Percolation Processes. I. Low-Density Expansion for the Mean Number of Clusters in a Random Mixture*

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A cluster expansion, valid at low densities, is derived for the mean number of clusters in a random mixture of sites or bonds on a graph. It is shown that only clusters without a cut-point (stars) are required, and a number of general theorems for determining the weights are proved.

1. INTRODUCTION

In this paper we examine the derivation of series expansions, valid at low densities, for the mean number of clusters in a random mixture. We have introduced this problem in a previous paper,[1] hereafter referred to as I. For a detailed introduction to the problem, and to the closely related percolation problem, reference should be made to I, to Fisher and Essam,[2] and for a general survey, to Hammersley and Frisch.[3] We make use of the general concepts of graph theory which we have described in a paper,[4] hereafter referred to as II.

We begin with a statement of the problem in the formal terminology of graph theory and then derive a cluster expansion for the mean number function. We show that the expansion depends only on connected clusters without cut-points (stars) and that the corresponding weights are particularly simple in the system of weak lattice constants. We prove a number of general theorems of use in determining the weights of individual clusters and, as an example, derive series expansions for the site and bond problem on the plane triangular lattice.

We subsequently apply these expansions to a study of the mean number function and extend the series developments for the mean size of clusters in a random mixture.

2. STATEMENT OF THE PROBLEM

(A) Site Problem

In site mixtures the sites, or vertices, of a linear graph G are supposed colored black with probability p or white with complementary probability q. We adopt the convention that, in such a random mixture of two species represented by the black and white sites, the primary species is the black, and we refer to small p as *low density*. The bonds, or edges, of G are regarded as colored black if they connect two black sites, white if they connect two white sites, uncolored if they connect sites of different colors.

Any realization R of the probability distribution on G defines two linear graphs R_B and R_W which are, respectively, the ensembles of black and white clusters. More precisely, R_B is the section[5] graph of G defined by all the sites of G that are black in R and the term black cluster is used to describe any connected component of R_B. In general, R_B has many connected components, and it is the expectation value of the number of these that we study. Denoting the number of connected components of R_B by $n(R_B)$, we define the mean number function K by

$$K(p;G) = \langle n(R_B)\rangle. \tag{2.1}$$

(B) Bond Problem

In bond mixtures the bonds, or edges, of a linear graph G are supposed colored black with probability $\bar{p}$ or white with complementary probability $\bar{q}$. A realization $\bar{R}$ of the bond probability distribution on G defines two linear graphs $\bar{R}_B$ and $\bar{R}_W$ which are, respectively, the ensemble of black and white bond clusters. More precisely, $\bar{R}_B$ is the subgraph of G defined by the edges of G that are black in $\bar{R}$, together with their end points, and a black bond cluster is a connected component of $\bar{R}_B$. We define the corresponding mean number function by

$$\bar{K}(\bar{p};G) = \langle n(\bar{R}_B)\rangle. \tag{2.2}$$

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[1] M. F. Sykes and J. W. Essam. J. Math. Phys. 5, 1117 (1964).
[2] M. E. Fisher and J. W. Essam, J. Math. Phys. 2, 609 (1961).
[3] H. L. Frisch and J. M. Hammersley, J. Soc. Indust. Appl. Math. 11, 894 (1963).
[4] M. F. Sykes, J. W. Essam, B. R. Heap, and B. J. Hiley, J. Math. Phys. 7, 1557 (1966).
[5] Defined in II, Sec. 2.

In this simple conceptual form the sites of the graph are not assigned a color. The bond problem may be studied as a site problem in which each bond is made to correspond to a site on a suitably defined covering graph.[2]

In many applications of bond mixtures the bonds are primarily considered as connections between the sites, and a simplification results if we adopt the following convention which we call the *null-cluster convention.*

Suppose that for any realization $\bar{R}$ two sites are defined as connected if they are joined by a black bond. We employ the term *black-connected* cluster to describe any connected component of the *partial*[5] graph, $\bar{P}_B$ of G whose edge set is the edge set of $\bar{R}_B$. Some black-connected clusters may reduce to isolated sites (null-clusters). We define the mean number function $\bar{K}_0$ for this convention to be the expectation value for the number of black-connected site clusters. We write

$$\bar{K}_0(\bar{p}; G) = \langle n(\bar{P}_B)\rangle, \tag{2.3}$$

where the suffix on $\bar{K}_0$ denotes the operation of the null-cluster convention. If we denote the mean number, or expectation value, of the isolated sites or null-clusters by $\langle\text{n.c.}\rangle$, then

$$\bar{K}_0(\bar{p}; G) = \bar{K}(\bar{p}; G) + \langle\text{n.c.}\rangle. \tag{2.4}$$

3. LOW-DENSITY CLUSTER EXPANSION FOR $K(p; G)$ (SITE PROBLEM)

We now describe a method of obtaining the mean number function which avoids a detailed specification of cluster perimeters[6] required by the alternative perimeter method. It has been outlined in II and is readily formalized.

