wet and dry wall models, by simply setting the wall occupancy probability to be $p_w = 1$ and $p_w = 0$, respectively.

1.2.3. Inclusion of wall sites. Note that in previous work on the wet wall case [14], the sites on the wall, at $x = -1$, were not considered to be part of the cluster since this would produce an infinite cluster each time, as all wall sites are wet with certainty. However, the damp wall model lends itself more naturally to including wall sites in the cluster, as these will vary between wet and dry and will affect the probability of a given cluster, so the cluster parameters in section 1.2.1 have been defined to include wall sites. These parameters could be altered to reflect the wet wall definition from [14] by defining $a_t \geqslant 0$ and $b_t \geqslant 0$, thus considering the cluster to be formed by occupied sites which do not lie on the wall.

In the dry wall case, it would never be possible for wall sites to form part of the cluster, as they are all dry with certainty. This will remain true for the dry wall extreme of the damp wall model, despite the definition allowing for wall sites to form part of the cluster, because the dry wall extreme (setting $p_w = 0$) will result in all wall sites being dry and hence not forming part of clusters, as required.

1.2.4. Probability of a cluster. Each cluster will have a particular probability of forming from a given seed, based on the rules of the directed compact growth model. Given a particular cluster, we can calculate the probability of it having formed by noting its cluster length, ℓ, the number of wet sites at $x = -1$ (on the wall) v_1 and the number of wet sites at $x = 0$ (next to the wall) v_2. The probability of such a configuration $\varphi_{\ell,v_1,v_2} \in \Omega_{\ell,v_1,v_2}$ is [16]

$$\pi(\varphi_{\ell,v_1,v_2}) = (pq)^{\ell-1} q^2 \left(\frac{p_w}{pq}\right)^{v_1} \left(\frac{q_w}{q}\right)^{v_2-v_1}, \tag{1.12}$$

where $(pq)^{\ell-1}$ corresponds to weighting with pq for each growth stage after the seed and q^2 corresponds to the probability of the cluster terminating—these two factors are equivalent to the probability of a cluster in the bulk, which is then adjusted to account for interactions with the damp wall. The factor of $\left(\frac{p_w}{pq}\right)^{v_1}$ corresponds to adjusting for the v_1 wet wall sites. For each of these, we include the wall occupancy probability p_w rather than the bulk occupancy probability p, and remove a factor of q since the adjacent site below the wall cannot become occupied. The factor of $(\frac{q_w}{q})^{v_2-v_1}$ corresponds to adjusting for interactions with dry wall sites, counted by the $(v_2 - v_1)$ wet sites at $x = 0$ with no adjacent wet wall site. For these sites, we include the probability of a dry wall site q_w rather than the bulk probability q.

So applying (1.12) to the particular example of a cluster in figure 1, which has $\ell = 8$, $v_1 = 2$ and $v_2 = 4$, we have

$$\pi(\varphi_{8,2,4}) = (pq)^7 q^2 \left(\frac{p_w}{pq}\right)^2 \left(\frac{q_w}{q}\right)^2 \tag{1.13}$$

$$= p^5 q^5 p_w^2 q_w^2. \tag{1.14}$$

1.3. Properties of interest

We define the properties we will consider in this paper with respect to a general cluster, using the cluster parameters introduced in section 1.2.1.

1.3.1. Percolation probability. In general, for any directed percolation model, the percolation probability P is defined as the probability that a cluster grown from a given seed never terminates, and hence $Q = 1 - P$ is the probability of a finite cluster. In terms of the cluster parameters, we can define

$$P = \Pr\big(\lim_{t\to\infty} m_t > 0\big). \tag{1.15}$$

In our particular model of directed compact percolation near a damp wall, P is a function of the bulk occupancy probability p and the wall occupancy probability p_w, so we denote the percolation probability by $P(p, p_w)$. We can calculate the probability of a finite cluster by summing over individual cluster probabilities, and so we have $Q(p, p_w)$, as given in [16],

$$Q(p, p_w) = \left(\frac{q}{p}\right) \sum_{\ell=1}^{\infty} \sum_{v_1,v_2} c_{\ell,v_1,v_2} \left(\frac{p_w}{pq_w}\right)^{v_1} \left(\frac{q_w}{q}\right)^{v_2} (pq)^{\ell}, \tag{1.16}$$

where $c_{\ell,v_1,v_2} = |\Omega_{\ell,v_1,v_2}|$.

We expect $P(p, p_w)$ to be zero below some value of $p = p_c$, which we call the percolation threshold, or critical probability. Approaching the curve from the $P(p, p_w) > 0$ side, $P(p, p_w)$ vanishes with critical exponent β. That is, as $p \to p_c$,

$$P(p, p_w) \sim |p - p_c|^{\beta}. \tag{1.17}$$

We note that the critical point has been found to occur at the same point, $p_c = \frac{1}{2}$, for all cases considered in this paper—that is, in the bulk [10], wet wall [14], dry wall [15] and damp wall [16] cases of directed compact percolation.

1.3.2. Mean cluster length. The length of the cluster is defined generally to be the number of sites in the shortest path from seed to terminal point, so we define the mean length L to be

$$L = \langle \ell \rangle, \tag{1.18}$$

where $\ell = T + 1$ is the number of occupied columns in a given cluster.

For the directed compact percolation model, we can define $L(p, p_w)$ in terms of the cluster probabilities and lengths, as

$$L(p, p_w) = \frac{\sum_{\ell=1}^{\infty} \sum_{v_1,v_2} \ell\, c_{\ell,v_1,v_2} \left(\frac{p_w}{pq_w}\right)^{v_1} \left(\frac{q_w}{q}\right)^{v_2} (pq)^{\ell}}{\sum_{\ell=1}^{\infty} \sum_{v_1,v_2} c_{\ell,v_1,v_2} \left(\frac{p_w}{pq_w}\right)^{v_1} \left(\frac{q_w}{q}\right)^{v_2} (pq)^{\ell}}, \tag{1.19}$$

where the denominator is equal to $Q(p, p_w)$, as $L(p, p_w)$ is the *normalized* mean length, and we can define the unnormalized mean length to be

$$\bar{L}(p, p_w) = L(p, p_w)Q(p, p_w). \tag{1.20}$$

We note that for $p < p_c$, $Q(p, p_w) = 1$ and thus $L(p, p_w) = \bar{L}(p, p_w)$ for the low-density region.

The associated critical exponent for the mean length, τ, tells us how the mean length behaves near the phase transition at $p_c = \frac{1}{2}$. We have, as $p \to p_c$,

$$L(p, p_w) \sim |p - p_c|^{-\tau}. \tag{1.21}$$

1.3.3. Mean number of wall contacts. We will define a 'wall contact' to mean that the cluster *includes a site on the wall*, and as such will be equivalent to counting the number of wet wall sites in a cluster, or the number of occupied sites at $x = -1$. In terms of cluster parameters, this corresponds to each time $b_t = -1$. So we can define the mean number of wall contacts to be

$$N = \left\langle \sum_{t=1}^{T} \delta_{b_t,-1} \right\rangle. \tag{1.22}$$

Note that for the dry wall model considered in [4], a contact with the wall was defined as a site in the cluster being *adjacent* to the wall, i.e. at $x = 0$, as this was the smallest x-value that could contain an occupied site in the dry wall case. To account for both definitions, and to be able to count the times the cluster is at any value of x, we define generally N_x, which is the mean number of times the cluster includes a site at a given x-value:

$$N_x = \left\langle \sum_{t=1}^{T} \sum_{c=b_t}^{a_t} \delta_{c,x} \right\rangle. \tag{1.23}$$

For the directed compact percolation model in the damp wall case, we define $N(p, p_w)$ as

$$N(p, p_w) := N_{-1}(p, p_w) = \langle v_1 \rangle, \tag{1.24}$$

where we recall that v_1 is the number of wet sites on the damp wall. Naturally if we apply this definition of a wall contact to the dry wall case, we expect our result to be equal to zero, since there will be no contacts with the wall. We can define

$$N_0(p, p_w) = \langle v_2 \rangle, \tag{1.25}$$

which counts the mean number of sites adjacent to the damp wall and agrees with the definition given in [4] for the dry wall case. However, in the damp wall model, it is preferable to define wall contacts as wet sites at $x = -1$, since wall sites will vary between wet and dry and we wish to include the wall sites as part of the cluster, as discussed in section 1.2.3. So we will not proceed with the calculation of $N_0(p, p_w)$, but the notation will be helpful in our comparison with the dry wall model.

