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Mean Length of Finite Clusters in Directed Compact Percolation Near a Damp Wall

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Abstract The mean length of finite clusters is derived exactly for the case of directed compact percolation near a damp wall. We find that the result involves elliptic integrals and exhibits similar critical behaviour to the dry wall case.

Keywords Compact percolation · Exact solution · Mean length

1 Introduction

The directed compact percolation model, introduced by Domany and Kinzel [4], is an exactly solvable model. Directed compact percolation provides us with a testing ground for the predictions concerning the phase and critical behaviour of percolation more generally as we can examine its behaviour in a variety of circumstances, such as the damp wall we examine here. It also allows us to understand the effect of directness on percolation, so providing a link between isotropic and directed, non-compact percolation. Directed compact percolation near a wall was reviewed in [7] and the percolation probability was calculated in the damp wall case. Expressions for the mean cluster length have already been found in the bulk [5], wet wall [6] and dry wall [1] cases, and this paper proceeds to find an expression in the damp wall case.

The model is defined on a directed square lattice, the sites of which are the points in the t, x plane with integer co-ordinates such that $t \in \mathbb{N} \cup 0$, $x \in \mathbb{Z}$ and $t + x$ is even. The growth rule is that the site (t, x) becomes wet with certainty if both the sites $(t-1, x \pm 1)$

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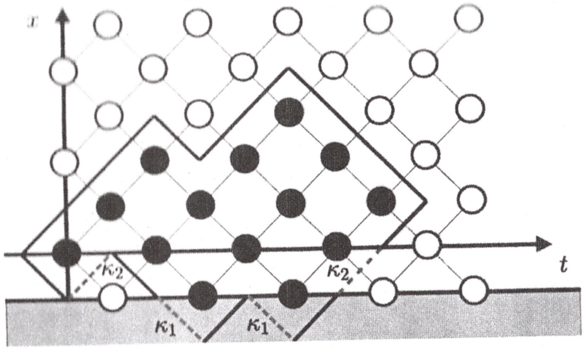

Fig. 1 An example of a directed compact percolation cluster near a damp wall, with the corresponding pair of weighted directed walks. This particular cluster has length 8

are wet, and with probability p if only one of these sites is wet, defining $q = 1 - p$ to be the probability of remaining dry in this case. Where both of the sites $(t-1, x \pm 1)$ are dry, the site (t, x) remains dry with certainty, ensuring a single compact cluster is produced from a *seed* of m contiguous sites at $t = 0$, $x \geq 0$, where we restrict ourselves to the case $m = 1$ in this work. We introduce a wall at $x = -1$, and consider the wall to be *damp*—that is, made up by some combination of wet and dry wall sites, governed by a wall occupancy probability p_w.

The problem can be mapped to pairs of directed walks, weighted with two fugacities κ_1 and κ_2 as shown in Fig. 1, and the generating function $G(z; \kappa_1, \kappa_2)$ for these walks, found in [7], allows us to solve the percolation problem.

2 Mean Cluster Length

We define the length of the cluster to be the number of sites in the shortest path from seed to terminal point, including the seed—it is hence equal to the number of occupied columns.

The generating function $G(z; \kappa_1, \kappa_2)$ for the walk problem [7] gives a weighting of z to each growth stage in the corresponding cluster, and so a cluster with t growth stages (and hence length $t+1$) will gain a weighting of z^t. So in order to find the mean cluster length we take the derivative with respect to z of $zG(z; \kappa_1, \kappa_2)$, and evaluate with the values corresponding to the percolation problem—noting that we must also adjust by a terminating factor of q^2 to bring the walk model in line with the percolation problem. So the unnormalised mean cluster length is given by

$$\bar{L}(p, p_w) = (1-p)^2 \frac{d}{dz}\left[zG\left(z; \frac{p_w}{pq}, \frac{q_w}{q}\right)\right]_{z=p(1-p)} \tag{2.1}$$

where $G(z; \kappa_1, \kappa_2)$ was found in [7]. In [8], after substituting the percolation values of κ_1 and κ_2 the generating function was re-expressed as:

$$G\left(z; \frac{p_w}{pq}, \frac{q_w}{q}\right) = \left(1 + \frac{\omega_c}{z}\right)\left[A_1 \frac{\omega_c}{z} - A_2(z)\left(1 - \frac{\omega_c}{z}\right)\right] \sum_{r=1}^{\infty} C_r Z_{2r}(c) z^{2r}$$
$$- \left(1 + \frac{\omega_d}{z}\right)\left[A_1 \frac{\omega_c}{z} - A_2(z)\left(\frac{\omega_c}{\omega_d} - \frac{\omega_c}{z}\right)\right] \sum_{r=1}^{\infty} C_r Z_{2r}(d) z^{2r}$$

