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Percolation Processes. II. The Pair Connectedness*

JOHN W. ESSAM†
Applied Mathematics Department, Brookhaven National Laboratory
Upton, L.I., New York 11973

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The probability that two sites of a crystal lattice belong to the same cluster (the pair connectedness) is shown to play an important role in percolation theory. Use of the linked-graph method to obtain low-density series expansions leads to the discussion of topological invariants for rooted graphs. These are related to the k-weights which arose in a previous investigation of the mean number of clusters. The mean size of clusters is related to the pair connectedness via sum rules.

1. INTRODUCTION AND DEFINITIONS

A discussion of the relevance of percolation theory to physics has been given in the review by Frisch and Hammersley.[1] We shall be more interested here in the graph theoretic aspects of the problem.

The pair connectedness in percolation theory is the analog of the pair correlation function in statistical mechanics. Just as summation of the spin correlation function over all pairs of lattice sites gives the susceptibility of an Ising ferromagnet the same summation of the pair connectedness[2] leads to the mean size of clusters.[3] In the absence of exact results, high-temperature series expansions of the susceptibility provide the most accurate determination of the Curie temperature.[4] The mean size of clusters has a similar strong divergence at the critical probability p_c and this led Domb[5] to suggest the use of low density expansions for its location. A fairly extensive study of these expansions was made by Sykes and Essam[6] with the conclusion that useful information about the critical region can be derived by this method but that the initial terms of the series are less regular than those for the susceptibility. Our long term objective will therefore be to extend the mean size expansions via the pair connectedness and also to study the latter in its own right. In the meantime a graph theoretic interpretation of the coefficients in the expansion will be given together with a prescription for finding the graph weights.

We begin by defining the functions of interest for a finite linear graph with the idea of proceeding to the limit of uniform infinite lattice graphs[7] for which the polynomials become infinite series. Two cases will be distinguished, the "site problem" and the "bond problem."

In the site problem, particles are distributed over vertices of the graph (sites of the lattice) subject to the constraint of at most one particle per site. The particles are otherwise independently distributed, the probability p of finding an occupied site being given. If $G = (V, E)$ is a linear graph with vertex set V and edge set E, then there is a possible state of the system for every subset V' of V, namely the one in which the vertices of V' are occupied but those of $V - V'$ are unoccupied. The assumption of independence means that the probability of occurrence of the state corresponding to V' is

$$\pi(V') = p^{|V'|}(1-p)^{|V-V'|}. \quad (1.1)$$

The expectation value of any function of state $A(V', G)$ is given by

$$\langle A; G\rangle = \sum_{V'\subseteq V} \pi(V')A(V', G); \quad (1.2)$$

for example, the mean value of the occupation number

$$\nu_i = \begin{cases} 1 & \text{for } i \in V' \\ 0 & \text{otherwise} \end{cases} \quad (1.3)$$

of the ith vertex is

$$\langle \nu_i; G\rangle = p. \quad (1.4)$$

It is also found that

$$\langle \nu_i\nu_j; G\rangle = p^2 \quad \text{for} \quad i \neq j, \quad (1.5)$$

which is consistent with the assumption of independence. There is therefore no correlation between particles on different vertices, as there would be if they were interacting.

In the bond problem it is the edges of the graph (bonds of the lattice) which are occupied by particles, and the above description may be taken over by replacing V by E. (Notice that occupied and unoccupied by a particle is just a way of thinking of the state of the vertex or edge, and we could equally well use spin up–spin down, black–white, open–closed, depending on the application.[1]) The site problem is more general than the bond problem, since the bond problem on any graph is isomorphic with the site problem on its covering graph.[7] It is nevertheless useful to retain the distinction since a covering graph is more complex than the graph from which it was derived (for example, the honeycomb covering lattice is the Kagomé lattice) and also since some theorems which are easily formulated for the bond problem are more obscure (or may not ever be true) when transcribed to the site problem.

We have seen that the positions of the particles are uncorrelated, but just by chance they form clusters as a result of the restricted space. In the case of a crystal lattice, the proximity of two particles may be measured in the geometrical sense, but in the general problem on a graph it is indicated by the interrelation between vertices and edges. We define the connectedness indicator $\gamma_{v,v'}$ for two vertices $v, v' \in V$ by

$$\gamma_{v,v'} = \begin{cases} 1 & \text{if } v, v' \text{ are connected by a chain of occupied edges}, \\ 0 & \text{otherwise.} \end{cases} \quad (1.6)$$

For the purpose of this definition, an edge is considered occupied in the site problem if both vertices it connects are occupied; in particular, v and v' must be occupied. The expectation value of $\gamma_{v,v'}$ will be called the vertex–vertex connectedness or, briefly, the *pair connectedness*. If v and v' are the same vertex, the expectation value is p for the site problem and unity for the bond problem. Two related quantities are $\gamma_{e,e'}$ and $\gamma_{v,e}$; these are defined as in (1.6), but $e, e' \in E$ are to be terminal links of the chain. Their mean values will be known as the edge–edge and vertex–edge connectedness, respectively, and are related to the pair connectedness by

$$\langle \gamma_{v,e'}; G \rangle = p \langle \gamma_{v,v'}; G_e^\gamma \rangle \quad (1.7)$$

and

$$\langle \gamma_{e,e'}; G \rangle = p^2 \langle \gamma_{v,v'}; G_{e,e'}^\gamma \rangle, \quad (1.8)$$

where G_e^γ and $G_{e,e'}^\gamma$ are the graphs obtained from G by contracting[7] the edges e and e' to yield the vertices v and v'. Thus G_e^γ and $G_{e,e'}^\gamma$ have one and two less vertices than G, respectively.