We define ψ to be the critical exponent for the mean number of wall contacts, such that as $p \to p_c$, we have

$$N(p, p_w) \sim |p - p_c|^{-\psi}. \tag{1.26}$$

As with mean length, we will work with the unnormalized mean number of contacts, which we define to be

$$\bar{N}(p, p_w) = N(p, p_w)Q(p, p_w) \tag{1.27}$$

again noting that $\bar{N}(p, p_w) = N(p, p_w)$ for the low-density region.

2. Previously considered cases

We review the results found previously for percolation in the bulk, near a wet wall and near a dry wall. These results will be compared to calculations for the damp wall model.

2.1. Bulk case

This is the case of directed compact percolation which is unrestricted, i.e. in the absence of any walls. As such the properties calculated do not depend on the wall occupation probability p_w, but only on the bulk occupancy probability p.

2.1.1. Percolation probability (bulk case). The percolation probability for a cluster in the bulk case grown from a seed of width m was found in [10] to be

$$P_m^{\text{bulk}}(p) = \begin{cases} 0 & p < \dfrac{1}{2} \\ 1 - \left(\dfrac{1-p}{p}\right)^{2m} & p \geqslant \dfrac{1}{2} \end{cases} \tag{2.1}$$

with critical exponent $\beta^{\text{bulk}} = 1$.

2.1.2. Mean cluster length (bulk case). Essam [10] showed the normalized mean cluster length for the bulk problem to be the same both above and below the percolation threshold p_c, and expressed simply as

$$L_m^{\text{bulk}}(p) = \frac{m}{|1 - 2p|}, \tag{2.2}$$

with critical exponent $\tau^{\text{bulk}} = 1$.

2.2. Wet wall

We introduce a wall at $x = -1$ and restrict the cluster to only those sites $x \geqslant 0$. Note that, in line with previous work on the wet wall [14], the results presented below do not consider sites on the wall to form part of the cluster in the wet wall case, as discussed in section 1.2.3.

We require that all sites on the wall be wet with certainty, equivalent to setting the wall occupancy probability $p_w = 1$, which results in our wet wall case. This, along with the rules for compactness, has the effect of causing the cluster to remain attached to the wall.

2.2.1. Percolation probability (wet wall). The percolation probability in the wet wall case for a cluster grown from a seed of a single site adjacent to the wall was found in [14] to be

$$P^{\mathrm{wet}}(p) = \begin{cases} 0 & p < \dfrac{1}{2} \\ 1 - \left(\dfrac{1-p}{p}\right) & p \geqslant \dfrac{1}{2}, \end{cases} \tag{2.3}$$

with critical exponent $\beta^{\mathrm{wet}} = 1$.

2.2.2. Mean cluster length (wet wall). The average cluster length in the wet wall case was found, for a seed of width 1 adjacent to the wall, by Essam and TanlaKishani [14] to be

$$L^{\mathrm{wet}}(p) = \frac{1}{|2p-1|}. \tag{2.4}$$

We note that the mean length of finite clusters for a wet wall is identical to that of the bulk case, with exponent $\tau^{\mathrm{wet}} = 1$.

2.3. Dry wall

The dry wall is the case where all sites on the wall, at $x = -1$, remain dry with certainty—with the effect that the clusters are not constrained to remain attached to the wall.

2.3.1. Comparing the general damp case to a dry wall. The general damp wall can be made into a dry wall in one of two ways: the natural way is to set $p_w = 0$, in which case the wall at $x = -1$ will be forced to be dry and the seed at the origin will start one step from the wall, only able to propagate upwards, away from the wall, in the first growth step.

Alternatively we can set $p_w = p$, allowing the wall sites to be filled with the bulk occupancy probability. In this way, the wall sites at $x = -1$ can be considered to be part of the bulk, as they are filled with the same probability, and we can consider the position of the wall to have 'shifted' down by one unit, to $x = -2$. We note that all sites at $x = -2$ will be unoccupied with certainty; thus, we have a dry wall at $x = -2$. Also, with the seed at the origin and the wall location shifted down one unit, we have the modification that the seed is starting two steps from the wall rather than adjacent to it.

This second mapping to a dry wall is convenient in reconciling the difference in definition for mean number of contacts discussed in section 1.3.3, which we will explore in section 2.3.4.

We will investigate both scenarios in our comparison of the damp wall model to the dry wall extreme for all properties calculated in this paper, and we expect that it should be possible to obtain the same dry wall result from both situations.

2.3.2. Percolation probability (dry wall). The percolation probability in the dry wall case, for a cluster grown from a seed of width 1 adjacent to the wall, was found in [15] to be

$$P^{\mathrm{dry}}(p) = \begin{cases} 0 & p < \dfrac{1}{2} \\ \dfrac{(2p-1)^2}{p^3} & p \geqslant \dfrac{1}{2}, \end{cases} \tag{2.5}$$

with critical exponent $\beta^{\mathrm{dry}} = 2$. Note that this differs from the exponent for the percolation probability found in the bulk and the wet wall cases $\beta^{\mathrm{bulk}} = \beta^{\mathrm{wet}} = 1$.

2.3.3. Mean cluster length (dry wall). In [4], an exact expression for the unnormalized mean length is obtained in the dry case:

$$\bar{L}^{\mathrm{dry}}(p) = \theta(p - p_c)\frac{q(3-2p)}{p^3} + L^*(p), \tag{2.6}$$

where $\theta(p - p_c)$ is the unit step function, which is zero for values of p below $p_c = \frac{1}{2}$, and

$$L^*(p) = \frac{q^2}{p}\sum_{r=0}^{\infty}[(2r+2)C_{r+1}(pq)^{r+1} + (2r+1)C_r(pq)^r]\sum_{s=r+1}^{\infty}C_s(pq)^s. \tag{2.7}$$

Also in [4], Zeilberger's algorithm [21] was used to express this in terms of the elliptic integrals $K(m)$ and $E(m)$:

$$L^*(p) = \frac{1}{8p^3}\left(-5 + 4z + 6\sqrt{1-4z} - \frac{8E(16z^2)}{\pi} + \frac{2(3-4z)(1+4z)K(16z^2)}{\pi}\right), \tag{2.8}$$

where $z = p(1-p)$. Note that the square root evaluates to $|1-2p|$.

We note that despite the dry wall percolation probability being a simple rational function, elliptic integrals are required to express the mean cluster length in the dry wall case. The asymptotic form as $p \to 1/2$ in the dry case, conjectured in [11] and confirmed in [4], is

$$\bar{L}^{\mathrm{dry}}(p) \cong B\log|2p-1| + C^{\pm}, \tag{2.9}$$

where $B = -\frac{8}{\pi}$ and $C^{\pm} = \frac{4\log 8 - 8}{\pi} \mp 4$. This implies that, effectively, $\tau^{\mathrm{dry}} = 0$, and again this is different to the exponent found in the bulk and wet wall cases.

2.3.4. Mean number of occupied sites at or near the wall (dry wall). In the dry wall case, unlike the wet wall, the cluster is not constrained to remain adjacent to the wall and may move away or towards the wall, so we are interested in the cluster's interaction with the wall. As discussed in section 1.3.3, the definition of *wall contact* differs between this paper and [4], due to the different requirements of the damp wall model. Based on our definition, we can say that the mean number of contacts in the dry case is

$$N^{\mathrm{dry}} := N_{-1}(p, 0) = 0.$$

However, the expression calculated in [4] is equal to the mean number of times the cluster is adjacent to the wall, at $x = 0$, so we denote this by N_0, corresponding to the mean number of occupied sites at $x = 0$ for the dry wall case $p_w = 0$:

$$N_0^{\mathrm{dry}} := N_0(p, 0).$$

The unnormalized value for this in the dry wall case was found in [4] to be

$$\bar{N}^{\mathrm{dry}}(p) = \theta(p - p_c)\frac{q(1-2q^3)}{p^4} - \frac{q}{p}Q(p) + N^*(p), \tag{2.10}$$

where

$$N^*(p) = \frac{(1-p)^3(1-2p)}{p}\sum_{r=0}^{\infty}(C_r(pq)^r + C_{r+1}(pq)^{r+1})\sum_{s=r+1}^{\infty}(s-r)C_s(pq)^{s-1}. \tag{2.11}$$

As with the mean length expression, Brak and Essam [4] used Zeilberger's algorithm [21] to express $N^*(p)$ in terms of elliptic integrals:

$$N^*(p) = \frac{(1-2p)}{8p^4}\left(1 - 4z - 2(1-2z)\sqrt{1-4z} + \frac{4z(1+2z)}{\sqrt{1-4z}} + \frac{8E(16z^2)}{\pi} - \frac{2(3-4z)(1+4z)K(16z^2)}{\pi}\right), \tag{2.12}$$

