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The Potts model and flows: III. Standard and subgraph break-collapse methods

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Abstract. An algorithm is developed for the exact calculation of the many-spin correlation functions of Potts model clusters which is more efficient than the standard break-collapse method successfully used in real space renormalisation group calculations. The improved performance is based on a relationship which, at any stage of the calculation, allows the replacement of certain subgraphs by single effective edges, thereby decreasing the number of iterations needed. Our method avoids, as in the standard one, the time-consuming summation over spin states and may be used to extend series expansion and real space renormalisation group calculations on crystal lattices. Both methods are based on a number of combinatorial formulae, the proofs of which are given in this paper.

1. Introduction

In papers I and II of this series (Essam and Tsallis 1986, de Magalhães and Essam 1986, hereafter referred to as PF1 and PF2 respectively) the many-spin correlation functions of the λ-state Potts model (Potts (1952); see Wu (1982) for a review) were expressed in terms of transmissivity functions. The latter were shown to have a simple combinatorial formulation in terms of flow polynomials (Tutte 1954). These relations are summarised in § 2 of this paper.

In electrical network theory, multiport impedances may be obtained either by direct solution of Kirchoff's equations or by progressive reduction of the network size using series and parallel combination rules and by replacing more complex subnetworks by effective impedances. The reduction method can have considerable computational advantage over the direct approach for large networks. Here we consider a Potts model on a network, or graph, in which the branches represent spin-spin interactions. An algorithm of the reduction type, which we shall call the subgraph break-collapse method (SBCM), will be developed for the flow polynomials and the transmissivity functions. The SBCM uses, in addition to the series, parallel and replacement rules, a relation called the break-collapse equation.

The break-collapse equation reduces the calculation of the spin correlation functions for a Potts model cluster with graph G to that for two smaller graphs obtained from G by deleting and contracting a chosen edge†. An example of this is given in figure 5(a) where the central edge of the Wheatstone bridge L is deleted and contracted to

† These graphs were called by Tsallis and Levy (1981) the 'broken and collapsed clusters', but we prefer to follow the nomenclature of graph theory and call them the deleted and contracted graphs respectively.

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give graphs G_1 and G_2 respectively. The open circles in figure 5(*a*) (known as roots) are the locations of the spins between which the correlation function is to be calculated. The correlation functions for G_1 and G_2 can be easily computed using the series and parallel equations. Therefore one can calculate, by a simple procedure, the pair correlation function for this five-edge graph without examining the λ^4 spin configurations or alternatively the $2^5 = 32$ bond percolation configurations represented by the subgraphs of L.

A convenient set of variables with which to describe the properties of a Potts model cluster is the 'thermal transmissivities' of the bonds (Tsallis and Levy 1981) which will be denoted collectively by t. For example, the transmissivity of the bond connecting spins 1 and 2 may be defined by considering the bond in isolation from the cluster, choosing a state for spin 1 and calculating the difference in probabilities of finding spin 2 in the same state and in a particular different state. For ferromagnetic interactions this probability is equal to unity at zero temperature and decreases monotonically to zero as the temperature tends to infinity. The precise expression for the thermal transmissitivy, t_e, of the bond corresponding to edge e of G, as a function of the temperature and its interaction parameter, is given in equation (2.2). In PFI the correlation function between spins 1 and 2 was proved to have the same probabilistic property as the thermal transmissivity provided that the probabilities are calculated taking into account all cluster interactions. This led to the identification of the pair correlation function with the equivalent transmissivity $t_{12}^{\mathrm{eq}}(t, G)$ between spins 1 and 2 of a cluster with graph G, introduced by Tsallis and Levy (1981).