It is convenient to define the *pair connectedness of a two-rooted*[7] *graph* G^{ii}, obtained from G by designating the vertices v and v' as root points, by

$$D(p; G^{\text{ii}}) = \langle \gamma_{v,v'}; G \rangle. \quad (1.9)$$

It was shown in Ref. 3 that, for the site problem,

$$D(p; G^{\text{ii}}) = \sum_m [c_m^{\text{ii}}; G^{\text{ii}}] D(c_m^{\text{ii}}) p^{v_m}, \quad (1.10)$$

where

$$D(c_m^{\text{ii}}) = \sum_j (-1)^{v_m - v_j} [c_j^{\text{ii}}; c_m^{\text{ii}}]^F. \quad (1.11)$$

Essentially this shows that, to obtain the coefficient of p^n in the polynomial $D(p; G^{\text{ii}})$, it is necessary to enumerate all connected two-rooted section graphs[7] C^{ii} of G^{ii} with n vertices, each of which contributes $D(C^{\text{ii}})$, its *strong pair connectedness weight*, to the coefficient. The graph c_i^{ii} is the ith member of a list of connected two-rooted graphs no two of which are isomorphic. The list must be complete in the sense that every connected two-rooted section graph of G^{ii} is isomorphic to some graph in the list, and $[c_m^{\text{ii}}; G^{\text{ii}}]$ is the number of such section graphs isomorphic with c_m^{ii}. The grouping together of isomorphic graphs in this way is of more value when working with an infinite lattice, and for this reason the square bracket quantities are called strong lattice constants.[7] To complete the explanation of the above formulas, v_m is the number of vertices in c_m^{ii}, and F restricts c_j^{ii} to be a section graph of c_m^{ii} having full vertex perimeter.[7] Similarly, for the bond problem it was shown that[3] the pair connectedness, which is now distinguished by a bar, may be written

$$\bar{D}(p; G^{\text{ii}}) = \sum_m (c_m^{\text{ii}}; G^{\text{ii}}) \bar{d}(c_m^{\text{ii}}) p^{e_m}, \quad (1.12)$$

where

$$\bar{d}(c_m^{\text{ii}}) = \sum_j (-1)^{e_m - e_j} (c_j^{\text{ii}}; c_m^{\text{ii}})^F. \quad (1.13)$$

These formulas are similar to those for the site problem, but the round brackets denote weak lattice constants (i.e., subgraphs are enumerated rather than section graphs); e_m is the number of edges in c_m^{ii} and F denotes full edge perimeter.[7]

Since every subgraph of a graph corresponds to a unique section graph with the same number of vertices (the one with the same vertex set), we may rewrite (1.10) in the form

$$D(p; G^{\text{ii}}) = \sum_m (c_m^{\text{ii}}; G^{\text{ii}}) d(c_m^{\text{ii}}) p^{v_m}, \quad (1.14)$$

where $d(c_m^{\text{ii}})$ denotes the *weak pair connectedness weight* of c_m^{ii}. We notice that if C^{ii} is a connected two-rooted graph, then, when $p = 1$, the roots must be connected so that $D(1; C^{\text{ii}}) = 1$, which leads to a recursion formula for the d's,

$$d(C^{\text{ii}}) = 1 - \sum_m{}' (c_m^{\text{ii}}; C^{\text{ii}}) d(c_m^{\text{ii}}), \quad (1.15)$$

where the prime denotes omission of the term $c_m^{\text{ii}} = C^{\text{ii}}$. For the bond problem, $\bar{D}(1; C^{\text{ii}}) = 1$ so that from (1.12) we see that $\bar{d}$ also satisfies (1.15) and therefore

$$\bar{d}(c_m^{\text{ii}}) = d(c_m^{\text{ii}}), \quad (1.16)$$

which allows us to express the pair connectedness for both bond and site problems in terms of the weak weights.

The strong weights are more useful than the weak weights for computational purposes on an infinite lattice, but the weak weights are of more interest from a graph-theoretic point of view since they are topological invariants of the graph (i.e., homeomorphic graphs[7] have the same weak weight). The latter property was established for the weak weights in the mean number expansion (k-weights) in the first paper of this series[8] (subsequently referred to as I).