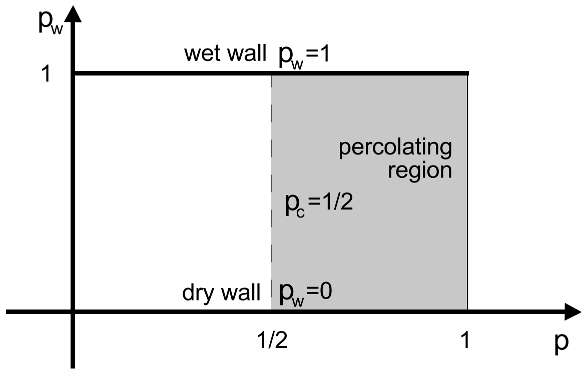

Figure 2. A visual representation of the damp wall parameters, with bulk occupancy p on the horizontal axis and wall occupancy p_w on the vertical axis.

where $z = p(1-p)$ and $\sqrt{1-4z} = |1-2p|$. The asymptotic form near $p_c = \frac{1}{2}$, also found in [4], is

$$\bar{N}^{\text{dry}}(p) \cong 2 + \frac{16}{\pi}(1-2p)\log|2p-1|. \tag{2.13}$$

These calculations from [4], despite using a different definition of *wall contact*, will still be helpful as a comparison for the dry wall case. As discussed in section 2.3.1, if we set $p_w = p$ in the damp wall case, this corresponds to the dry wall case with a shifted wall location, effectively at $x = -2$. As such, counting the sites at $x = -1$ becomes equivalent to counting sites adjacent to a dry wall. So we expect that

$$N_{-1}(p, p) \sim N_0(p, 0) \tag{2.14}$$

although we note that the different seed location relative to the wall will cause some minor difference between the two, but they should display the same critical behaviour.

2.4. Damp wall: phase diagram and percolation probability

2.4.1. Phase diagram. A phase diagram for the model of directed compact percolation near a damp wall is given in figure 2. Most significantly, we note that the location of the critical point at $p = \frac{1}{2}$ does not change on varying p_w.

We see that the extremes $p_w = 1$ and $p_w = 0$ correspond to the wet and dry wall cases, respectively. However, as discussed in section 2.3.1, $p_w = p$ also becomes equivalent to the dry wall case with the location of the wall effectively shifted to $x = -2$.

2.4.2. Percolation probability. In the case of a damp wall, the percolation probability was solved in [16] by mapping the clusters to pairs of directed walks and found to be

$$P(p, p_w) = \begin{cases} \dfrac{(1-2p)^2}{p^2(p - p_w + pp_w)} & p > \dfrac{1}{2} \\ 0 & p \leqslant \dfrac{1}{2}, \end{cases} \tag{2.15}$$

with critical exponent $\beta^{\text{damp}} = 2$, except in the wet wall limit $p_w = 1$ where $\beta = 1$. We will explore this mapping to walks and the associated generating function found in [16], and use similar methods to find expressions for the other properties of interest in the damp case.

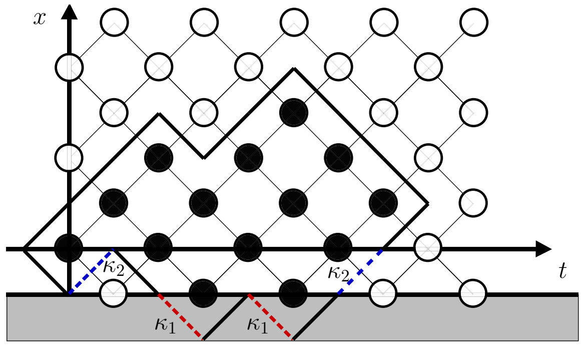

Figure 3. Mapping the cluster to pairs of directed walks. The probability of this cluster forming is $p^5q^5p_w^2q_w^2$. The weighting of the associated damp vesicle is $\kappa_1^2\kappa_2^2z^9$ and the weighting of the corresponding pair of non-intersecting walks (where we truncate the first and last steps of the walks which make up the vesicle) is $g(z, \kappa_1, \kappa_2) = \kappa_1^2\kappa_2^2z^7$. We note the relationship between the weighting and the cluster probability, i.e. $q^2g\left(pq, \frac{p_w}{pq}, \frac{q_w}{q}\right) = q^2\left(\frac{p_w}{pq}\right)^2\left(\frac{q_w}{q}\right)^2(pq)^7 = p^5q^5p_w^2q_w^2$.

3. Damp wall: generating function

We review the mapping to pairs of directed walks used for the damp wall case in [16] and express the properties of interest in terms of the generating function for directed walks. We then go on to rearrange this generating function to a form which will be more convenient to calculate the properties needed.

3.1. Mapping to pairs of directed walks

For clusters of seed width $m = 1$, which were the focus of the work in [16], we can uniquely map each cluster to a staircase polygon, or vesicle, that encloses the cluster. We define two walks to begin at $(-1, a_0)$ and $(-1, b_0)$, and terminate at $(T + 1, a_T)$ and $(T + 1, b_T)$. Noting that $a_T = b_T$, and that for $m = 1$ we also have $a_0 = b_0$, this has the effect of enclosing the cluster in a vesicle, which will be a staircase polygon. We weight the vesicle with parameters that we will later relate to the percolation problem.

We add a weighting of z^t to a vesicle with perimeter $2t$—that is, weighting according to its half-perimeter. We also add two different weightings, or fugacities, for wall interaction: we add a factor of κ_1 for each step from $x = -1$ to $x = -2$ and a factor of κ_2 for each step from $x = -1$ to $x = 0$. See figure 3 for an example of the mapping for an individual cluster, with corresponding probability and weighted walks.

As in [16], we denote by d_{t',u_1,u_2} the number of staircase polygons of half-perimeter t', with u_1 steps from $x = -1$ to $x = -2$ and u_2 steps from $x = -1$ to $x = 0$. So we can express the generating function for these *damp vesicles* as

$$G_{dv}(z; \kappa_1, \kappa_2) = \sum_{t'=2}\sum_{u_1,u_2} d_{t',u_1,u_2}\kappa_1^{u_1}\kappa_2^{u_2}z^{t'}. \tag{3.1}$$

In fact it is convenient for us to truncate the vesicle at each end—resulting in pairs of strictly avoiding walks, starting and ending one step apart. The length of each such walk corresponds to the number of time steps in the corresponding cluster. However, we do not wish to lose the weighting of κ_1 or κ_2 that might be present on one of these truncated steps. So we simply define the generating function we are interested in relating to the percolation problem,

$G(z; \kappa_1, \kappa_2)$, to be the damp vesicle generating function with two factors of z removed:

$$G(z; \kappa_1, \kappa_2) = \frac{1}{z^2} G_{dv}(z; \kappa_1, \kappa_2). \tag{3.2}$$

An expression for this generating function was found in [16], and we will go on to express this in a more convenient form in section 3.5.

3.2. Percolation probability in terms of $G(z; \kappa_1, \kappa_2)$

We note that if we can relate the weighted walks to the percolation problem, then the generating function for these walks will lead us to the sum of the probabilities of all finite clusters grown from a seed of width 1 adjacent to the wall. This is equivalent to $Q(p, p_w)$, and hence will lead to an expression for $P(p, p_w)$ which is the probability of an infinite cluster being grown from the seed.

In relating the weighted walks to the percolation problem, we note that a step weighted with κ_1 is equivalent to a wet wall site in the cluster, and κ_2 is equivalent to a dry wall site. With this in mind, we find the values of z, κ_1 and κ_2 corresponding to the percolation problem to be

$$z = pq, \qquad \kappa_1 = \frac{p_w}{pq}, \qquad \kappa_2 = \frac{q_w}{q}. \tag{3.3}$$

We can hence express $Q(p, p_w)$ in terms of the generating function,

$$Q(p, p_w) = q^2 G\left(pq, \frac{p_w}{pq}, \frac{q_w}{q}\right). \tag{3.4}$$

We note that the extra factor of q^2 is to account for the probability of a cluster terminating. Hence, we have the following relationship between the percolation probability and the generating function:

$$P(p, p_w) = 1 - q^2 G\left(pq, \frac{p_w}{pq}, \frac{q_w}{q}\right) \tag{3.5}$$

$$= \begin{cases} \dfrac{(1-2p)^2}{p^2(p - p_w + pp_w)} & p > \dfrac{1}{2} \\ 0 & p \leqslant \dfrac{1}{2}. \end{cases} \tag{3.6}$$

This result, found in [16], leads us to investigate other properties of the percolation cluster and how we can relate them to the generating function for directed walks.