$$+ A_2(z)\left(1 - \frac{\omega_c}{\omega_d}\right)\sum_{r=1}^{\infty} C_r^2 z^{2r} - A_2(z)\omega_c(c-d)\kappa_2 \tag{2.2}$$

where

$$c = \frac{p}{q}, \qquad d = \frac{p_w - p}{p} \tag{2.3}$$

$$\omega_c = pq, \qquad \omega_d = \frac{p(p_w - p)}{p_w^2} \tag{2.4}$$

$$A_1 = \frac{1}{(c-d)\omega_c} \tag{2.5}$$

$$A_2(z) = \frac{cd}{\omega_c \kappa_1^2 (c-d)(z^2 - \omega_c\omega_d)} \tag{2.6}$$

Also note that $Z_{2r}(d)$ is a sum over Dyck paths of length $2r$ weighted with a factor $\kappa = 1+d$ for each contact with the axis (the contact polynomial, denoted $\hat{Z}^S_{2r}$ in [3]). It was shown in (3.19) and (3.23) of [3] that

$$Z_{2r}(d) = (1+d)\sum_{m=0}^{r} B_{2r,2m} d^m \tag{2.7}$$

where $B_{t,h}$ is the Ballot number, as defined in [2]:

$$B_{t,h} = \frac{(h+1)t!}{(\frac{1}{2}(t+h)+1)!(\frac{1}{2}(t-h))!} \tag{2.8}$$

2.1 Calculation

We can write the mean length as:

$$\bar{L}(p, p_w) = Q(p, p_w) + p(1-p)^3 \frac{d}{dz} G\left(z; \frac{p_w}{pq}, \frac{q_w}{q}\right)\bigg|_{z=p(1-p)} \tag{2.9}$$

where we recall from [7] that $Q(p, p_w)$, the probability of a finite cluster, is given by:

$$Q(p, p_w) = q^2 G\left(pq; \frac{p_w}{pq}, \frac{q_w}{q}\right) = \begin{cases} 1 & \text{for } p < \frac{1}{2} \\ \frac{q^2(1-2q)+qq_w(1-q)^2}{(1-q)^2(1-2q+qq_w)} & \text{for } p > \frac{1}{2} \end{cases} \tag{2.10}$$

We differentiate (2.2), using the following relationships to simplify the expressions:

$$A_2 := A_2(\omega_c) = \frac{A_1\omega_d}{\omega_c - \omega_d}, \qquad A_2'(\omega_c) = -\frac{2}{\omega_c - \omega_d} A_2 \quad \text{and} \quad \frac{A_2}{\omega_d} = \frac{A_1 + A_2}{\omega_c} \tag{2.11}$$

and hence write the derivative of the generating function as:

$$\frac{d}{dz} G\left(z; \frac{p_w}{pq}, \frac{q_w}{q}\right)\bigg|_{z=p(1-p)} = \Gamma_0 + \Gamma_1 + \Gamma_2 \tag{2.12}$$

where

$$\Gamma_0 = \frac{2(1+c)^4 d(1+d)^2}{(c-d)^2(1-cd)} \tag{2.13}$$

$$\Gamma_1 = \frac{4A_1}{\omega_c}\sum_{r=1}^{\infty} rC_r Z_{2r}(c)\omega_c^{2r} - \frac{2A_1}{\omega_c}\sum_{r=1}^{\infty} rC_r^2\omega_c^{2r}$$

$$+\frac{A_1}{\omega_c}\sum_{r=0}^{\infty} C_r\omega_c^r \sum_{s=r+1}^{\infty} C_s\omega_c^s \tag{2.14}$$

$$\Gamma_2 = -\frac{A_1}{\omega_c}\sum_{r=0}^{\infty} C_r\omega_c^r \sum_{s=r+1}^{\infty} C_s\omega_c^s - \frac{3A_1+2A_2}{\omega_c}\sum_{r=1}^{\infty} C_r Z_{2r}(c)\omega_c^{2r}$$

$$+\frac{A_1+A_2}{\omega_c}\left[2\sum_{r=1}^{\infty} C_r^2\omega_c^{2r} - \left(1+\frac{\omega_d}{\omega_c}\right)\sum_{r=1}^{\infty} C_r Z_{2r}(d)\omega_c^{2r}\right] \tag{2.15}$$

Note that the last term of Γ_1 cancels the first term of Γ_2. These terms have been introduced to obtain the simplification below.