If $E' \subseteq E$ is the set of occupied edges in a given state for the bond problem on a graph $G = (V, E)$, then the mean number of clusters is defined by

$$\bar{K}_0(p; G) = \langle n; G \rangle, \qquad (1.17)$$

where $n(E', G)$ is the number of components in the subgraph $G' = (V, E')$. The zero subscript serves to remind us that isolated vertices of G' are counted as clusters (see I). In the site problem the mean number of clusters $K(p; G)$ is defined similarly, but G' is the section graph of G defined by the subset $V' \subseteq V$ of occupied vertices. In I it was shown that both mean number functions were determined by the weak k-weights

$$\bar{K}_0(p; G) = \sum_m (c_m; G) k(c_m) p^{e_m} \qquad (1.18)$$

and

$$K(p; G) = \sum_m (c_m; G) k(c_m) p^{v_m}, \qquad (1.19)$$

although again for computational purposes it is more useful to use the strong lattice constant expansion

$$K(p; G) = \sum_m [c_m; G] K(c_m) p^{v_m} \qquad (1.20)$$

in the case of the site problem. The properties of the mean number weights have been discussed in I, but subsequently the weak weights were investigated independently by Crapo[9] in the more general context of matroids. Two useful properties which were obtained by Crapo but which did not appear in I are

$$|k(G^{\mathrm{D}})| = |k(G)| \qquad (1.21)$$

(where G^{D} is a dual of the planar graph G) and

$$k(G) = k(G_e^\gamma) - k(G_e^\delta), \qquad (1.22)$$

where G_e^γ and G_e^δ are the graphs obtained from G by contracting and deleting the edge e of G. The deletion of an edge from a connected graph may separate it into one or more components. In I the k-weights were defined for connected graphs only, but for a general graph we adopt the definition[7]

$$\sum_{E' \subseteq E} k(G') = n(G),$$

where $G' = (V, E')$. This relation may be inverted to yield

$$k(G) = \sum_{E' \subseteq E} (-1)^{|E-E'|} n(G'). \qquad (1.23)$$

If the number of components $n(G)$ in G is greater than one, the k-weight is zero unless all but one of the components are isolated vertices. When G is the trivial graph with n vertices but no edges, (1.23) shows that $k = n$. For the graph G with one arbitrary connected component C and $n - 1$ isolated vertices, we find $k(G) = k(C)$. Combining these results, we see that the definition of k for connected graphs may be written (in agreement with I) as

$$\sum_{C' \subseteq G}{}' k(C') = -r(G),$$

where C' is a connected subgraph of G, the rank $r(G) = |V| - n(G)$, and the prime on the sum indicates omission of subgraphs with one vertex. By using k rather than $\beta = |k|$, as in Crapo's work, we see that Eq. (1.22) holds for any edge including loops.

The main result of this paper is to relate the mean number and pair connectedness weights. Theorems are then developed which enable the pair connectedness weights of a general two-rooted graph, and hence the mean number weights, to be expressed in terms of the pair connectedness weights of the elementary[7] two-rooted graphs. This work parallels that of Van Leeuwen, Groeneveld, and de Boer[10] on the pair correlation in an imperfect classical gas. The technique used is to develop theorems for the mean number and pair connectedness functions and then use the following proposition.

Proposition 1: Consider the weight factors in the weak and strong lattice constant expansions of the function $\langle A; G \rangle$. The weak weight of a graph g is the coefficient of $p^{e(g)}$ in the polynomial $\langle A; g \rangle$ for the bond problem, and the strong weight is the coefficient of $p^{v(g)}$ in the same polynomial for the site problem.

This results from the fact that the only subgraph of g with $e(g)$ edges is g itself and the only section graph with $v(g)$ vertices is g itself. The proposition was used in I, and as a further example of its use we note that (1.22) follows from the result of Kasteleyn and Fortuin[11] for the bond problem

$$\langle n; G \rangle = p \langle n; G_e^\gamma \rangle + (1 - p) \langle n; G_e^\delta \rangle. \qquad (1.24)$$

The above authors also note that (1.24) is valid for the bond problem pair connectedness, and so

$$d(G^{\mathrm{ii}}) = d(G_e^{\mathrm{ii}\gamma}) - d(G_e^{\mathrm{ii}\delta}). \qquad (1.25)$$

The rest of the paper is broken down as follows. Section 2 is concerned with the site problem, Sec. 3 deals with modifications required for the bond problem, and Sec. 4 relates the mean size of clusters to the pair connectedness. In the latter section the mean-size weight factors which, even in the weak case, are not topological invariants are expressed as a sum over the pair connectedness weights for all possible two-rootings.

2. SITE PROBLEM

A. Pair Connectedness

In general, the pair connectedness $D(p; G^{ii})$ is a polynomial of degree $v(G^{ii})$, but under certain conditions some of the coefficients vanish. The most obvious case is when the roots are adjacent. In this case $D(p; G^{ii}) = p^2$, and, applying Proposition 1, we have our first theorem.

Theorem 1: If G^{ii} is a two-rooted graph with three or more vertices and adjacent roots, then the strong pair connectedness weight is zero [i.e., $D(G^{ii}) = 0$].

Clearly, if D_n is the coefficient of p^n, then

$$D_n = 0 \quad \text{for} \quad 0 \le n < v_{\min}, \tag{2.1}$$

where $v_{\min}$ is the number of vertices in the shortest chain connecting the roots of G^{ii}. A more useful result for the computation of weight factors is the following:

$$D_n = 0 \quad \text{for} \quad v_{\max} < n \le v(G^{ii}), \tag{2.2}$$

where $v_{\max}$ is the number of vertices in the maximal 1-irreducible[7] two-rooted subgraph S^{ii} of G^{ii}. This results from the fact that the connectedness of the roots of G^{ii} is unaltered by changing the state of occupation of vertices not in S^{ii}, and so $D(p; G^{ii}) = D(p; S^{ii})$.