3.3. Mean length in terms of $G(z; \kappa_1 \kappa_2)$

We recall the definition of the cluster length as the number of particles in the shortest path from the seed to the terminal point, including the seed. The generating function gives a weighting of z for each 'time step' in a cluster's growth, not including the seed. As a result, a cluster with t time steps (and hence length $t + 1$) will gain a weighting of z^t. By differentiating $zG(z; \kappa_1, \kappa_2)$ by z we effectively multiply that coefficient of z^t by $t + 1$. To find the unnormalized mean length we than evaluate at the values corresponding to the percolation problem—remembering that we must also adjust by a factor of q^2 to bring the walk model in line with the percolation problem.

So the unnormalized mean cluster length (noting that above p_c the average is taken only over finite clusters) is given by

$$\bar{L}(p, p_w) = (1 - p)^2 \frac{\mathrm{d}}{\mathrm{d}z}(zG(z; \kappa_1, \kappa_2)) \tag{3.7}$$

evaluated at $z = p(1 - p)$, $\kappa_1 = \frac{p_w}{p(1-p)}$, $\kappa_2 = \frac{1-p_w}{1-p}$.

3.4. Mean number of wall contacts in terms of $G(z;\kappa_1,\kappa_2)$

The number of wall contacts is defined to be the number of sites on the wall which are included in the cluster, which corresponds to the number of wet wall sites in the cluster. As the generating function $G(z;\kappa_1,\kappa_2)$ gives a weighting of κ_1 for each wet wall site, we can define the unnormalized mean number of contacts with the wall as

$$\bar{N}(p)=(1-p)^2\kappa_1\frac{\partial}{\partial\kappa_1}(G(z;\kappa_1,\kappa_2))|_{z=p(1-p),\ \kappa_1=\frac{pw}{p(1-p)},\ \kappa_2=\frac{1-pw}{1-p}}. \tag{3.8}$$

3.5. Re-expressing the generating function

Before we use the generating function to find the above properties of interest, we first manipulate it into a more convenient form. In [16], the variables c and d were introduced by

$$\kappa_1=(1+c)(1+d)\qquad\text{and}\qquad\kappa_2=1-cd \tag{3.9}$$

in terms of which we further define

$$\omega_c=\frac{c}{(1+c)^2}\qquad\text{and}\qquad\omega_d=\frac{d}{(1+d)^2}. \tag{3.10}$$

We take the generating function found in [16], $G(z;\kappa_1,\kappa_2)$, and split its terms up according to the presence of partition functions with respect to c, d or neither, so it is of the form

$$G(z;\kappa_1,\kappa_2)=\Gamma_0(z)+\Gamma_c(z)-\Gamma_d(z), \tag{3.11}$$

where

$$\Gamma_0(z)=-A_2(z)\omega_c(c-d)\kappa_2 \tag{3.12}$$

$$\Gamma_c(z)=A_1\frac{\omega_c}{z}\sum_{r=1}^{\infty}C_r[Z_{2r-2}(c)z^{2r-1}+Z_{2r}(c)z^{2r}] \\ -A_2(z)\omega_c\sum_{r=1}^{\infty}C_r[Z_{2r+2}(c)z^{2r}+Z_{2r-2}(c)z^{2r-2}] \tag{3.13}$$

$$\Gamma_d(z)=A_1\frac{\omega_c}{z}\sum_{r=1}^{\infty}C_r[Z_{2r-2}(d)z^{2r-1}+Z_{2r}(d)z^{2r}] \\ -A_2(z)\omega_c\sum_{r=1}^{\infty}C_r[Z_{2r+2}(d)z^{2r}+Z_{2r-2}(d)z^{2r-2}], \tag{3.14}$$

where we have defined for convenience of expression

$$A_1=\frac{1}{(c-d)\omega_c} \tag{3.15}$$

$$A_2(z)=\frac{cd}{\omega_c\kappa_1^2(c-d)(z^2-\omega_c\omega_d)}. \tag{3.16}$$

Also note that $Z_{2r}(d)$ is a sum over Dyck paths of length $2r$ weighted with a factor $\kappa=1+d$ for each contact with the axis (the contact polynomial, denoted $\hat{Z}^S_{2r}$ in [5]). It was shown in equations (3.19) and (3.23) of [5] that

$$Z_{2r}(d)=(1+d)\sum_{m=0}^{r}B_{2r,2m}d^m, \tag{3.17}$$

where $B_{t,h}$ is the Ballot number, as defined in [3],

$$B_{t,h} = \frac{(h+1)t!}{\left(\frac{1}{2}(t+h)+1\right)!\left(\frac{1}{2}(t-h)\right)!}. \tag{3.18}$$

The expression in (3.17) is also valid for $Z_{2r}(c)$, but in the subsequent application to percolation theory an expansion in terms of ω_c will be more useful. We note that $Z_{2r}(c)$ satisfies the recurrence relation, found in [3],

$$\omega_c Z_{2r}(c) = -C_{r-1} + Z_{2r-2}(c), \tag{3.19}$$

which allows us to re-express $\Gamma_c(z)$ and $\Gamma_d(z)$ as follows:

$$\begin{aligned}\Gamma_c(z) = {} & \left(1+\frac{\omega_c}{z}\right)\left[A_1\frac{\omega_c}{z} - A_2(z)\left(1-\frac{\omega_c}{z}\right)\right]\sum_{r=1}^{\infty} C_r Z_{2r}(c) z^{2r} \\ & + \frac{\omega_c}{z^2}\left(A_1 + A_2(z)\right)\sum_{r=1}^{\infty} C_r C_{r-1} + A_2(z)\sum_{r=1}^{\infty} C_r^2 z^{2r}\end{aligned} \tag{3.20}$$

$$\begin{aligned}\Gamma_d(z) = {} & \left(1+\frac{\omega_d}{z}\right)\left[A_1\frac{\omega_c}{z} - A_2(z)\left(\frac{\omega_c}{\omega_d}-\frac{\omega_c}{z}\right)\right]\sum_{r=1}^{\infty} C_r Z_{2r}(d) z^{2r} \\ & + \frac{\omega_c}{z^2}\left(A_1 + A_2(z)\right)\sum_{r=1}^{\infty} C_r C_{r-1} + A_2(z)\frac{\omega_c}{\omega_d}\sum_{r=1}^{\infty} C_r^2 z^{2r}.\end{aligned} \tag{3.21}$$

So we note that the terms involving $C_r C_{r-1}$ will cancel in $\Gamma_c(z) - \Gamma_d(z)$,

$$\begin{aligned}\Gamma_c(z) - \Gamma_d(z) = {} & \left(1+\frac{\omega_c}{z}\right)\left[A_1\frac{\omega_c}{z} - A_2(z)\left(1-\frac{\omega_c}{z}\right)\right]\sum_{r=1}^{\infty} C_r Z_{2r}(c) z^{2r} \\ & - \left(1+\frac{\omega_d}{z}\right)\left[A_1\frac{\omega_c}{z} - A_2(z)\left(\frac{\omega_c}{\omega_d}-\frac{\omega_c}{z}\right)\right]\sum_{r=1}^{\infty} C_r Z_{2r}(d) z^{2r} \\ & + A_2(z)\left(1-\frac{\omega_c}{\omega_d}\right)\sum_{r=1}^{\infty} C_r^2 z^{2r}.\end{aligned} \tag{3.22}$$

This manipulation has been done to make the following calculations simpler, and also to assist in making contact with previously studied cases.

4. Mean cluster length

We recall that the cluster length is the number of particles in the shortest path from the seed to the terminal point, including the seed, and the unnormalized mean cluster length is given by

$$\bar{L}(p, p_w) = (1-p)^2 \frac{\mathrm{d}}{\mathrm{d}z}\left(zG(z;\kappa_1,\kappa_2)\right)_{z=pq,\ \kappa_1=\frac{p_w}{pq},\ \kappa_2=\frac{q_w}{q}}. \tag{4.1}$$

Recalling from equation (3.4) that $Q(p, p_w)$, the probability of a finite cluster given in [16], is simply related to the generating function at the percolation values, we can hence write the mean length as

$$\bar{L}(p, p_w) = Q(p, p_w) + p(1-p)^3 \left.\frac{\mathrm{d}}{\mathrm{d}z} G(z;\kappa_1,\kappa_2)\right|_{z=p(1-p)}. \tag{4.2}$$

4.1. Calculating the mean cluster length

We proceed with the calculation using the rearranged generating function found in section 3.5. In this and the following percolation calculations, c and d are functions of p and p_w given by

$$c = \frac{p}{q} \qquad \text{and} \qquad d = \frac{p_w - p}{p}. \tag{4.3}$$

Using (3.9), this is in agreement with the values of κ_1 and κ_2 in (3.3). Further using (3.10) gives

$$\omega_c = pq \qquad \text{and} \qquad \omega_d = \frac{p(p_w - p)}{p_w^2}. \tag{4.4}$$