To recapitulate, any realization, R, of the probability distribution defines a section graph R_B of G which is the graph of the black sites and bonds. In the notation of II, G contains $[g_i; G]$ section graphs isomorphic with g_i and the probability of any one of these being R_B is just

$$p^{v_i}(1-p)^{v_G - v_i}. \tag{3.1}$$

Thus if g_i has n_i connected components

$$K(p; G) = \sum_i n_i p^{v_i}(1-p)^{v_G - v_i}[g_i; G], \tag{3.2}$$

where the summation is taken over all the strong lattice constants of G. Now these latter can be expressed in terms of the connected constants only, and because mean number is an extensive property it follows, by the arguments of Sec. 5 of II, that the resultant expression is linear in the connected constants and that a cluster expansion may be developed for $K(p; G)$. Thus we may write, denoting connected graphs by c_i,

$$K(p; G) = \sum_i W_i(p)[c_i; G], \tag{3.3}$$

where W_i is the appropriate weight function of c_i.

It is evident from the form of (3.2) that the weight functions are polynomials in p. They can be evaluated by carrying out the substitutions for the separated lattice constants, and the coefficient of any connected constant $[c_i; G]$ will come from two sources (II, Theorem II):

(1) from c_i itself. The presence of the factor p^{v_i} in (3.1) ensures that this contribution is always of degree at least v_i in p.

(2) from separated constants such as, for example, a three component graph $c_r \cup c_s \cup c_t$. In the reduction of these it is evident that c_i cannot occur as an overlap partition unless $c_r \cup c_s \cup c_t$ has at least v_i vertices, and therefore again by (3.1) the contribution is of degree at least v_i in p.

Alternatively, the weight functions may be evaluated from the mean number functions of the individual connected constants. Since the weight of any graph c_i is expressible in terms of the $K(p; G)$ of c_i and all its connected subgraphs [Eqs. (5.19) and (5.20) of II], and this expression is linear, and further, each function is a polynomial in p of degree at most v_i, it follows that $W_i(p)$ is of degree at most v_i in p. Thus by virtue of the previous result, $W_i(p)$ can only have one nonzero coefficient—that of the v_ith power of p, and we state the result as a theorem.

Theorem I: The strong weight function $W_i(p)$ of $[c_i; G]$ can be written $K_i p^{v_i}$, where K_i is independent of p.

The strong weight functions arise quite naturally in the site problem since the clusters studied are all section graphs. However, we often find it convenient to work with the corresponding weak weight functions which we introduce by the following theorem.

Theorem II: The weak weight function $w_i(p)$ of $(c_i; G)$ can be written $k_i p^{v_i}$, where k_i is independent of p.

Proof: The result follows at once by conversion of the strong weight functions $K_i p^{v_i}$ into the weak weight functions by means of the conversion matrix for weights, which is just the transpose of the reciprocal conversion matrix for the connected con-

[6] Defined in I, Sec. 2, and in Ref. 2, Sec. 2.

stants, and which relates constants with the same number of *vertices*.

We now define $K(c_i) = K_i$ and $k(c_i) = k_i$ to be the (strong) K-wt and (weak) k-wt of c_i, respectively. Following the convention of II, Sec. 2, we also abbreviate $K(s_i)$, $k(s_i)$ to K_i, k_i whenever it is clear from the context which graph dictionary is being used, and further, we sometimes write K_G, k_G for $K(G)$, $k(G)$. For any graph G we have

$$K(p; G) = \sum_i [c_i; G] K_i p^{v_i} \tag{3.4}$$

$$= \sum_i (c_i; G) k_i p^{v_i}, \tag{3.5}$$

where the summations are taken over all the connected constants of G. The form of (3.4) and (3.5) which results from Theorems I and II makes it possible, when G is an infinite graph, to derive series developments in powers of p as appropriate to low densities. By the methods of this section, the weights of the site and the bond are found to be $+1$ and -1, respectively (in both systems), and we use this result as a lemma to prove the next theorem.

Theorem III: For any graph G

$$\sum_{i>2} [c_i; G] K_i = \sum_{i>2} (c_i; G) k_i = C(G), \tag{3.6}$$

where $C(G)$ denotes the cyclomatic number (or circuit rank) of G and the summation is taken over all the connected constants of G except the site and the bond. (In graph dictionary order these will have suffixes 1 and 2.)