If G^{ii} itself is 1-irreducible, then we learn nothing about the polynomial from (2.2), but it may still be that $D(p; S^{ii})$ has degree less than $v(S^{ii})$. The characteristic of a 1-reducible two-rooted graph is that either it is not connected or, if it is connected, then there must be at least one articulation point (vertex, the deletion of which separates the graph into two or more components at least one of which has no roots). A 1-irreducible graph may still have an articulation set V_x of higher order, and, if there is such a set, the corresponding section graph of which is complete, then (2.2) may be generalized. Suppose that S_x^{ii} is the maximal two-rooted section graph of S^{ii} having no such articulation set; then $D(p; S^{ii}) = D(p; S_x^{ii})$. If V_x contains both roots, the result is trivial, but otherwise it follows by the previous argument that the occupation of vertices not in S_x^{ii} is irrelevant to the connectedness of the roots. Thus (2.2) holds with $v_{\max}$ equal to the number of vertices in S_x^{ii}. Using Proposition 1 now leads to the following theorem.

Theorem 2: If G^{ii} is a two-rooted graph which either is disconnected or contains an articulation set the section graph of which is complete, then its strong pair connectedness weight is zero.

In calculating strong pair connectedness weights, we are thus led to consider only 1-irreducible graphs, and even some of their weights may vanish by the above theorems. In a case where the weight is not expected to be zero, it is still possible to simplify the calculation if the graph may be formed by series–parallel combination of smaller graphs.

If S^{ii} is the 1-irreducible two-rooted graph obtained by identifying the roots of the graphs S_1^{ii} and S_2^{ii} (i.e., connecting them in parallel), then

$$1 - p^{-2}D(p; S^{ii}) = [1 - p^{-2}D(p; S_1^{ii})][1 - p^{-2}D(p; S_2^{ii})]. \tag{2.3}$$

This is true since $p^{-2}D$ is the probability that if the roots are occupied, then there is a chain of occupied vertices between the roots. The probability that there is no such chain is the product of the probabilities of no chain for each of the parallel components. The equation is trivially satisfied when the roots of either component are adjacent. A *composite* graph[7] $\mathcal{C}$ has at least two parallel components, but the root points are not adjacent. By repeated factorization and use of Proposition 1, it is possible to express the weight of a composite graph as a product over the weights of its *simple*[7] components. If $\mathcal{C}$ is the parallel combination of n simple two-rooted graphs $\mathcal{S}_1, \cdots, \mathcal{S}_n$, then

$$D(\mathcal{C}) = (-1)^{n+1} \prod_{t=1}^{n} D(\mathcal{S}_t). \tag{2.4}$$

The pair connectedness of a simple graph may be further factorized if it has a vertex through which all paths between the roots must pass. Such a vertex is called a nodal point; a simple two-rooted graph with one or more nodal points is called *nodal*.[7] Nodal graphs can thus be formed by series combination of smaller graphs. If S^{ii} is a 1-irreducible two-rooted graph obtained by series combination of S_1^{ii} and S_2^{ii}, then

$$D(p; S^{ii}) = p^{-1}D(p; S_1^{ii})D(p; S_2^{ii}). \tag{2.5}$$

The strong pair connectedness weight of a nodal graph $\mathcal{N}$, which is the series combination of n nonnodal graphs $S_1^{ii}, \cdots, S_n^{ii}$, may be written, by repeated use of (2.5) and then Proposition 1, as

$$D(\mathcal{N}) = \prod_{t=1}^{n} D(S_t^{ii}). \tag{2.6}$$

If the nonnodal graph S_t^{ii} is simple and therefore *elementary*,[7] no further reduction is possible, but, if it is composite, (2.4) may be applied again. Finally,

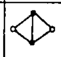
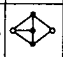
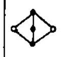
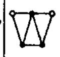
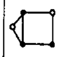
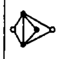
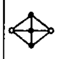
n	e_n	$D(e_n)$	$d(e_n)$	n	e_n	$D(e_n)$	$d(e_n)$
1		-1	+2	6		0	-3
2		0	+2	7		-1	-3
3		-1	+2	8		0	+4
4		0	-2	9		+1	+4
5		0	-2	10		+1	-6

FIG. 1. Pair connectedness weights for the elementary graphs with four and five vertices.

if it has a root-connecting edge or multi-edge, it contributes a factor p^2 to the pair connectedness and hence a factor of zero to the strong weight unless it is just a two-rooted multi-edge, in which case it contributes a factor of one to the weight. By repeated application of (2.4) and (2.6) the weight of any graph may be written, apart from a sign, as the product of the weights of its elementary constituents. These weights for graphs with five or less vertices are given in Fig. 1.