In an earlier version of the SBCM, known as the break-collapse method or BCM (Tsallis and Levy 1981, Tsallis 1987), edges in series or parallel were replaced by 'effective edges'. In the SBCM replacement by effective edges is also made in graphs which are 'non-reducible' by series/parallel combination. In the BCM for $t_{12}^{\mathrm{eq}}(t, G)$ presented by Tsallis and Levy (1981), it is implicitly assumed that, in *any* stage of calculation, each edge e has the thermal transmissivity t_e. This hypothesis restricts the application of the break-collapse equation to graphs which do not result from previous use of series and/or parallel equations. In order to remove this restriction, Tsallis (1987) conjectured a BCM which involves *effective edges* whose thermal transmissivities are ratios of multilinear functions of the t_e, thus extending Tsallis and Levy's conjecture (1981). It allowed the exact calculation of equivalent transmissivities for complex graphs such as, for example, the two-rooted graph shown in figure 1(*h*) of da Silva *et al* (1984) which has 35 edges and 20 independent cycles. The computing time of $t_{12}^{\mathrm{eq}}(t_x, t_y, t_z, G)$ for this graph calculated by the BCM was, for example, 200 min for $\lambda = 3$ (da Silva, private communication) on the IBM-370 (model 158; 4Mb memory) computer. Notice that it would be practically impossible to calculate it from its definition as spin trace which would involve the examination of λ^{16} configurations. This is just one example, among many others, of graphs with many independent cycles which appear frequently in real space renormalisation group calculations. In fact the BCM for a general graph has been successfully applied (Chaves *et al* 1979, de Oliveira *et al* 1980, Tsallis and Levy 1981 and references therein, Chao 1981, de Magalhães *et al* 1982, de Oliveira and Tsallis 1982, Tsallis and dos Santos 1983, Lam and Zhang 1983, da Silva *et al* 1984, Costa and Tsallis 1984, Tsallis 1987) to the calculation of critical frontiers and critical exponents by the renormalisation group procedure. It has been applied to the pure as well as the randomly bond-diluted (isotropic or anisotropic) Potts model for arbitrary and specific values of λ. It is one of the main objectives of this paper to provide a proof of Tsallis' conjecture (1987).

The calculation of the m-spin correlation function with $m \geq 3$, described in PF2, requires the introduction of a generalisation of $t_{12}^{eq}(t, G)$ to m roots, namely the partitioned m-rooted equivalent transmissivity $t_{\mathbb{P}}^{eq}(G)$, where $\mathbb{P}$ refers to any partition of the roots $1, 2, \ldots, m$ of G into blocks. It was shown that the m-spin correlation function is a linear combination of the $t_{\mathbb{P}}^{eq}(G)$ corresponding to all possible partitions of the m roots of G. Therefore, once one calculates the partitioned equivalent transmissivities, then one can easily obtain all the correlation functions and the partition function through the use of expressions derived in PF1 and PF2 (and summarised in § 2.2).

The partitioned m-rooted equivalent transmissivities were shown (PF1, PF2) to have a simple combinatorial formulation in terms of the flow polynomials of graph theory (Tutte 1954, 1984) and of their corresponding extensions for m-rooted graphs, namely, the partitioned m-rooted flow polynomials (PF2). In this paper we extend the deletion-contraction technique for flow polynomials described by Tutte (1954) from edges to subgraphs. This forms the basis of the SBCM for the above polynomials and is the starting point for our derivation of the SBCM for $t_{\mathbb{P}}^{eq}(G)$.

This paper is divided into seven sections and one appendix. In § 2 we summarise results from PF2 concerning the partition function and the multispin correlation function. In § 3, we derive properties of the flow polynomials and partitioned flow polynomials and in § 4 they are used to derive the corresponding results for the partitioned equivalent transmissivity. In § 5 we describe the computer algorithms (SBCM and the BCM) which may be applied to both flow polynomials and transmissivities; explicit illustrations of both algorithms are given. In § 6, we study the $\lambda \to 1$ limit of our results. Consideration of this limit enables the SBCM to be extended to the partitioned m-rooted connectedness function (this function contains as a particular case the usual pair connectedness of percolation theory). In § 7 we summarise our results. Finally, in the appendix, we quote the SBCM formulae in terms of the p variable of Kasteleyn and Fortuin (1969) and also give its extension to the Whitney rank function and to its generalisation, the partitioned m-rooted rank function.

2. Main results of papers I and II

In this section we summarise the formulae for the equivalent transmissivities and flow polynomials derived in PF1 and PF2 and also give expressions for the multispin correlation functions in terms of the transmissivity functions. Since the t variable has been shown to be more convenient than the p variable (see PF2), we shall restrict ourselves, throughout the main body of this paper, to the t variable. The corresponding results in the p variable will be quoted in the appendix.