Proof: In both the weak and the strong systems, the site has weight $+1$ and the bond, weight -1. Thus from (3.4)

$$K(p; G) = v_G p - l_G p^2 + \sum_{i>2} [c_i; G] K_i p^{v_i}. \tag{3.7}$$

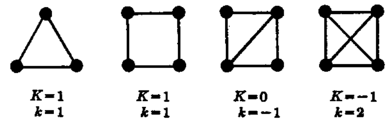
But for $p = 1$ the expected number of clusters must reduce to the number of components in G or $K(1; G) = n_G$, and on setting $p = 1$ in (3.7),

$$\sum_{i>2} [c_i; G] K_i = n_G - v_G + l_G = C(G) \tag{3.8}$$

by definition. Likewise the result holds for weak weights.

Theorem III may be used to derive the weak and strong weights by successive application to the connected constants arranged in a suitable graph dictionary order (i.e., in order of ascending cyclomatic number). *We may thus take (3.6) as defining the quantities K_i, k_i associated with a graph c_i.*

The weights of all graphs with three and four vertices are found to contain only four with nonzero contributions,



(3.9)

which suggests that only stars have nonzero weight—a result we now prove.

Theorem IV: If c_i has a cut point, $K_i = k_i = 0$.

Proof: We prove the result in the strong system, and the result for the weak system follows by changing to weak weights throughout.

Suppose a graph G has a cut point at the vertex A. Then by definition the deletion of A, together with all its incident edges, leave a graph with at least two connected components. Denote the vertex set of any one of these by V' and that of all the others by V''. Denote by G' the section graph of G with the vertex set $V' + A$ and by G'' that with vertex set $V'' + A$. Then $G = G' + G''$. By application of (3.6) to G, G' and G'',

$$\sum_{i>2} [c_i; G] K_i = C(G), \tag{3.10}$$

$$\sum_{i>2} [c_i; G'] K_i = C(G'), \tag{3.11}$$

$$\sum_{i>2} [c_i; G''] K_i = C(G''), \tag{3.12}$$

and since A is an articulation point

$$C(G) = C(G') + C(G''). \tag{3.13}$$

The constants of G result from embeddings of each c_i in G, and these may be grouped into three mutually disjoint classes: those that lie wholly in G', those that lie wholly in G'', and those that lie neither wholly in G nor wholly in G' and which necessarily correspond to those c_i with cut points. Denoting the contribution of this third class of embeddings by an asterisk, we must have [from (3.10)–(3.13)]

$$\sum_{i>2} [c_i; G]^* K_i = 0, \tag{3.14}$$

and since the only connected graph of three points with a cut point has weight zero, the result follows inductively by successive application of (3.14) to all graphs with cut points.

Theorem IV enables the definitive equation (3.6) for weights to be restricted to multiply connected graphs, and we may write, for any such graph M,

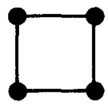
$$\sum_{i>1} [s_i; M] K_i = \sum_{i>1} (s_i; M) k_i = C(M), \tag{3.15}$$

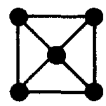
where the summation is taken over all stars except the bond (s_1). For theoretical purposes it is most convenient to study the weak weights; we do so in Sec. 5 after we have developed the cluster expansion for the bond problem to which, as we show, the weak weights also apply. For practical purposes the strong weights are useful in the actual derivation of series expansions, and we prove two theorems, restricted to strong weights, that simplify the derivation of individual weights.

Theorem V: The strong K-wt of a graph G is the coefficient of the v_Gth power of p in $K(p; G)$.

Proof: The result is evident since there is only one nonzero constant with v_G vertices in the strong system—the graph itself.

As examples of the application of this theorem we quote

$G =$  $$K(p; G) = 4p - 4p^2 + p^4, \qquad K_G = +1, \tag{3.16}$$

$G =$  $$K(p; G) = 5p - 8p^2 + 4p^3 + p^4 - p^5, \qquad K_G = -1. \tag{3.17}$$

We remark, parenthetically, that the corresponding k-wts can be found by use of the conversion matrix. The method is capable of elaboration and an appropriate technology can be developed to obtain $K(p; G)$ for an individual graph by the use of recurrence relations. It is found that many stars have zero K-wt, and most of these are accounted for by the next theorem.

Theorem VI: If G is a graph with a cut set of n vertices, and further the section graph which has these n vertices as vertex set is a complete graph of n vertices, then $K_G = 0$.

Proof: Denote the vertex set of the cut set by N. Denote the vertex set of one of the connected components that result from the deletion of N and its incident edges by V', and that of all the others by V''. Denote by G' the section graph with vertex set $N + V'$, and by G'' that with vertex set $V'' + N$. If in a realization R of the distribution any of the vertices in N are black, they must all be members of the same component of R_B since the section graph of G defined by N is a complete graph. But $G = G' + G''$ and the number of components in the sum graph G is the sum of the components in G' and G'' if the cut set contains no components (probability q^n). If the cut set contains a component it contributes both to G' and G''. Thus

$$K(p; G' + G'') = K(p; G') + K(p; G'') + q^n - 1. \tag{3.18}$$

By Theorem V the required weight is the coefficient of the $(v_{G'} + v_{G''} - n)$th power of p, and this exceeds max $(v_{G'}, v_{G''})$ since $v_{G'} > n$, $v_{G''} > n$. Therefore $K_G = 0$.