B. Relationship between the Pair Connectedness and Mean Number

Consider graphs G^{ii} and G_n, where G^{ii} is obtained by rooting two vertices of a graph G, and G_n is obtained by connecting the same vertices of G by a chain of n edges. It will be shown that

$$K(p; G_n) = K(p; G) + (n-1)p - np^2 + p^{n-1}D(p; G^{ii}) \quad \text{for} \quad n \geq 1. \tag{2.7}$$

As a first step we establish the result for $n = 1$ which involves finding the change in mean number when two vertices of G are connected by an additional edge. The only case in which a change occurs is when both roots are occupied but do not already belong to the same cluster. This happens with probability $p^2 - D(p; G^{ii})$ and reduces the number of clusters by one. Hence (2.7) is true with $n = 1$. Suppose now we wish to go from $n = 1$ to $n = 2$ by insertion of an additional vertex. The number of clusters is increased by one if either

(i) the inserted vertex is occupied and both roots are unoccupied [probability $p(1-p)^2$]

or

(ii) the inserted vertex is unoccupied, the roots are both occupied and do not belong to the same cluster on G {probability $(1-p)[p^2 - D(p; G^{ii})]$},

but is otherwise unchanged. Thus

$$K(p; G_2) = K(p; G_1) + p(1-p) - (1-p)D(p; G^{ii}) \tag{2.8}$$

and applying this result taking G to be G_{n-1} with one edge deleted from the chain

$$K(p; G_n) = K(p; G_{n-1}) + p(1-p) - (1-p)p^{n-2}D(p; G^{ii}) \quad \text{for} \quad n \geq 2. \tag{2.9}$$

Iterating (2.9), we find

$$K(p; G_n) = K(p; G_1) + (n-1)p(1-p) - (1-p^{n-1})D(p; G^{ii}) \quad \text{for} \quad n \geq 1, \tag{2.10}$$

which together with the result for $n = 1$ establishes (2.7) for $n > 1$. In fact, (2.7) holds for $n = 0$, but the proof is omitted.

Using Proposition 1, we find, from (2.7) the following relationships involving the strong K-weights:

$$K(G_1) = K(G) + D(G^{ii}) \quad \text{for} \quad v(G) > 2, \tag{2.11}$$

$$K(G_n) = D(G^{ii}) \quad \text{for} \quad n \geq 2 \quad \text{and} \quad v(G) > 2. \tag{2.12}$$

From (2.12) it follows that the strong mean number weights are independent of the number of edges in a given bridge[7] provided that there are at least two.

Combining (2.11) and (2.12) results in the following relation between K-weights:

$$K(G_n) = K(G_1) - K(G) \quad \text{for} \quad n \geq 2 \quad \text{and} \quad v(G) > 2 \tag{2.13}$$

e.g.,

$$-1 = 0 - +1$$

In practice, strong pair connectedness weights are determined by breaking down the graph into its elementary constituents and then using (2.11) or (2.12) to determine the weights of the elementary graphs from a list of K-weights. Such a list will eventually be much shorter than a list of pair connectedness weights since there are far more two-rooted graphs. Equations (2.11) and (2.12) are sometimes useful in hand computation of K-weights as is (2.13).

3. BOND PROBLEM

This section runs parallel to the previous section, and for this reason detailed arguments will not always be given. Formulas for weak pair connectedness weights will be obtained via the bond problem functions and Proposition 1, but, as we saw in the Introduction, they also determine the site problem pair connectedness.

A. Pair Connectedness

Theorem 1 is no longer valid for weak weights, and Theorem 2 holds only for disconnected graphs and graphs with articulation sets of order one. This is because even when G^{ii} has an articulation set V_x of order greater than two corresponding to a complete section graph, the vertices of V_x can be connected either directly by an occupied edge or indirectly by a chain of occupied edges which lies outside S_x^{ii}. The latter edges are therefore relevant to the connectedness of the roots in the case of the bond problem.

Equation (2.3) takes on a simpler form for the bond problem since the occupation of the roots need not be considered:

$$1 - \bar{D}(p; S^{ii}) = [1 - \bar{D}(p; S_1^{ii})][1 - \bar{D}(p; S_2^{ii})]. \tag{3.1}$$

Adjacent roots no longer give a trivial result so that we now consider the factorization of the weak weight of a ladder graph[7] $\mathfrak{L}$, which is similar to a composite graph, but the roots may be connected by a multi-edge. If e is the multiplicity of the multi-edge, it contributes a factor $(1 - p)^e$ to $1 - \bar{D}$ and so

$$d(\mathfrak{L}) = (-1)^{e+n+1} \prod_{t=1}^{n} d(\mathcal{S}_t), \tag{3.2}$$

where $\mathfrak{L}$ is the parallel combination of n simple graphs and e edges. In particular a two-rooted multi-edge of multiplicity e has weak weight $(-1)^{e+1}$.

The rule for series combination is also very simple, that is,

$$\bar{D}(p; S^{ii}) = \bar{D}(p; S_1^{ii})\bar{D}(p; S_2^{ii}). \tag{3.3}$$

Using Proposition 1, we may write the weak weight of a nodal graph $\mathcal{N}$ as the product of the weights for its nonnodal constituents:

$$d(\mathcal{N}) = \prod_{t=1}^{n} d(S_t^{ii}). \tag{3.4}$$

The nonnodal graphs this time fall into three classes, elementary, ladder (includes multi-edge of multiplicity two or more), and single edge with two roots (contributes a factor of unity). The ladder graph contributions may be further factorized using (3.2) and, as before, repeated use of (3.2) and (3.4) enables the weight of any graph to be expressed as a product of the weights of elementary graphs, the first few of which are listed in Fig. 1 beside the strong weights.