Using (3.11) and (4.2), we can write the mean length of finite clusters as

$$\bar{L}(p, p_w) = Q(p, p_w) + p(1-p)^3(\Gamma_0'(\omega_c) + \Gamma_c'(\omega_c) - \Gamma_d'(\omega_c)). \tag{4.5}$$

Differentiating (3.12) with respect to z, and evaluating at the percolation values, gives

$$\Gamma_0'(\omega_c) = -A_2'(\omega_c)\omega_c(c-d)\kappa_2 = 2c(1+d)^2 A_2 = \frac{2(1+c)^4 d(1+d)^2}{(c-d)^2(1-cd)}, \tag{4.6}$$

where we define

$$A_2 := A_2(\omega_c). \tag{4.7}$$

Differentiating (3.22) gives a much more complicated expression, but if we then substitute $z = \omega_c$ and use the relationships

$$A_2(\omega_c) = \frac{A_1\omega_d}{\omega_c - \omega_d}, \qquad A_2'(\omega_c) = -\frac{2}{\omega_c - \omega_d}A_2 \qquad \text{and} \qquad \frac{A_2}{\omega_d} = \frac{A_1 + A_2}{\omega_c}, \tag{4.8}$$

we have

$$\begin{aligned}\Gamma_c'(\omega_c) - \Gamma_d'(\omega_c) = {} & \frac{4A_1}{\omega_c}\sum_{r=1}^{\infty} rC_r Z_{2r}(c)\omega_c^{2r} - \frac{2A_1}{\omega_c}\sum_{r=1}^{\infty} rC_r^2\omega_c^{2r} - \frac{3A_1 + 2A_2}{\omega_c}\sum_{r=1}^{\infty} C_r Z_{2r}(c)\omega_c^{2r} \\ & + \frac{A_1 + A_2}{\omega_c}\left[2\sum_{r=1}^{\infty} C_r^2\omega_c^{2r} - \left(1 + \frac{\omega_d}{\omega_c}\right)\sum_{r=1}^{\infty} C_r Z_{2r}(d)\omega_c^{2r}\right].\end{aligned} \tag{4.9}$$

In order to make contact with the dry wall limit analysed in [4], we write the partition function in the form

$$Z_{2r}(c) = \omega_c^{-r}\left(1 + c - \sum_{s=0}^{r-1} C_s\omega_c^s\right) \tag{4.10}$$

$$= \omega_c^{-r}\left((c - c^*) + \sum_{s=r}^{\infty} C_s\omega_c^s\right), \tag{4.11}$$

where

$$c^*(z) = \sum_{r=1}^{\infty} C_r z^r = \frac{1 - 2z - \sqrt{1-4z}}{2z} \tag{4.12}$$

$$c^* := c^*(\omega_c) = \begin{cases} c & \text{for} \quad c \leqslant 1 \\ \dfrac{1}{c} & \text{for} \quad c > 1. \end{cases} \tag{4.13}$$

4.2. Result for the mean length of compact clusters

After extensive manipulations described in [12], the final result is

$$\bar{L}(p, p_w) = A_L(p, p_w)\sum_{r=1}^{\infty} C_r^2 (pq)^{2r} + B_L(p, p_w)\sum_{r=1}^{\infty} C_r Z_{2r}(d)(pq)^{2r} + D_L(p, p_w)L^*(p) + E_L(p, p_w), \tag{4.14}$$

where we recall $L^*(p)$ from the mean length in the dry wall case, given in section 2.3.3 in terms of elliptic integrals, and

$$A_L(p, p_w) = \frac{p_w^2(1-p)^3}{(1-p_w)(p-p_w+pp_w)^2} \tag{4.15}$$

$$B_L(p, p_w) = -\frac{(1-p)^2(p_w^2(1-p)+p_w-p)}{(1-p_w)(p-p_w+pp_w)^2} \tag{4.16}$$

$$D_L(p, p_w) = \frac{p}{p-p_w+pp_w} \tag{4.17}$$

$$E_L(p, p_w) = \begin{cases} \dfrac{p(p_w(1-p_w)-p+p^2p_w^2)}{(1-p_w)(p-p_w+pp_w)^2} & \text{for} \quad p \leqslant \frac{1}{2} \\ -\dfrac{q^2\big(1-p_w-(2-3p_w)(1+p_w)q+\big(1-5p_w^2\big)q^2+p_w^2q^3\big)}{(1-p_w)(1-q)^2(1-(1+p_w)q)^2} & \text{for} \quad p > \frac{1}{2}. \end{cases} \tag{4.18}$$

4.3. Analysis of result

We see from (4.14) that the mean length has been grouped into four terms, each with rational coefficients. We consider each of these terms to analyse the result for mean length.

We note that the term with coefficient $A_L(p, p_w)$ can be expressed in terms of elliptic integrals, using a result from [20]

$$\sum_{r=1}^{\infty} C_r^2 (pq)^{2r} = \left[\frac{E(16p^2q^2)}{\pi p^2q^2} - \frac{(1-16p^2q^2)K(16p^2q^2)}{2\pi p^2q^2} - \frac{1}{4p^2q^2} - 1\right], \tag{4.19}$$

noting that the coefficient of $K(16p^2q^2)$ in (4.19) goes to zero as $p \to \frac{1}{2}$, so this term does not contribute to the dominant critical behaviour near $p_c = \frac{1}{2}$.

We now analyse the term in (4.14) with coefficient $B_L(p, p_w)$, making the definition

$$a_r = C_r Z_{2r}(d)(pq)^{2r}. \tag{4.20}$$

We can use the results from [5] for the single walk partition function $Z_{2r}(d)$, along with the asymptotic form of the Catalan numbers, to write

$$a_r \sim \begin{cases} \left(\dfrac{4^r}{r^{3/2}\sqrt{\pi}}\right)\left(\dfrac{2^r}{r^{3/2}}\right)(pq)^{2r} & \text{for} \quad d < 1 \\ \left(\dfrac{4^r}{r^{3/2}\sqrt{\pi}}\right)\left(\dfrac{2^r}{r^{1/2}}\right)(pq)^{2r} & \text{for} \quad d = 1 \\ \left(\dfrac{4^r}{r^{3/2}\sqrt{\pi}}\right)\left(\dfrac{d+1}{\sqrt{d}}\right)^r(pq)^{2r} & \text{for} \quad d > 1, \end{cases} \tag{4.21}$$

where we recall that $d = \frac{p_w - p}{p}$.

We can now simply use the ratio test to prove that $\sum a_r$ converges regardless of $p \in [0, 1]$, $p_w \in [0, 1)$. For $d \leqslant 1$, that is, $p_w \leqslant 2p$, we have

$$\left|\frac{a_{r+1}}{a_r}\right| \sim 8(pq)^2 \leqslant 1/2 \tag{4.22}$$

for $p \leqslant 1$, since $pq \leqslant 1/4$, and so the sum converges. Note that since $p_w < 1$, we have $p_w \leqslant 2p$ automatically when $p > 1/2$. For $d > 1$, that is, $(p_w - p) > p$,

$$\left|\frac{a_{r+1}}{a_r}\right| \sim \frac{4p^2q^2p_w}{\sqrt{p(p_w - p)}} < 4pp_wq^2 < 4pq^2 < 1, \tag{4.23}$$

so once again the sum converges. Thus, despite not being simply expressed, the term with coefficient $B_L(p, p_w)$ always converges—including near $p = \frac{1}{2}$. As a result, it does not contribute to the divergent critical behaviour near $p = \frac{1}{2}$ of the mean length, which we see via the term with coefficient $D_L(p, p_w)$.

Looking at the term with coefficient $D_L(p, p_w)$, we recall that $L^*(p)$ comes from the mean length expression in the dry wall case [4] and can be expressed in terms of elliptic integrals as in equation (2.8). The logarithmic divergence of this term will dominate the critical behaviour of $\bar{L}(p, p_w)$, as we note that the final term is simply $E_L(p, p_w)$, a rational function in p and p_w. So the damp wall result can be expressed in terms of elliptic integrals and is of a similar form to the dry wall case, as we would expect, although it also involves additional terms (the double sum with coefficient $B_L(p, p_w)$) which have not been simply expressed.