4. LOW-DENSITY CLUSTER EXPANSION FOR $\bar{K}(\bar{p}; G)$ (BOND PROBLEM)

The cluster expansion method applies to bond mixtures in an analogous manner to the treatment of site mixtures. If g_i is any subgraph of G, with no isolated vertices, the probability of g_i being a realization of the bond distribution probability is now

$$\bar{p}^{l_i}(1 - \bar{p})^{l_G - l_i}, \tag{4.1}$$

and the argument proceeds formally as for the site problem. Thus we now write

$$\bar{K}(\bar{p}; G) = \sum_i (c_i; G)\bar{w}_i(\bar{p}) \tag{4.2}$$

and obtain in place of Theorem I:

Theorem VII: The bond weight function $\bar{w}_i(\bar{p})$ for $(c_i; G)$ can be written $\bar{k}_i\bar{p}^{l_i}$, where $\bar{k}_i$ is independent of $\bar{p}$.

Because the conversion matrix for weights converts from constants with r vertices to constants with *r vertices* (and not *edges*), the *strong* bond weight function of a graph, $\bar{W}_i(\bar{p})$, is in general a polynomial in $\bar{p}$.

By including the subgraphs of G with isolated vertices we obtain corresponding results for the null-cluster convention, and we denote the corresponding weights, independent of $\bar{p}$ in Theorem VII by $\bar{k}_i^0$. The bond weights of the site and the bond are found to be, for the site $\bar{k} = 0$, $\bar{k}^0 = +1$ and, for the bond $\bar{k} = +1$, $\bar{k}^0 = -1$. The null-cluster convention leads directly to:

Theorem VIII: For any graph c,

$$\bar{k}_i^0 = k_i, \tag{4.3}$$

or with the null cluster convention, the bond and site weights are identical.

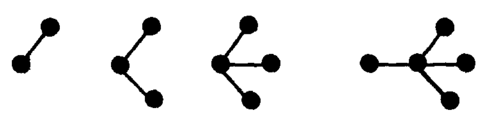
Proof: The result follows because under the null-cluster convention the site has weight $+1$, the bond, weight -1, and the number of clusters for $p = 1$ is the number of components in G. Therefore by the arguments of Theorem III the weights are defined by

$$\sum_{i>2} (c_i; G)\bar{k}_i^0 = C(G), \tag{4.4}$$

which is identical with the definitive equation (3.6) for weak weights.

We examine the properties of weak weights in Sec. 5 and conclude this section with a closer study of the relation between $\bar{k}_i$ and $\bar{k}_i^0$.

To relate the two systems of weak bond weights, we first observe that corresponding to Theorem V we have the result that the weak $\bar{k}$-wt of a graph G is the coefficient of the l_Gth power of $\bar{p}$ in $\bar{K}(\bar{p}; G)$. By an argument closely parallel to that of Theorem VI, it can be shown that graphs with a cut-vertex whose deletion, together with all its incident edges, leaves a graph with at least one edge have zero $\bar{k}$-wt. The only graphs with cut-vertices that do not satisfy this latter condition are those with the obvious general topology



a_1 a_2 a_3 a_4 $\cdots$ (4.5)

To determine the weight of the general graph a_s of this type, we use

$$\bar{K}(\bar{p}; a_s) = 1 - \bar{q}^s = \binom{s}{1}\bar{p} - \binom{s}{2}\bar{p}^2 + \cdots + (-1)^{s+1}\binom{s}{s}\bar{p}^s, \tag{4.6}$$

and therefore from the last coefficient the $\bar{k}$-wt of $a_s = (-1)^{s+1}$.

For a general graph with no repeated bonds (i.e., not a multigraph) the contribution of graphs of type (4.5) is

$$\sum_{\substack{\text{all}\\ \text{sites}}} \sum_s (-1)^{s+1}\binom{z}{s}\bar{p}^s - l_G\bar{p}, \tag{4.7}$$

where the sum runs over all the sites of G and z is the number of edges incident on a site. Every a_s can be associated with its center point except the bond a_1, which is counted twice.