B. Relationship between the Pair Connectedness and Mean Number

The basic formula is almost the same as (2.7):

$$\bar{K}_0(p; G_n) = \bar{K}_0(p; G) + (n - 1) - np + p^n \bar{D}(p; G^{ii}) \quad \text{for} \quad n \geq 0. \tag{3.5}$$

The case $n = 0$ is easily proven since G_0 is formed by identifying the roots of G^{ii} and a cluster is lost whenever the roots of G^{ii} are not already connected by a chain of occupied edges [probability $1 - \bar{D}(p; G^{ii})$]. The result

$$\bar{K}_0(p; G_n) = \bar{K}_0(p; G_{n-1}) + (1 - p) \times [1 - p^{n-1}\bar{D}(p; G^{ii})], \tag{3.6}$$

which allows the length of the chain to be increased, follows from the fact that insertion of a new link only changes the mean number if it is unoccupied and then only if its incident vertices are not already connected through the rest of the graph [probability $1 - p^{n-1}\bar{D}(p; G^{ii})$]. Iteration of (3.6) yields

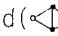
$$\bar{K}_0(p; G_n) = \bar{K}_0(p; G_0) + n(1 - p) - (1 - p^n)\bar{D}(p; G^{ii}) \quad \text{for} \quad n \geq 0, \tag{3.7}$$

which together with the result for $n = 0$ establishes (3.5) for general n.

Using Proposition 1 on (3.5) gives the basic relation between the weak mean number and pair connectedness weights

$$d(G^{ii}) = k(G_n) \quad \text{for} \quad e(G^{ii}) \geq 1 \quad \text{and} \quad n \geq 1. \tag{3.8}$$

Since (3.8) is true for all n, the k-weights of homeomorphic graphs are equal (see also I), and consequently the same equation implies that $d(G^{ii})$ is also a topological invariant of G^{ii}. Theorem IV of I also follows from (3.8) and the previous result that $d(G^{ii})$ is zero unless G^{ii} is 1-irreducible. As an example of (3.8), we have

d() = k() = 2.

C. Generalization of the Contraction-Deletion Rule (1.23) and the Effect of Edge Substitution on Weak Weights

Consider the graph

$$G_{e \to G_1{}^{ii}}$$

obtained from G by replacing the edge e of G by the two-rooted graph G_1^{ii}. A simple, but nevertheless useful, extension of (1.24) is

$$\bar{K}(p; G_{e \to G_1{}^{ii}}) = \bar{D}(p; G_1^{ii})\bar{K}(p; G_e^{\gamma}) + [1 - \bar{D}(p; G_1^{ii})]\bar{K}(p; G_e^{\delta}). \tag{3.9}$$

Using Proposition 1, we find

$$k(G_{e \to G_1{}^{ii}}) = d(G_1^{ii})[k(G_e^{\gamma}) - k(G_e^{\delta})] = d(G_1^{ii})k(G), \tag{3.10}$$

where we have used (1.22) to introduce $k(G)$. Thus the effect of replacing an edge of G by G_1^{ii} is to multiply the weak k-weight by the weak pair-connectivity weight of G^{ii}. This together with (3.8) enables the k-weight of any 2-reducible graphs to be expressed in terms of the k-weights of graphs with fewer vertices. A similar result is valid for the pair connectivity, namely

$$\bar{D}(p; G^{\text{ii}}_{e\to G_1}{}^{\text{ii}}) = \bar{D}(p; G_1^{\text{ii}})\bar{D}(p; G_e^{\text{ii}\gamma}) + [1 - \bar{D}(p; G_1^{\text{ii}})]\bar{D}(p; G_e^{\text{ii}\delta}), \tag{3.11}$$

which leads to

$$d(G^{\text{ii}}_{e\to G_1}{}^{\text{ii}}) = d(G_1^{\text{ii}})d(G^{\text{ii}}). \tag{3.12}$$

For example, graphs 2, 3, 4, and 5 in Fig. 1 are 2-reducible. The d-weights of 2 and 3 may be obtained from 1 by replacing an edge by a chain of length two. Similarly replacing the edges of 1 by a two-rooted triangle, for which $d = -1$, yields the d-weights of 4 and 5. Finally, by taking G_1^{ii} in (3.10) and (3.12) to be a two-rooted double bond,[7] for which $d = -1$, we find that doubling an edge changes the sign of the weak weights.

4. THE MEAN SIZE OF CLUSTERS AND HIGHER MOMENTS

A. Definitions

The mean number of clusters was defined in (1.17) as the mean number of components in the graph G', which is either the subgraph defined by the occupied edges or the section graph defined by the occupied vertices depending on the problem. We now define moments of the cluster size distribution by

$$M_{rs}(p; G) = \left\langle \sum_{i=1}^{n} e_i^r v_i^s; G \right\rangle, \tag{4.1}$$

where e_i and v_i are the numbers of edges and vertices in the ith component of G'. The bond problem moments are denoted by $\bar{M}_{rs}$. Clearly M_{00} is the mean number of clusters previously discussed. M_{01} and M_{10} are the mean number of vertices and edges respectively, and are simple functions

$$M_{01} = |V|\, p, \quad M_{10} = |E|\, p^2,$$
$$\bar{M}_{01} = |V|, \qquad \bar{M}_{10} = |E|\, p. \tag{4.2}$$