In order to make contact with the dry wall limit analysed in [1] we write the partition function in the form:

$$Z_{2r}(c) = \omega_c^{-r}\left(1+c-\sum_{s=0}^{r-1} C_s\omega_c^s\right) \tag{2.16}$$

$$= \omega_c^{-r}\left((c-c^*)+\sum_{s=r}^{\infty} C_s\omega_c^s\right) \tag{2.17}$$

where

$$c^* = \sum_{r=1}^{\infty} C_r\omega_c^r = \frac{1-2\omega_c-\sqrt{1-4\omega_c}}{2\omega_c} = \begin{cases} c & \text{for } c \le 1 \\ \frac{1}{c} & \text{for } c > 1 \end{cases} \tag{2.18}$$

Substituting (2.17) into (2.14) gives:

$$\Gamma_1 = \frac{4A_1}{\omega_c}(c-c^*)\sum_{r=1}^{\infty} rC_r\omega_c^r + \frac{4A_1}{\omega_c}\sum_{r=1}^{\infty} rC_r\omega_c^r \sum_{s=r}^{\infty} C_s\omega_c^s$$

$$-\frac{2A_1}{\omega_c}\sum_{r=1}^{\infty} rC_r^2\omega_c^{2r} + \frac{A_1}{\omega_c}\sum_{r=0}^{\infty} C_r\omega_c^r \sum_{s=r+1}^{\infty} C_s\omega_c^s \tag{2.19}$$

We split the second term into two halves, replace r by $r+1$ in the first half and combine the second half with the remaining two terms to give

$$\Gamma_1 = 4A_1(c-c^*)\sum_{r=1}^{\infty} rC_r\omega_c^{r-1} + \frac{2A_1}{\omega_c}\sum_{r=0}^{\infty}(r+1)C_{r+1}\omega_c^{r+1}\sum_{s=r+1}^{\infty} C_s\omega_c^s$$

$$+\frac{A_1}{\omega_c}\sum_{r=0}^{\infty}(2r+1)C_r\omega_c^r \sum_{s=r+1}^{\infty} C_s\omega_c^s \tag{2.20}$$

So we have:

$$\Gamma_1 = 4A_1(c - c^*)\frac{d}{dz}\left(\sum_{r=1}^{\infty} C_r z^r\right)_{z=\omega_c} + \frac{A_1}{\omega_c}\left(\frac{p}{q^2}L^*(p)\right) \tag{2.21}$$

where

$$L^*(p) = \frac{q^2}{p}\sum_{r=0}^{\infty}[(2r+2)C_{r+1}\omega_c^{r+1} + (2r+1)C_r\omega_c^r]\sum_{s=r+1}^{\infty} C_s\omega_c^s$$

which comes from the mean length calculation in the dry wall case, found in [1] to be:

$$\bar{L}^{dry}(p) = \theta(p - p_c)\frac{q(3-2p)}{p^3} + L^*(p) \tag{2.22}$$

where $\theta(p - p_c)$ is the unit step function, which is zero for values of p below $p_c = \frac{1}{2}$.

We split Γ_2 in (2.15) into two parts

$$\Gamma_2 = \Gamma_{21} + \Gamma_{22} \tag{2.23}$$

where

$$\Gamma_{21} = \frac{A_1 + A_2}{\omega_c}\left[\sum_{r=1}^{\infty} C_r^2\omega_c^{2r} - \left(1 + \frac{\omega_d}{\omega_c}\right)\sum_{r=1}^{\infty} C_r Z_{2r}(d)\omega_c^{2r}\right] \tag{2.24}$$

and

$$\Gamma_{22} = -\frac{A_1}{\omega_c}\sum_{r=0}^{\infty} C_r\omega_c^r \sum_{s=r+1}^{\infty} C_s\omega_c^s - \frac{3A_1 + 2A_2}{\omega_c}\sum_{r=1}^{\infty} C_r Z_{2r}(c)\omega_c^{2r}$$
$$+ \frac{A_1 + A_2}{\omega_c}\sum_{r=1}^{\infty} C_r^2\omega_c^{2r} \tag{2.25}$$