For the null cluster convention we must add the number of isolated clusters, and with the same restriction on multigraphs,

$$\langle\text{n.c.}\rangle = \sum \bar{q}^z. \tag{4.8}$$

On adding (4.8) and (4.7) we are left with

$$1 - l_G\bar{p}, \tag{4.9}$$

and the contribution from terms of type (4.5) cancel except for the site and the bond which now have weights $+1$ and -1, respectively, and we may write

$$\bar{k}_i^0 = \bar{k}_i = k_i \quad \text{for all stars} \quad i > 2. \tag{4.10}$$

On a regular lattice of coordination number z, with N sites, we may write

$$\langle\text{n.c.}\rangle = Nq^z. \tag{4.11}$$

5. PROPERTIES OF THE WEAK WEIGHTS

We now establish a number of theorems applicable to the weak k-wts defined for multiply connected graphs by

$$\sum_{i\geq 2} (s_i; M)k_i = C(M). \tag{5.1}$$

Theorem IX: If two graphs are homeomorphic they have equal k-wts.

Proof: The result is more or less obvious from the definitive equation (5.1). A tedious proof is readily constructed, but we confine our treatment to examples from which it is evident that the result will follow inductively.

First, every Jordan[7] curve has weight $+1$ since there is only one multiply connected subgraph, the graph itself, and the cyclomatic number is 1. [The site and bond are excluded by (5.1).] Thus

$$k\text{-wt of} \quad (n)_p = +1. \tag{5.2}$$

For stars of cyclomatic number 2 there is only one topological type—the θ graph. Any θ graph $(r, s, t)_\theta$ has three Jordan subgraphs:

$$(r+s)_p, \quad (r+t)_p, \quad (s+t)_p, \tag{5.3}$$

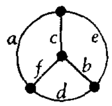
all with weight $+1$, and therefore

$$k\text{-wt of} \quad (r, s, t)_\theta = -1. \tag{5.4}$$

For stars of cyclomatic number 3, there are

[7] The various types of graph are described in II, Sec. 7.

four topological types, and we work with only one, the tetrahedral or α graph:



or $(a, b; c, d; e, f)_\alpha$,

of which the relevant subgraphs are the seven Jordan curves

$$(a + c + f)_p, \quad (a + d + e)_p, \quad (b + c + e)_p,$$
$$(b + d + f)_p, \quad (a + b + c + d)_p,$$
$$(a + b + e + f)_p, \quad (c + d + e + f)_p, \tag{5.5}$$

which contribute $+7$ and the six θ graphs

$$(a, c + f, d + e)_\theta, \quad (b, c + e, d + f)_\theta,$$
$$(c, a + f, b + e)_\theta, \quad (d, a + e, b + f)_\theta, \tag{5.6}$$
$$(e, a + d, b + c)_\theta, \quad (f, a + c, b + d)_\theta,$$

which contribute -6. Therefore

$$k\text{-wt of } (a, b; c, d; e, f)_\alpha = +2. \tag{5.7}$$

Theorem IX effects a great reduction in the number of individual k-wts that need be worked, and we now state the k-wts of all stars with $C(S) = 3$:

$$\begin{array}{ll} \alpha \text{ graphs} & k\text{-wt} = +2, \\ \beta \text{ graphs} & k\text{-wt} = +1, \\ \gamma \text{ graphs} & k\text{-wt} = +1, \\ \delta \text{ graphs} & k\text{-wt} = +1. \end{array} \tag{5.8}$$

For $C(S) = 4$ there are 17 topological types and we list these, together with their weak weights in the Appendix.

In practice the determination of the weight of a graph from (5.1) becomes heavy as the cyclomatic number increases. Numerous results can be established to effect a reduction in the calculation, and we quote two theorems,[8] which we apply in a subsequent paper to the problem of high-density expansions.

Theorem X: If S is a planar topological star, that is, a planar star drawn to conform with the planar condition, and $\sum^*$ denotes summation over all substars that are not finite faces, then

$$\sum_{i \geq 2}{}^* (s_i; S) k_i = 0. \tag{5.9}$$

To avoid a special notation we adopt the convention that the restriction $\sum^*$ imposed on the summation implies the corresponding restriction on the lattice constants. Thus in (5.9) the constant $(s_i; S)$ is the constant for the embeddings of s_i that are not finite faces. Likewise, in the next theorem $\sum_B$ implies that $(s_i; S)$ is the constant for embeddings that contain the boundary.

Proof: The result follows from Euler's law of the edges which states that for a planar topological graph the cyclomatic number is equal to the number of finite faces. Each of these has weight unity in (5.1), and therefore the total contribution from stars that are not finite faces must be zero.

Theorem XI: If S is a planar topological star and $\sum_B$ denotes summation over all substars which contain the contour of the infinite face of S (the boundary of S), then

$$\sum_{i \geq 2}{}_B (s_i; S) k_i = 0, \quad C(S) > 1. \tag{5.10}$$

Proof: We show that if the result holds for $C(s_i) < n$ it will hold for $C(S) = n$. It is true for $C(S) = 2$. If S is a planar topological star, then so are all its substars, and we may divide these into mutually disjoint categories by the contours of their infinite faces. The members of any category are the substars of the graph bounded by the contour that contain the contour, and this graph must have cyclomatic number less than $C(S)$ unless the contour is the contour of S. Assuming the result holds for $C(s_i) < C(S)$, the contribution from each category is zero unless the contour reduces to a finite face. If we exclude these,

$$\sum{}^* (s_i; S) k_i = 0 \tag{5.11}$$

by the previous theorem, and therefore

$$\sum{}_B^* (s_i; S) k_i = 0, \tag{5.12}$$

but the asterisk, which excludes finite faces from the summation, is now redundant since $C(S) > 1$ and no finite face contains the boundary. Thus the result is proved.