The second moments yield various measures of the mean size of clusters. If the size of a cluster is defined by vertex content, then we may either choose a vertex, calculate the mean size of clusters containing that vertex, and average over all vertices, which gives

$$|V|^{-1} M_{02}(p; G), \tag{4.3}$$

or choose an edge, calculate the mean size of clusters containing that edge, and average over all edges, which gives

$$|E|^{-1} M_{11}(p; G). \tag{4.4}$$

Alternatively, it may be useful to measure size by edge content, in which case the two methods of computation yield

$$|V|^{-1} M_{11}(p; G) \quad \text{and} \quad |E|^{-1} M_{20}(p; G), \tag{4.5}$$

respectively. To establish contact with previous notation,[6]

$$S(p)^S = M_{02}(p; G)/M_{10}(p; G) \tag{4.6}$$

and

$$S(p)^B = \bar{M}_{20}(p; G)/\bar{M}_{10}(p; G). \tag{4.7}$$

B. Relationship between Mean Size and Pair Connectedness

It was shown in Ref. 3 that $S(p)^S$ could be expressed as a sum over the pair connectedness. This sum rule may be generalized to all second moments for site and bond problems. The essential observation is that

$$\sum_{\substack{v,v'\in V\\ v\neq v'}} \gamma_{v,v'} = \sum_i v_i(v_i - 1), \tag{4.8}$$

which when averaged gives

$$M_{02}(p; G) = M_{01}(p; G) + 2 \sum_t D(p; G_t^{\text{ii}}), \tag{4.9}$$

where the sum is over all two rootings of G. Similarly,

$$\sum_{\substack{v\in V\\ e\in E}} \gamma_{v,e} = \sum_i v_i e_i, \tag{4.10}$$

so that

$$M_{11}(p; G) = \sum_{\substack{v\in V\\ e\in E}} \langle \gamma_{v,e}; G\rangle \tag{4.11}$$

$$= p \sum_{v,e} \langle \gamma_{v,v'}; G_e^{\gamma}\rangle, \tag{4.12}$$

which is again a sum over the pair connectedness, and finally

$$M_{20}(p; G) = M_{00}(p; G) + \sum_{\substack{e,e'\in E\\ e\neq e'}} \langle \gamma_{e,e'}; G\rangle \tag{4.13}$$

$$= M_{10}(p; G) + p^2 \sum_{\substack{e,e'\in E\\ e\neq e'}} \langle \gamma_{v,v'}, G_{e,e'}^{\gamma}\rangle. \tag{4.14}$$

All the above equations hold for the corresponding bond problem quantities.

The equations may be generalized to higher moments; for example,

$$M_{03}(p; G) = M_{01}(p; G) + 6 \sum D(p; G^{\text{ii}}) + 6 \sum D(p; G^{\text{iii}}) \tag{4.15}$$

and

$$M_{04}(p; G) = M_{01}(p; G) + 14 \sum D(p; G^{\text{ii}}) + 36 \sum D(p; G^{\text{iii}}) + 24 \sum D(p; G^{\text{iv}}), \quad (4.16)$$

where $D(p; G^{\text{iii}})$ and $D(p; G^{\text{iv}})$ are the connectedness functions appropriate to three- and four-rooted graphs and the sums are over all possible rootings.

C. Graphical Expansions

1. Strong Lattice Constant Expansion for the Site Problem

Substitution of (1.10) in (4.9) provides a graphical expansion for $M_{02}(p; G)$:

$$M_{00}(p; G) = |V|\, p + 2 \sum_t \sum_m [c_m^{\text{ii}}; G_t^{\text{ii}}] D(c_m^{\text{ii}}) p^{v_m} \quad (4.17)$$

$$= \sum_r [c_r; G] M_{02}(c_r) p^{v_r}, \quad (4.18)$$

where in going from (4.17) to (4.18) we have grouped together contributions from c_m^{ii} which are isomorphic with c_r when the roots are ignored. By applying Proposition 1, the weight functions may be related

$$M_{02}(G) = \begin{cases} 2 \sum_t D(G_t^{\text{ii}}) & \text{for } |V| > 1 \\ 1 & \text{for } |V| = 1 \end{cases}. \quad (4.19)$$

The sum over rootings in (4.19) is more conveniently replaced by a sum over 1-irreducible two-rooted graphs no two of which are isomorphic:

$$M_{02}(G) = 2 \sum_t ((s_t^{\text{ii}}; G)) D(s_t^{\text{ii}}) \quad (4.20)$$

where $((s_t^{\text{ii}}; G))$ is the number of rootings[7] of G isomorphic with s_t^{ii}. A similar analysis of M_{11} and M_{20} via Eqs. (4.12) and (4.14) gives the following expressions for the strong weights:

$$M_{11}(G) = \sum_t D(H_t^{\text{ii}}) \quad \text{for} \quad |V| > 2, \quad (4.21)$$

where H_t^{ii} is obtained by rooting any vertex and any nonincident edge of G and then contracting the edge. We have

$$M_{20}(G) = 2 \sum_t D(H_t^{\text{ii}}) \quad \text{for} \quad |V| > 3, \quad (4.22)$$

where H_t^{ii} is obtained by choosing a pair of nonincident edges of G and contracting. Each pair is counted once only.