We note that, for our later convenience, we have split the term with a coefficient 2 up into two halves, one in each of the components. Substituting (2.16) into (2.25), we have:

$$\Gamma_{22} = -\frac{A_1}{\omega_c}\sum_{r=0}^{\infty} C_r\omega_c^r \sum_{s=r+1}^{\infty} C_s\omega_c^s + \frac{3A_1 + 2A_2}{\omega_c}\sum_{r=1}^{\infty} C_r\omega_c^r \sum_{s=0}^{r-1} C_s\omega_c^s$$
$$+ \frac{A_1 + A_2}{\omega_c}\sum_{r=1}^{\infty} C_r^2\omega_c^{2r} - \frac{3A_1 + 2A_2}{\omega_c}(1 + c)\sum_{r=1}^{\infty} C_r\omega_c^r \tag{2.26}$$

Interchanging the limits on the first term it cancels with part of the second term, leaving

$$\Gamma_{22} = \frac{A_1 + A_2}{\omega_c}\left[2\sum_{r=1}^{\infty} C_r\omega_c^r \sum_{s=0}^{r-1} C_s\omega_c^s + \sum_{r=1}^{\infty} C_r^2\omega_c^{2r}\right] - \frac{3A_1 + 2A_2}{\omega_c}(1 + c)c^* \tag{2.27}$$

Now if we split the first term in half, reverse the order of summation, and manipulate the sums, we get:

$$\Gamma_{22} = \frac{A_1 + A_2}{\omega_c} c^*(c^* + 2) - \frac{3A_1 + 2A_2}{\omega_c} c^*(1 + c) \tag{2.28}$$

$$= \frac{A_1 + A_2}{\omega_c} c^*(c^* - 2c) - \frac{A_1}{\omega_c} c^*(1 + c) \tag{2.29}$$

and so we have simplified Γ_2 to the form:

$$\Gamma_2 = \frac{A_1 + A_2}{\omega_c} c^*(c^* - 2c) - \frac{A_1}{\omega_c} c^*(1 + c) + \frac{A_1 + A_2}{\omega_c} \left[\sum_{r=1}^{\infty} C_r^2 \omega_c^{2r} - \left(1 + \frac{\omega_d}{\omega_c}\right) \sum_{r=1}^{\infty} C_r Z_{2r}(d) \omega_c^{2r} \right] \tag{2.30}$$

2.2 Exact Result

The final result for the mean length of compact clusters, using (2.9) with (2.10) and (2.12) and grouping coefficients, is:

$$\bar{L}(p, p_w) = A_L(p, p_w) \sum_{r=1}^{\infty} C_r^2 (pq)^{2r} + B_L(p, p_w) \sum_{r=1}^{\infty} C_r Z_{2r}(d)(pq)^{2r} + D_L(p, p_w) L^*(p) + E_L(p, p_w) \tag{2.31}$$

where

$$A_L(p, p_w) = \frac{p_w^2 q^3}{q_w(1 - q(1 + p_w))^2} \tag{2.32}$$

$$B_L(p, p_w) = -\frac{q^2(p_w^2 q + p_w - p)}{q_w(1 - q(1 + p_w))^2} \tag{2.33}$$

$$D_L(p, p_w) = \frac{p}{(1 - q(1 + p_w))} \tag{2.34}$$

$$E_L(p, p_w) = \begin{cases} \frac{p(p_w(1-p_w)-p+p^2 p_w^2)}{(1-p_w)(p-p_w+pp_w)^2} & \text{for } p \le \frac{1}{2} \\ -\frac{q^2(1-p_w-(2-3p_w)(1+p_w)q+(1-5p_w^2)q^2+p_w^2 q^3)}{(1-p_w)(1-q)^2(1-q(1+p_w))^2} & \text{for } p > \frac{1}{2} \end{cases} \tag{2.35}$$

2.3 Critical Behaviour

The asymptotics of the mean length, calculated in [8], show that the behaviour near the critical point $p = \frac{1}{2}$ is dominated by the elliptic integral $K(16p^2q^2)$ from the $L^*(p)$ term, and we have:

$$\bar{L}(p, p_w) \cong B \log|2p - 1| + C \tag{2.36}$$

where

$$B = \frac{-8}{\pi(1 - p_w)} \tag{2.37}$$

$$C = \frac{1}{(1-p_w)}\left[\frac{4(3\log 2 - 2)}{\pi} \mp 4\right] + \frac{8p_w^2}{\pi(1-p_w)^3} + C^*(p_w) \tag{2.38}$$

$$+ \frac{1}{(1-p_w)^3}\begin{cases} p_w + 2p_w^2 - p_w^3 & \text{for } p < \frac{1}{2} \\ 8 - 17p_w + 6p_w^2 + p_w^3 & \text{for } p > \frac{1}{2} \end{cases} \tag{2.39}$$

where

$$C^*(p_w) = -\frac{p_w^2 + 2p_w - 1}{2(1-p_w)^3}\sum_{r=1}^{\infty} C_r Z_{2r}(2p_w - 1)\left(\frac{1}{4}\right)^{2r} \tag{2.40}$$