As an example we can now simplify the calculation of the k-wt of the tetrahedral graph $(a, b; c, d; e, f)_\alpha$. The subgraphs which contain the boundary are: the graph itself and

$$(a + d + e)_p \quad \text{contribution } +1$$

$$\left.\begin{array}{l} (a, c + f, d + e)_\theta \\ (d, a + e, b + f)_\theta \\ (e, a + d, c + b)_\theta \end{array}\right\} \text{contribution } -3,$$

[8] In these theorems we use the terms star and substar in place of multiply connected graph and multiply connected subgraph. It is clear from the context that the bond is not intended.

and therefore by (5.10)

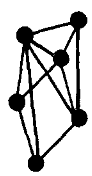
$$k\text{-wt of } \quad (a, b; c, d; e, f)_\alpha = +2. \tag{5.13}$$

For completeness we now state the theorem for weak weights which corresponds to Theorem VI for strong weights.

Theorem XII: If G is a graph with a cut set of n vertices, and further the section graph G''' which has these n vertices as vertex set is a complete graph of n vertices, then with G' and G'' defined as in Theorem VI if k_G, $k_{G'}$, $k_{G''}$, $k_{G'''}$ denote the weak weights of G, G', G'', G''', respectively:

$$k_{G'''}k_G = k_{G'}k_{G''}. \tag{5.14}$$

We omit the proof of this theorem since the result is most easily established by techniques we shall describe in a subsequent paper; the direct proof is long. It is a result of great practical use. For example, the k-wt of the graph formed by placing a tetrahedron on one of the triangular faces of a square-based pyramid as drawn,



(5.15)

is found by taking G' as the pyramid ($k = -3$), G'' as the tetrahedron ($k = 2$), and G''' as the triangle they have in common ($k = 1$), and therefore from (5.14), $k = -6$.

The scope of Theorem XII is much extended by Theorem IX, and all homeomorphs of (5.15) will have $k = -6$. A particularly useful application is to the large class of graphs which have a cut-set with $n = 2$, together with their homeomorphs. Then g''' is the bond, and the required weight is just the product $-k_{G'}k_{G''}$. Thus the weight of a θ-graph is obtained as -1 since G' and G'' are polygons. The weights for β- and γ-graphs follow from the product of a polygon and a θ-graph.

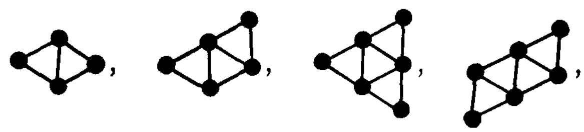
Theorems X and XI can be extended to K-wts, but in the present paper we do not elaborate further the theory of strong weights since the theory of weak weights is the more elegant and more generally useful.

6. EXPANSION FOR $K(p; G)$ FOR THE TRIANGULAR LATTICE (SITE PROBLEM)

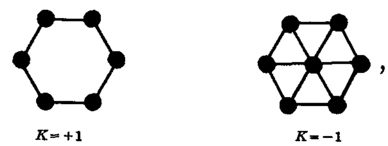
For an infinite graph we neglect edge effects and denote by $k(p; G)$ the mean number of clusters per site. The first contributions to a low-density expansion, valid for small p, come from the site, the bond, and the triangle. Denoting the triangular lattice by T, we have

$$k(p; T) = p - 3p^2 + 2p^3 + \cdots . \tag{6.1}$$

To extend this series we examine contributions from star graphs with more than three sites. The number of stars embeddable in the triangular lattice increases very rapidly with the number of sites, but many of these have zero K-wt. This is because of the presence of a cut-set of the special type described in Theorem VI; such cut-sets are of frequent occurrence on the triangular lattice. Thus for example the graphs



all have zero K-wt. There are no strong embeddings of $(4)_p$ or $(5)_p$ and the next contributions to (6.1) arise from



of which there are one per site, respectively, to give $+p^6 - p^7$. We summarize in Table I the graphs with 8, 9, and 10 sites required to extend (6.1) to the term in p^{10} and so obtain

$$k(p; T) = p - 3p^2 + 2p^3 + p^6 - p^7 + 3p^8 - 4p^9 + 9p^{10} + \cdots . \tag{6.2}$$