4.3.1. Asymptotic behaviour. The behaviour near the critical point $p = \frac{1}{2}$ is dominated by the elliptic integral $K(16p^2q^2)$ from the $L^*(p)$ term, and we have

$$\bar{L}(p, p_w) \cong B \log|2p - 1| + C, \tag{4.24}$$

where

$$B = \frac{-8}{\pi(1 - p_w)} \tag{4.25}$$

$$C = \frac{1}{(1 - p_w)}\left[\frac{4(3\log 2 - 2}{\pi} \mp 4\right] + \frac{8p_w^2}{\pi(1 - p_w)^3} + C^*(p_w) + \frac{1}{(1 - p_w)^3}\begin{cases} p_w + 2p_w^2 - p_w^3 & \text{for} \quad p < \dfrac{1}{2} \\ 8 - 17p_w + 6p_w^2 + p_w^3 & \text{for} \quad p > \dfrac{1}{2}, \end{cases} \tag{4.26}$$

where

$$C^*(p_w) = -\frac{p_w^2 + 2p_w - 1}{2(1 - p_w)^3}\sum_{r=1}^{\infty} C_r Z_{2r}(2p_w - 1)\left(\frac{1}{4}\right)^{2r}. \tag{4.27}$$

So we can see that the damp wall case exhibits similar critical behaviour to the dry wall case. We will explore the relationship between the damp wall and dry wall further in section 4.4.

4.3.2. Expansion. Expanding (4.14) in powers of p and q gives series which agree with those obtained by applying (3.7) to the expression for the generating function found in [16], equation (4.69), both yielding the following expansions:

$$\begin{aligned}\bar{L}(p, p_w) = 1 &+ p_w + \left(1 + p_w + 2p_w^2\right)p + \left(2 + p_w + 5p_w^3\right)p^2 \\ &+ \left(3 + 3p_w + 3p_w^2 - 7p_w^3 + 14p_w^4\right)p^3 \\ &+ \left(6 + 3p_w + 2p_w^2 + 21p_w^3 - 42p_w^4 + 42p_w^5\right)p^4 \\ &+ \left(9 + 11p_w + 9p_w^2 - 37p_w^3 + 138p_w^4 - 198p_w^5 + 132p_w^6\right)p^5 + \cdots\end{aligned} \tag{4.28}$$

$$\bar{L}(1-q, p_w) = (1-p_w)q + \left(5 - 3p_w^2\right)q^2 + \left(13 + 5p_w - 5p_w^2 - 5p_w^3\right)q^3 + \left(26 + 18p_w - p_w^2 - 14p_w^3 - 7p_w^4\right)q^4 + \left(46 + 44p_w + 17p_w^2 - 19p_w^3 - 27p_w^4 - 9p_w^5\right)q^5 + \cdots. \quad (4.29)$$

4.3.3. Behaviour of the mean length near the curve $q = 1/(1+p_w)$ in the low-density region. The amplitudes A_1 and A_2 are divergent at $q = 1/(1+p_w)$, a point in the low-density region $q \geqslant \frac{1}{2}$. This appears to lead to a divergence in the mean length below the critical probability, which is impossible on physical grounds. As discussed in [12], this apparent divergence must be cancelled out by other factors in these terms.

4.4. Comparison to the mean length of a dry wall

We compare the mean length result we have found for the damp wall model to the previously found result for the dry wall case. We consider each of the two mappings to the dry wall, $p_w = 0$ and $p_w = p$, as introduced in section 2.3.1.

4.4.1. The case $p_w = 0$.

When $p_w = 0$,

$$d = -1, \qquad A_L = 0, \qquad Z_{2r}(d) = 0, \qquad D_L(p, 0) = 1 \quad (4.30)$$

and

$$E_L(p, 0) = \begin{cases} 0 & \text{for} \quad p < \dfrac{1}{2} \\ \dfrac{q(1+2q)}{(1-q)^3} & \text{for} \quad p > \dfrac{1}{2}; \end{cases} \quad (4.31)$$

thus,

$$\bar{L}(p, 0) = L^*(p) + \begin{cases} 0 & \text{for} \quad p < p_c \\ \dfrac{q(1+2q)}{(1-q)^3} & \text{for} \quad p > p_c \end{cases} \quad (4.32)$$

$$= \bar{L}^{\text{dry}}(p) \quad (4.33)$$

as expected. So we see that for $p_w = 0$, the damp wall expression describes the dry wall case.

4.4.2. The case $p_w = p$.

When $p_w = p$,

$$d = 0, \qquad A_L(p, p) = -B_L(p, p) = -\frac{q^2}{p^2} \qquad \text{and} \qquad Z_{2r}(d) = C_r, \quad (4.34)$$

so the first two terms of (4.14) cancel. Further $D_L(p, p) = 1/p$ and

$$E_L(p, p) = \begin{cases} -\dfrac{1}{p} & \text{for} \quad p < 1/2 \\ \dfrac{q^2(3+q)}{(1-q)^4} & \text{for} \quad p > 1/2; \end{cases} \quad (4.35)$$

thus,

$$\bar{L}(p,p) = \frac{L^*(p)}{p} + \begin{cases} -\dfrac{1}{p} & \text{for} \quad p < p_c \\ \dfrac{q^2(q+3)}{(1-q)^4} & \text{for} \quad p > p_c \end{cases} \tag{4.36}$$

$$= \frac{L^{\text{dry}}(p) - Q^{\text{dry}}(p)}{p}. \tag{4.37}$$

This relation is expected since with $p_w = p$ the dry wall problem has source at $(0, 1)$ instead of the usual $(0, 0)$. The clusters are therefore one unit shorter and there is one less factor of p.

So we see we are able to equally derive the dry wall result from the damp wall by setting $p_w = p$, although the shift in wall location means that the relationship is not as clear immediately as in the $p_w = 0$ case.

5. Mean number of wall contacts

We now consider the problem of finding the mean number of wall contacts, defined as the number of times the cluster makes contact with the wall. We expect the evaluation of this to yield a similar form of solution to the mean length, as the dry wall case for this property also involves elliptic integrals and the calculation proceeds in a similar way to the mean length calculation.

5.1. Calculating mean number of wall contacts

We recall that the generating function $G(z; \kappa_1, \kappa_2)$ gives a weighting of κ_1 for each wet wall site, and so we can define the unnormalized mean number of contacts with the wall as

$$\bar{N}(p) = (1-p)^2 \kappa_1 \frac{\partial}{\partial \kappa_1} (G(z; \kappa_1, \kappa_2))\bigg|_{z=p(1-p),\ \kappa_1 = \frac{p_w}{p(1-p)},\ \kappa_2 = \frac{1-p_w}{1-p}}. \tag{5.1}$$

We recall the expression for $G(z; \kappa_1, \kappa_2)$ given in (3.11), (3.12) and (3.22) and first attempt to re-express all variables in terms of κ_1 and κ_2. We note the relationships

$$c + d = \kappa_1 + \kappa_2 - 2 \tag{5.2}$$

$$cd = 1 - \kappa_2 \tag{5.3}$$

$$\omega_c \omega_d = \frac{1-\kappa_2}{\kappa_1^2}. \tag{5.4}$$

We can re-express Γ_0 from (3.12) explicitly in terms of z, κ_1 and κ_2 only,

$$\Gamma_0(\kappa_1) = \frac{-\kappa_2(1-\kappa_2)}{\kappa_1^2 z^2 + \kappa_2 - 1}. \tag{5.5}$$

Similarly we will express $\Gamma_c - \Gamma_d$ from (3.22) in terms of z, κ_1 and κ_2, but as it is a cumbersome expression to work with, we will express its terms separately, defining

$$\Gamma_c(\kappa_1) - \Gamma_d(\kappa_1) = \alpha(\kappa_1)\beta(\kappa_1) + \gamma(\kappa_1)\delta(\kappa_1) + \Gamma_3(\kappa_1), \tag{5.6}$$

where (noting that c, d, ω_c and ω_d are functions of κ_1 and κ_2)

$$\alpha(\kappa_1) = \left(1 + \frac{\omega_c}{z}\right) \left[\frac{c\kappa_1^2 z - (1-\kappa_2)(1+c)^2}{c(c-d)(\kappa_1^2 z^2 + \kappa_2 - 1)}\right] \tag{5.7}$$

$$\beta(\kappa_1) = \sum_{r=1}^{\infty} C_r Z_{2r}(c) z^{2r} \tag{5.8}$$

$$\gamma(\kappa_1) = -\left(1 + \frac{\omega_d}{z}\right)\left[\frac{d\kappa_1^2 z - (1-\kappa_2)(1+d)^2}{d(c-d)(\kappa_1^2 z^2 + \kappa_2 - 1)}\right] \tag{5.9}$$

$$\delta(\kappa_1) = \sum_{r=1}^{\infty} C_r Z_{2r}(d) z^{2r} \tag{5.10}$$

$$\Gamma_3(\kappa_1) = \frac{-\kappa_2}{\kappa_1^2 z^2 + \kappa_2 - 1} \sum_{r=1}^{\infty} C_r^2 z^{2r}. \tag{5.11}$$

We have suppressed the dependence on κ_2 and z which are to be held constant when taking the κ_1 derivative, denoted by $'$, below.