2. Weak Lattice Constant Expansions for Site and Bond Problems

Following the arguments of I, we may expand the moments for site and bond problems in terms of weak lattice constants. Thus

$$M_{rs}(p; G) = \sum_t (c_t; G) m_{rs}(c_t) p^{v_t} \quad (4.23)$$

and

$$\bar{M}_{rs}(p; G) = \sum_t (c_t; G) \bar{m}_{rs}(c_t) p^{e_t}. \quad (4.24)$$

Now for a connected graph C we have

$$M_{rs}(1; G) = \bar{M}_{rs}(1; G) = |E|^r |V|^s, \quad (4.25)$$

and so in analogy with (1.15) we obtain the recursive definition of the site-problem weak weights

$$m_{rs}(C) = |E|^r |V|^s - \sum_t{}' (c_t; C) m_{rs}(c_t). \quad (4.26)$$

Clearly, $\bar{m}_{rs}(c_t)$ satisfies the same equation, and so

$$\bar{m}_{rs}(c_t) = m_{rs}(c_t). \quad (4.27)$$

There is therefore only one set of weak weights to be determined, and these will be derived from bond problem formulas together with Proposition 1. Substitution of (1.12) into the barred versions of (4.9), (4.12), and (4.14) enables the relations

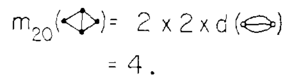
$$m_{02}(G) = 2 \sum d(G_t^{\text{ii}}) \quad \text{for} \quad |E| > 2 \quad (4.28)$$

$$m_{11}(G) = \sum_t d(H_t^{\text{ii}}) \quad \text{for} \quad |E| > 1, \quad (4.29)$$

and

$$m_{20}(G) = 2 \sum_t d(H_t^{\text{ii}}) \quad \text{for} \quad |E| > 2 \quad (4.30)$$

to be obtained; e.g.,



The sums in (4.29) and (4.30) are over the same graphs as in (4.21) and (4.22).

5. CONCLUSION

The pair connectedness, which is an interesting concept in its own right, has been shown to provide an important link between the previously discussed concepts of mean number and mean size. It is hoped that the formulas of Sec. 4, which relate the mean size weights to the pair connectedness weights and hence to the mean number weights, will enable a significant extension of the mean size expansions to be made.

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† Present address: Westfield College, University of London, Hampstead, London N.W.3, England.

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Discrete Lorentz and Singleton Representations of the Universal Covering Group of the $3 + 2$ de Sitter Group

LEON JAFFE
Center for Particle Theory, Department of Physics, University of Texas, Austin, Texas 78712

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We consider the relationship between the discrete reduction of $\bar{S}O(3, 2)$ with respect to $\bar{S}O(3, 1)$ and the singleton reduction of $\bar{S}O(3, 2)$ with respect to $\bar{S}O(3) \otimes \bar{S}O(2)$.

INTRODUCTION

The unitary irreducible representations (UIR's) of $S = \bar{S}O(3, 2)$ [$\bar{S}O(p, q)$ denotes the universal covering group of $SO(p, q)$] have been considered in Ref. 1 for those UIR's which have a singleton reduction with respect to its maximal pseudocompact subgroup $K = \bar{S}O(3) \otimes \bar{S}O(2)$. [A singleton reduction of a representation of group with respect to a subgroup means that each irreducible representation of the subgroup occurs once only in the reduction.] These will simply be referred to as singleton UIR's.

However, for many physical applications, we are interested in S, for example, because it contains L_+, the covering group of the proper Lorentz group [$L_+ \cong \bar{S}O(3, 1)$], as a subgroup. It therefore becomes useful to know the reduction of representations of S with respect to L_+. This has been done by the author[2] for unitary and nonunitary irreducible representations of S with a discrete singleton reduction with respect to L_+, which we shall refer to as the "discrete Lorentz" representations of S.

We arrive at the remarkable result that all the discrete Lorentz UIR's are nothing but singleton representations [i.e., with a singleton reduction with respect to K]. There is no *a priori* reason for supposing that this should be so.

We thus independently arrive at many of the singleton UIR's obtained by Ehrman. In fact, we obtain all those singleton UIR's in which for each value of the angular momentum the eigenvalues of the generator of $\bar{S}O(2)$, Γ_0, are bounded.

There is reason to believe that a discrete reduction of UIR's of S with respect to L_+ must of necessity be a singleton reduction.[3] If this is correct, then we have considered all UIR's of S with a discrete reduction with respect to L_+. It follows that all other UIR's of S have a nondiscrete reduction with respect to L_+. Hence, all the nonsingleton UIR's of S do not have a discrete reduction with respect to $\bar{S}O(3, 1)$. This applies also to those singleton representations in which Γ_0 has an unbounded spectrum of eigenvalues within each angular momentum subspace.

The discrete reduction of the UIR's of S with respect to L_+ is quite simple. We give a brief review of this. (The complete analysis can be found in Ref. 2, which includes nonunitary representations as well.) The bases of the representation spaces of the restricted class of irreducible representations we thus obtain are diagonal with respect to the Casimir operators of L_+. We effect a similarity transformation which takes this "Lorentz" basis into a "maximal compact" basis that is diagonal with respect to the Casimir operators of K, and determine the spectrum of eigenvalues of Γ_0 in each subspace with definite angular momentum. This determines the reduction with respect to K. This procedure is in many ways simpler than a direct determination of the reduction as in Ref. 1.

Class II(c) and V representations (see text) are the Majorana representations.

Class IV(a) and IV(b) representations (see text) provide a nautral generalization of the finite nonunitary Dirac representation to unitary representation