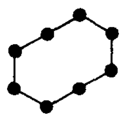
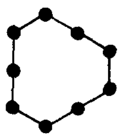
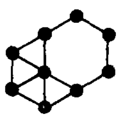
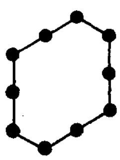
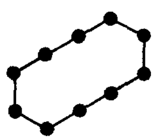
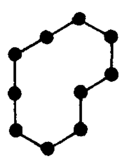
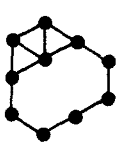
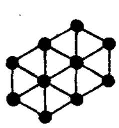
Further coefficients may be derived by direct enumeration, but this particular series is more readily extended by exploiting the exact matching relation which holds for the triangular lattice (I Sec. 3). That (6.2) is correct may be verified by comparison with Eq. (3.4) of I. For three-dimensional lattices no such matching relations exist, and the methods of this section enable the corresponding expansions for mean number for the simple cubic, body-centered cubic, and face-centered cubic lattices to be derived. We shall show in a subsequent paper that these expansions are of use in extending series expansions for the mean size of clusters at low densities.

7. EXPANSION FOR $\bar{K}(\bar{p}; G)$ FOR THE TRIANGULAR LATTICE (BOND PROBLEM)

For an infinite lattice we derive the expansion for the mean number of bond clusters *per site.* The expansion for the triangular lattice starts with the contribution from the graphs of type (4.5) which, per site, amounts to

$$3\bar{p} - 15\bar{p}^2 + 20\bar{p}^3 - 15\bar{p}^4 + 6\bar{p}^5 - \bar{p}^6. \qquad (7.1)$$

TABLE I. Some graphs on the triangular lattice that contribute to $k(p; T)$

Graph[a]	Number (per site)	K-weight	Contribution
	3	+1	$+3p^8$
	2	+1	$+2p^9$
	6	−1	$-6p^9$
	3	+1	$+3p^{10}$
	3	+1	$+3p^{10}$
	6	+1	$+6p^{10}$
	6	−1	$-6p^{10}$
	3	+1	$+3p^{10}$

[a] We illustrate the individual space-types.

TABLE II. Summary of stars on the triangular lattice and their contributions to $k(\bar{p}; T)$.

	k-wt	$l = 7$	$l = 8$	$l = 9$	$l = 10$
polygons (Jordan curves)	+1	42	123	380	1212
θ-graphs	−1	42	165	609	2283
α-graphs	+2	0	0	20	120
β-graphs	+1	0	6	54	375
γ-graphs	+1	6	36	162	666
δ-graphs	+1	0	0	0	6
F-graphs	−3	0	0	0	15
H-graphs	−1	0	0	0	24
K-graphs	−1	0	0	6	48
L-graphs	−1	0	0	0	12
N-graphs	−1	0	0	6	48
O-graphs	−1	0	0	2	12
Contribution:		$+6\bar{p}^7$	$+0\bar{p}^8$	$+13\bar{p}^9$	$+27\bar{p}^{10}$

TABLE III. Weak lattice constants for the triangular lattice.

θ-graphs.	$l = 7$	$(1,2,4)_\theta =$	30
		$(1,3,3)_\theta =$	12
	$l = 8$	$(1,2,5)_\theta =$	96
		$(1,3,4)_\theta =$	60
		$(2,2,4)_\theta =$	6
		$(2,3,3)_\theta =$	3
	$l = 9$	$(1,2,6)_\theta =$	312
		$(1,3,5)_\theta =$	168
		$(1,4,4)_\theta =$	69
		$(2,2,5)_\theta =$	30
		$(2,3,4)_\theta =$	30
	$l = 10$	$(1,2,7)_\theta =$	1068
		$(1,3,6)_\theta =$	516
		$(1,4,5)_\theta =$	378
		$(2,2,6)_\theta =$	132
		$(2,3,5)_\theta =$	126
		$(2,4,4)_\theta =$	63
α-graphs	$l = 9$	$(1,1;1,1;1,4)_\alpha =$	6
		$(1,1;1,2;1,3)_\alpha =$	12
		$(1,2;1,2;1,2)_\alpha =$	2
	$l = 10$	$(1,1;1,1;1,5)_\alpha =$	24
		$(1,1;1,2;1,4)_\alpha =$	48
		$(1,1;1,3;1,3)_\alpha =$	24
		$(1,2;1,2;1,3)_\alpha =$	24
β-graphs	$l = 8$	$(1,2;1,2;1,1)_\beta =$	6
	$l = 9$	$(1,2;1,2;1,2)_\beta =$	30
		$(1,2;1,3;1,1)_\beta =$	24
	$l = 10$	$(1,2;1,2;1,3)_\beta =$	102
		$(1,2;1,2;2,2)_\beta =$	51
		$(1,2;1,3;1,2)_\beta =$	120
		$(1,2;1,4;1,1)_\beta =$	60
		$(1,2;2,2;1,2)_\beta =$	12
		$(1,2;2,3;1,1)_\beta =$	6
		$(1,3;1,3;1,1)_\beta =$	24
γ-graphs	$l = 7$	$(1,2;1,2;1)_\gamma =$	6
	$l = 8$	$(1,2;1,2;2)_\gamma =$	12
		$(1,2;1,3;1)_\gamma =$	24
	$l = 9$	$(1,2;1,2;3)_\gamma =$	30
		$(1,2;1,3;2)_\gamma =$	48
		$(1,2;1,4;1)_\gamma =$	60
		$(1,3;1,3;1)_\gamma =$	24
	$l = 10$	$(1,2;1,2;4)_\gamma =$	96
		$(1,2;1,3;3)_\gamma =$	108
		$(1,2;1,4;2)_\gamma =$	108
		$(1,2;1,5;1)_\gamma =$	168
		$(1,2;2,2;3)_\gamma =$	12
		$(1,2;2,3;2)_\gamma =$	12
		$(1,2;2,4;1)_\gamma =$	12
		$(1,3;1,3;2)_\gamma =$	42
		$(1,3;1,4;1)_\gamma =$	108
δ-graphs	$l = 10$	$(1,2,2,5)_\delta =$	6