5.1.1. Calculating $(\partial/\partial\kappa_1)_{\kappa_2,z}$. Recalling the relationships given in (3.9) and (3.10), we can calculate the partial derivatives with respect to κ_1 as

$$\left(\frac{\partial c}{\partial \kappa_1}\right)_{\kappa_2} = \frac{c}{c-d} \qquad \left(\frac{\partial d}{\partial \kappa_1}\right)_{\kappa_2} = \frac{d}{d-c} \tag{5.12}$$

$$\left(\frac{\partial \omega_c}{\partial \kappa_1}\right)_{\kappa_2} = \frac{c(1-c)}{(1+c)^3(c-d)} \qquad \left(\frac{\partial \omega_d}{\partial \kappa_1}\right)_{\kappa_2} = \frac{d(1-d)}{(1+d)^3(d-c)}. \tag{5.13}$$

Differentiating (5.5) gives

$$\Gamma_0'(\kappa_1) = \frac{2\kappa_1\kappa_2 z^2(1-\kappa_2)}{(\kappa_1^2 z^2 + \kappa_2 - 1)^2}. \tag{5.14}$$

Evaluating this at the percolation values, we have

$$\Gamma_0'\left(\frac{p_w}{pq}\right) = \frac{2pp_w(1-p)(p_w-p)}{(1-p_w)(p-p_w+pp_w)^2}. \tag{5.15}$$

The derivative of (5.6) gives

$$\Gamma_c'(\kappa_1) - \Gamma_d'(\kappa_1) = \alpha(\kappa_1)\beta'(\kappa_1) + \alpha'(\kappa_1)\beta(\kappa_1) + \gamma(\kappa_1)\delta'(\kappa_1) + \gamma'(\kappa_1)\delta(\kappa_1) + \Gamma_3'(\kappa_1), \tag{5.16}$$

where we are interested in the value of this at $\kappa_1 = \frac{p_w}{pq}$. We can calculate expressions for each term in (5.16), with the following results:

$$\alpha\left(\frac{p_w}{pq}\right) = \frac{2}{p - p_w + pp_w} \tag{5.17}$$

$$\alpha'\left(\frac{p_w}{pq}\right) = \frac{p(1-p)(1-2p)}{(p-p_w+pp_w)^2} - \frac{2pp_w(1-p)^2(2p-p_w)}{(1-p_w)(p-p_w+pp_w)^3} \tag{5.18}$$

$$\beta\left(\frac{p_w}{pq}\right) = \frac{1}{1-p}\sum_{r=1}^{\infty} C_r u^r - \sum_{r=1}^{\infty} C_r u^r \sum_{s=0}^{r-1} C_s u^s, \qquad \text{where } u = p(1-p), \tag{5.19}$$

$$\beta'\left(\frac{p_w}{pq}\right) = \frac{p^2}{p - p_w + pp_w}\left[\sum_{r=1}^{\infty} C_r u^r - (1-p)(1-2p)\frac{\mathrm{d}}{\mathrm{d}u}\left(\sum_{r=1}^{\infty} C_r u^r\right)\right.$$
$$\left. - (1-p)^2(1-2p)\sum_{r=1}^{\infty}\sum_{s=0}^{r-1}(s-r)C_r C_s u^{r+s-1}\right] \tag{5.20}$$

$$\gamma\left(\frac{p_w}{pq}\right) = 0 \tag{5.21}$$

$$\gamma'\left(\frac{p_w}{pq}\right) = \frac{p(1-p)(1-2p)(p_w - p + p_w^2(1-p))}{(1-p_w)(p-p_w+pp_w)^3} \tag{5.22}$$

$$\delta\left(\frac{p_w}{pq}\right) = \sum_{r=1}^{\infty} C_r Z_{2r}\left(\frac{p_w - p}{p}\right)(pq)^{2r} \tag{5.23}$$

$$\Gamma_3'(\kappa_1) = \frac{2pp_w(1-p)^2}{(1-p_w)(p-p_w+pp_w)^2}\sum_{r=1}^{\infty} C_r^2 (pq)^{2r}. \tag{5.24}$$

It can be noted that $\delta'(\kappa_1)$ is finite, and so the $\gamma(\kappa_1)\delta'(\kappa_1)$ term goes to zero, since $\gamma(\kappa_1) = 0$.

5.1.2. Simplifying the expressions. We now simplify some of the terms given in section 5.1.1, so that we will be able to express the mean number of contacts in a simpler form, in terms of known functions. We can show by manipulation (splitting the sum up into two parts and reversing the order of integration on one sum) that

$$2\sum_{r=1}^{\infty}\sum_{s=0}^{r-1} C_r C_s u^{s+r} = \left(\sum_{r=1}^{\infty} C_r u^r\right)^2 + 2\sum_{r=1}^{\infty} C_r u^r - \sum_{r=1}^{\infty} C_r^2 u^{2r}. \tag{5.25}$$

We have now expressed this double summation in terms of single sums, all of which can be re-expressed further by known relationships. We recall from (4.19) that the final term in (5.25) can be expressed in terms of elliptic integrals. Also we have from [4] that

$$\sum_{r=1}^{\infty} C_r (pq)^r = \begin{cases} \dfrac{p}{1-p} & p < \dfrac{1}{2} \\ \dfrac{1-p}{p} & p > \dfrac{1}{2}. \end{cases} \tag{5.26}$$

We use similar methods of manipulation as in (5.25) to obtain the relationship

$$2\sum_{r=1}^{\infty}\sum_{s=0}^{r-1}(s-r)C_r C_s u^{r+s-1} = \frac{-p}{q^3(1-2p)}N^*(p) + \sum_{s=1}^{\infty}\sum_{r=1}^{s} C_r C_s u^{s+r-1} - \sum_{s=1}^{\infty} s C_s u^{s-1}, \tag{5.27}$$

where we recall $N^*(p)$ from equation (2.11), thus making contact with the dry wall expression for mean number of contacts. We note that the double sum in (5.27) can be re-expressed in terms of known functions using (5.25), and that

$$\sum_{s=1}^{\infty} s C_s u^{s-1} = \frac{\mathrm{d}}{\mathrm{d}u}\left(\sum_{s=1}^{\infty} C_s u^s\right). \tag{5.28}$$

So we have

$$\sum_{s=1}^{\infty} s\,C_s(pq)^{s-1} = \begin{cases} \dfrac{1}{(1-2p)(1-p)^2}, & p < \dfrac{1}{2} \\ \dfrac{1}{p^2(2p-1)}, & p > \dfrac{1}{2}, \end{cases} \tag{5.29}$$

and in this way we have expressed simply all terms which have arisen that differ from those encountered in the mean length calculation, so we will be able to similarly group coefficients.

5.2. Result for the mean number of wall contacts

We combine (3.8), (5.15) and (5.16) to obtain the (unnormalized) mean number of wall contacts:

$$\bar{N}(p, p_w) = A_N(p, p_w)\sum_{r=1}^{\infty} C_r^2(pq)^{2r} + B_N(p, p_w)\sum_{r=1}^{\infty} C_r Z_{2r}(d)(pq)^{2r} + D_N(p, p_w)N^*(p) + E_N(p, p_w), \tag{5.30}$$

where

$$A_N(p, p_w) = -\frac{p_w^3(1-p)^3(1-2p)}{(1-p_w)(p-p_w+pp_w)^3} \tag{5.31}$$

$$B_N(p, p_w) = \frac{p_w(1-p)^2(1-2p)(p_w-p+p_w^2(1-p))}{(1-p_w)(p-p_w+pp_w)^3} \tag{5.32}$$

$$D_N(p, p_w) = \frac{p_w p^2}{(p-p_w+pp_w)^2} \tag{5.33}$$

$$E_N(p, p_w) = \begin{cases} \frac{p_w(1-p)(2p^3p_w^2-2p_w^3+pp_w(1+3p_w+4p_w^2)-p^2(1+2p_w+4p_w^2+2p_w^3))}{(1-p_w)(p-p_w+pp_w)^3}, & p < \frac{1}{2} \\ \frac{(1-p)^2p_w(p_w(1-2p_w)-p(1+3p_w-9p_w^2)+p^2(5-5p_w-7p_w^2-2p_w^3)-p^3(1-2p_w-4p_w^2-2p_w^3)-2p^4p_w^2)}{p^2(1-p_w)(p-p_w+pp_w)^3}, & p > \frac{1}{2}. \end{cases} \tag{5.34}$$

Refer to figure 4 for a graph of the mean number of contacts when $p_w = 0.4$.

5.3. Analysis of results

5.3.1. Asympotics behaviour. As in the case of mean length, the only term which is not expressed in terms of standard functions is the sum containing $Z_{2r}(d)$—but this does not contribute to the behaviour near the critical point, which is dominated by the $N^*(p)$ term, with constant contribution from $E_N(p, p_w)$.