So we can see that the damp wall case exhibits similar critical behaviour to the dry wall case.

2.4 Comparison to Dry Wall

We compare the mean length result we have found for the damp wall model to the previously found result for the dry wall case. Setting $p_w = 0$, we find that:

$$d = -1, \qquad A_L = 0, \qquad Z_{2r}(d) = 0, \qquad D_L(p, 0) = 1 \tag{2.41}$$

and

$$E_L(p, 0) = \begin{cases} 0 & \text{for } p < \frac{1}{2} \\ \frac{q(1+2q)}{(1-q)^3} & \text{for } p > \frac{1}{2} \end{cases} \tag{2.42}$$

thus:

$$\bar{L}(p, 0) = L^*(p) + \begin{cases} 0 & \text{for } p < p_c \\ \frac{q(1+2q)}{(1-q)^3} & \text{for } p > p_c \end{cases} \tag{2.43}$$

$$= \bar{L}^{dry}(p) \tag{2.44}$$

as expected. So we see that for $p_w = 0$ the damp wall expression describes the dry wall case found in [1].

2.5 Further Analysis

2.5.1 Expansion

Expanding (2.31) in powers of p and q gives series which agree with those obtained by applying (2.1) to the expression for the generating function found in [7], (4.69), both yielding the following expansions:

$$\begin{aligned}\bar{L}(p, p_w) =& 1 + p_w + (1 + p_w + 2p_w^2)p + (2 + p_w + 5p_w^3)p^2 \\ &+ (3 + 3p_w + 3p_w^2 - 7p_w^3 + 14p_w^4)p^3 \\ &+ (6 + 3p_w + 2p_w^2 + 21p_w^3 - 42p_w^4 + 42p_w^5)p^4\end{aligned}$$

$$+ (9 + 11p_w + 9p_w^2 - 37p_w^3 + 138p_w^4 - 198p_w^5 + 132p_w^6)p^5 + \cdots \tag{2.45}$$

$$\begin{aligned} L(1-q, p_w) =& (1-p_w)q + (5 - 3p_w^2)q^2 + (13 + 5p_w - 5p_w^2 - 5p_w^3)q^3 \\ &+ (26 + 18p_w - p_w^2 - 14p_w^3 - 7p_w^4)q^4 \\ &+ (46 + 44p_w + 17p_w^2 - 19p_w^3 - 27p_w^4 - 9p_w^5)q^5 + \cdots \end{aligned} \tag{2.46}$$

2.5.2 *Behaviour of the Mean Length Near the Curve $q = 1/(1 + p_w)$ in the Low Density Region*

An interesting mathematical peculiarity found in the solution is the apparent divergence in the low density region, as the amplitudes A_1 and A_2 are divergent at $q = 1/(1 + p_w)$. This appears to lead to a divergence in the mean length below the critical probability, which is impossible on physical grounds.

Numerically the mean length calculated from (2.31) shows no divergence at q^*. The strongest divergence is from the A_2 terms and the coefficient of A_2 evaluated at q^* contains a factor:

$$\frac{1}{2}\left(p_w^2 + \sum_{r=1}^{\infty} C_r^2 \left(\frac{p_w}{(1+p_w)^2}\right)^{2r}\right) - \sum_{r=1}^{\infty} C_r Z_{2r}(p_w) \left(\frac{p_w}{(1+p_w)^2}\right)^{2r}$$

We can show using Mathematica that this coefficient vanishes, although this is not sufficient to eliminate the divergence which is quadratic. We find that the resulting simple pole cancels that arising from A_1, and hence there is no divergence in the low density region, as expected.

We note that at $q = q^*$, $c = d = p_w$ and the origin of the divergent amplitudes came from using the formula in (4.40) of [7] for Z_{2r}^S. It may be that this could be avoided by instead using the expression in terms of Ballot numbers found in [2].

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