Up to six bonds the only star graphs are the following:

Graph	k-weight	Number	Contribution
$(3)_p$	$+1$	2	$+\ 2\bar{p}^3$
$(4)_p$	$+1$	3	$+\ 3\bar{p}^4$
$(5)_p$	$+1$	6	$+\ 6\bar{p}^5$
$(1, 2, 2)_\theta$	-1	3	$-\ 3\bar{p}^5$
$(6)_p$	$+1$	15	$+15\bar{p}^6$
$(1, 2, 3)_\theta$	-1	12	$-12\bar{p}^6$

and on adding these contributions to (7.1)

$$\bar{k}(\bar{p}; T) = 3\bar{p} - 15\bar{p}^2 + 22\bar{p}^3 - 12\bar{p}^4 + 9\bar{p}^5 + 2\bar{p}^6 + \cdots . \qquad (7.2)$$

We have extended (7.2) by enumerating every star graph on the triangular lattice with 7, 8, 9, and 10 lines, thus adding the terms

$$+6\bar{p}^7 + 0\bar{p}^8 + 13\bar{p}^9 + 27\bar{p}^{10}. \qquad (7.3)$$

We give in Table II a summary of the contributions from different topological types, and in Table III we list the lattice constants for all graphs of cyclomatic number 3 and 4.

In an analogous manner expansions can be obtained for other lattices. The graphs required are numerous but not too difficult to count in the *weak* system on a computer .The weight problem is made manageable by the results of Sec. 5. We shall subsequently apply these series developments to a study of the mean number function and also to extending the corresponding mean size series.

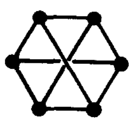
We have verified (7.3) by deriving the high-density expansion for the matching lattice; that is, the high-density expansion for the bond problem on the honeycomb lattice.

ACKNOWLEDGMENTS

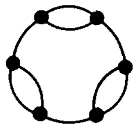
We are grateful to Professor C. Domb for suggesting the problem and to Professor M. E. Fisher for much constructive criticism.

APPENDIX: THE 17 TOPOLOGICAL TYPES OF STAR WITH $C(S) = 4$

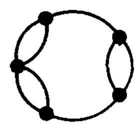
A graph $k = -5$



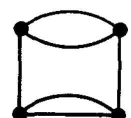
E graph $k = -1$



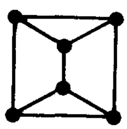
I graph $k = -1$



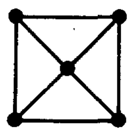
M graph $k = -1$



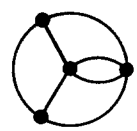
B graph $k = -4$



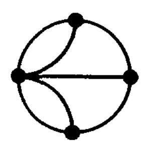
F graph $k = -3$



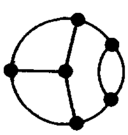
J graph $k = -2$



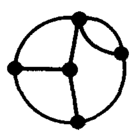
N graph $k = -1$



C graph $k = -2$

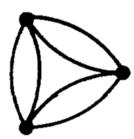


G graph $k = -2$

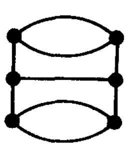


K graph $k = -1$

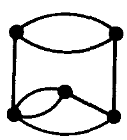
O graph $k = -1$



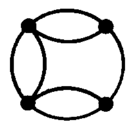
D graph $k = -1$



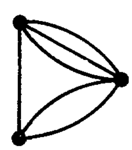
H graph $k = -1$



L graph $k = -1$



P graph $k = -1$



Q graph $k = -1$