From [4], we have the asymptotic form of $N^*(p)$,

$$N^*(p) \cong \pm 3 + \frac{16}{\pi}(1-2p)\log|1-2p|, \tag{5.35}$$

where the positive and negative correspond to approaching the critical point from above and below, respectively.

Hence, we have the asymptotic form for the mean number of contacts, as $p \to \frac{1}{2}$,

$$\bar{N}(p, p_w) \cong A(p_w) + B(p_w)(1-2p)\log|1-2p|, \tag{5.36}$$

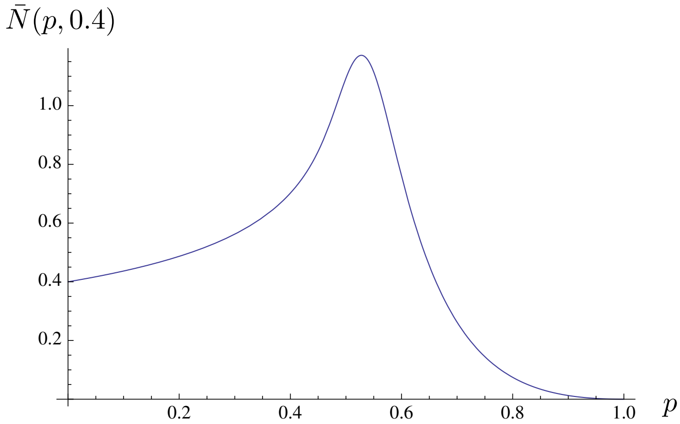

Figure 4. This is a graph of $\bar{N}(p, 0.4)$, plotted by expanding the sums in (5.30) up to $r = 100$.

where

$$A(p_w) = \frac{2p_w}{1 - p_w} \tag{5.37}$$

$$B(p_w) = \frac{16p_w}{\pi(1 - p_w)^2} \tag{5.38}$$

and so we note that the asymptotic form is similar to the dry wall case.

5.3.2. Expansion. Expanding (5.30) in powers of p gives the low-density expansion, for which the first five terms are

$$\bar{N}(p, p_w) = p_w + 2p_w^2 p + p_w\left(1 - 2p_w + 5p_w^2\right)p^2 + 2p_w^2\left(3 - 6p_w + 7p_w^2\right)p^3 \\ + p_w\left(3 - 10p_w + 37p_w^2 - 56p_w^3 + 42p_w^4\right)p^4 + O(p^5), \tag{5.39}$$

and in powers of q we have the high-density expansion,

$$\bar{N}(1 - q, p_w) = p_w(3 - 2p_w)q^2 - 4p_w(-2 + p_w^2)q^3 + p_w\left(13 + 12p_w - 8{p_w}^2 - 6{p_w}^3\right)q^4 \\ + 2p_w\left(8 + 18p_w + {p_w}^2 - 10{p_w}^3 - 4{p_w}^4\right)q^5 \\ + p_w\left(14 + 70p_w + 47{p_w}^2 - 28{p_w}^3 - 36{p_w}^4 - 10{p_w}^5\right)q^6 + O(q^7). \tag{5.40}$$

5.4. Dry wall comparison

5.4.1. The case $p_w = p$. As previously noted, the $p_w = p$ case corresponds to a similar situation to the dry wall case considered in [4], with the seed shifted up by one unit. Evaluating the coefficients in section 5.2 gives

$$B_N(p, p) = -A_N(p, p) = \frac{(1 - p)^2(1 - 2p)}{p^3}, \qquad D_N(p, p) = \frac{1}{p} \tag{5.41}$$

and

$$E_N(p, p) = \begin{cases} -\dfrac{1}{p^2} & \text{for} \quad p < \dfrac{1}{2} \\ \dfrac{q^2(1 + 2q)}{(1 - q)^4} & \text{for} \quad p > \dfrac{1}{2}. \end{cases} \tag{5.42}$$

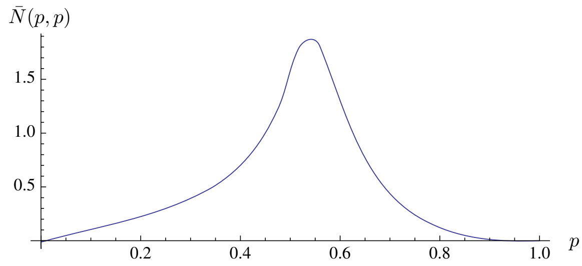

Figure 5. This is a graph of the dry wall case, $\bar{N}(p, p)$, plotted by expanding the sums in (5.30) up to $r = 100$.

Also when $p = p_w$, we have $d = 0$ and $Z_{2r}(0) = C_r$ so that the first two terms in (5.30) cancel which leads to

$$\bar{N}(p, p) = \frac{1}{p}N^*(p) + E_N(p, p) = \frac{\bar{N}_0^{\mathrm{dry}}(p) - Q^{\mathrm{dry}}(p)}{p}. \tag{5.43}$$

Refer to figure 5 for a graph of the mean number of contacts in this dry wall comparison.

5.4.2. The case $p_w = 0$. In the case $p_w = 0$, there are no cluster contacts with the wall. Nevertheless if we remove the first p_w factor, which is always present, the formula will count the mean number of wet sites in the $x = 0$ row, adjacent to the wall. Thus, for $p < \frac{1}{2}$,

$$\left.\frac{A_N(p, p_w)}{p_w}\right|_{p_w=0} = 0 \tag{5.44}$$

$$\left.\frac{B_N(p, p_w)}{p_w}\right|_{p_w=0} = -\frac{p(1-p)^2(1-2p)}{p^3} \tag{5.45}$$

$$\left.\frac{D_N(p, p_w)}{p_w}\right|_{p_w=0} = 1 \tag{5.46}$$

$$\left.\frac{E_N(p, p_w)}{p_w}\right|_{p_w=0} = -\frac{1-p}{p}. \tag{5.47}$$

Also when $p_w = 0$, $d = -1$ and $Z_{2r}(d) = 0$ so, by comparison with (2.10),

$$\left.\frac{\bar{N}(p, p_w)}{p_w}\right|_{p_w=0} = N^*(p) - \frac{1-p}{p} = \bar{N}^{\mathrm{dry}}(p). \tag{5.48}$$

So we have derived the dry wall result for the mean number of contacts, using our damp wall expression, in both the $p_w = p$ and $p_w = 0$ cases.

6. Conclusion

We have derived an exact expression for both the mean length and mean number of contacts with the wall in the model of directed, compact percolation near a damp wall. From each we can see how the limiting case of the dry wall arises. Our results show that the critical behaviour for a damp wall for each of these properties mimics that of the dry wall situation but

Table 2. Summary of critical exponents for each case of directed compact percolation.

	Exponent for	Bulk	Wet	Dry	Damp
β	percolation probability	1	1	2	2
τ	mean length	1	1	log	log
ψ	mean wall contacts	–	–	log	log

differs from that for a wet wall and in the bulk. We provide a summary of critical exponents in table 2.

We note that the generating function for the pairs of walks related to our problem is D-finite though not elementary or even algebraic. We have used this generating function to obtain expressions for our cluster properties. Previously [16], we found that surprisingly the percolation probability was a rational function of the two parameters of the model. However, we find, in contrast, that the mean length and mean number of contacts give expressions involving elliptic integrals and more complicated functions. Further to this we note that the functions involved in the damp wall case, although involving elliptic integrals similar to the dry wall case, are in a more complicated form. Naturally, there is the inclusion of the additional variable p_w which leads to more cumbersome coefficients, but in addition to this we note the presence of extra terms in the calculation of both the mean length and mean number of contacts, of the form of double sums which apparently cannot be expressed or evaluated into simpler expressions.

Our expressions show that the dry wall case is quite naturally expressed from the damp wall by setting $p_w = 0$, but we can in fact also reach the dry wall problem by setting $p_w = p$, thus shifting the location of the wall. We note that our expressions for properties calculated in this paper have a singular limit for the wet wall case, $p_w = 1$, due to the damp wall definition including sites on the wall, which was not previously the case for the wet wall considered in [14].

7. Future work

It would be interesting future work on directed compact percolation near a wall to calculate the mean size of clusters near a damp wall. The mean size and mean length are not simply related, and we will not be able to use similar methods to solve for the mean size. To use a generating function method, we would require an area generating function for weighted vesicles, rather than simply $G(z; \kappa_1, \kappa_2)$ which is a perimeter generating function.

We have begun to investigate mean size by exploring the recurrence relations and associated functional equations, with work to be published at a future time [13]. From preliminary results, it is not clear at this stage whether the solution for the mean size generating function is even D-finite, and it can be seen that the form of solution differs considerably from the previously studied cases.

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