2.1. *Definitions and summary of previous results*

We consider the Potts model for a graph G with vertex set V and edge set E. With each vertex i of V we associate a spin vector s_i of length s which can take on one of λ values $e_1, \ldots, e_\lambda$ which are the position vectors of the corners of a $(\lambda - 1)$-dimensional hypertetrahedron relative to its centre. The Hamiltonian of the model is

$$\mathscr{H}(G) = -\sum_{e \in E} J_e s_i \cdot s_j \tag{2.1}$$

where J_e is a given interaction parameter for the edge e. The 'thermal transmissivity' of the edge e, as defined in the introduction, is

$$t_e = \frac{1-\exp(-\lambda J_e/k_B T)}{1+(\lambda-1)\exp(-\lambda J_e/k_B T)}. \tag{2.2}$$

The corresponding bond percolation model on the graph G is one in which the edge e has probability t_e of being present. Let G' be a partial graph of G, i.e. a subgraph of G having the same vertex set and the subset E' of edges. If $Q(G')$ is a function defined for each G' its expected value is given by:

$$\langle Q\rangle_{G,t} = \sum_{G'\subseteq G} Q(G') \prod_{e\in E'} t_e \prod_{e\in E\setminus E'} (1-t_e) \tag{2.3}$$

and will be known as the 'percolation average' of Q.

Now suppose that m of the vertices of G are designated as roots and labelled $1, 2, \ldots, m$ and let $\mathbb{P}$ be a given partition of these roots into blocks. The partitioned equivalent transmissivity is defined by:

$$t^{\mathrm{eq}}_{\mathbb{P}}(t, G) = N_{\mathbb{P}}(t, G)/D(t, G) \tag{2.4}$$

with

$$D(t, G) \equiv \langle \lambda^{c}\rangle_{G,t} \tag{2.5}$$

and

$$N_{\mathbb{P}}(t, G) \equiv \langle \lambda^{c}\gamma_{\mathbb{P}}\rangle_{G,t} \tag{2.6a}$$

where

$$\gamma_{\mathbb{P}}(G') = \begin{cases} 1 & \text{if roots in the same block of the partition } \mathbb{P} \text{ are connected among themselves in } G' \text{ and if roots of different blocks are not connected} \\ 0 & \text{otherwise} \end{cases} \tag{2.6b}$$

and $c(G')$ is the number of independent cycles in the subgraph G'. When $\gamma_{\mathbb{P}}(G') = 1$ the roots are said to be $\mathbb{P}$-partitioned by G'. We write $\mathbb{P} = \{B_1, B_2, \ldots, B_b\}$ and the block B_i will be said to have l_i roots of type i. For example in figure 4(c) where $\mathbb{P} = \{\{1, 2\}; \{3, 4\}; \{5, 6, 7, 8, 9\}\}$, the roots of type 1, 2 and 3 are represented by squares, triangles and circles respectively. In the case $m = 2$ and $\mathbb{P}$ has a single block $\{1, 2\}$ it was shown (PF1) that this definition of $t^{\mathrm{eq}}_{\mathbb{P}}(t, G)$ agrees with that of Tsallis and Levy (1981) discussed in the introduction.

It was shown in PF1 that $D(t, G)$ is a multilinear form in the t_e variables:

$$D(t, G) = \sum_{G'\subseteq G} F(\lambda, G') \prod_{e\in E'} t_e \tag{2.7}$$

where $F(\lambda, G)$ is the flow polynomial of G given by

$$F(\lambda, G) = \sum_{G'\subseteq G} (-1)^{|E\setminus E'|}\lambda^{c(G')}. \tag{2.8}$$

Notice that if $t_e = t$ for all edges e of E then $D(t, G)$ becomes a polynomial in t.

$F(\lambda, G)$ has the physical interpretation of being the number of proper integer mod-λ flows on G. In PF1 the definition of such flows was given and illustrated. Briefly, G is given an arbitrary directing and a flow with integer value in the range 1 to $\lambda - 1$ is assigned to each edge subject to the condition that the signed sum of the flows at any vertex must be zero mod λ. This immediately implies that F is zero if G has any vertex of degree one. In the simplest non-trivial case of a polygonal graph $F(\lambda, G) = \lambda - 1$ and in general F is a topological invariant. In the case of the Ising model ($\lambda = 2$) the flow in any edge must be unity and the vertex condition can only be satisfied if the vertex has even degree. Thus $F(2, G) = 1$ if G has all even vertices but is otherwise zero. In this case (2.7) is the usual hyperbolic tangent expansion.

The corresponding multilinear form of $N_{\mathbb{P}}(t, G)$ is (PF2):

$$N_{\mathbb{P}}(t, G) = \sum_{G' \subseteq G} F_{\mathbb{P}}(\lambda, G') \prod_{e \in E'} t_e \tag{2.9}$$

where the partitioned m-rooted flow polynomial is

$$F_{\mathbb{P}}(\lambda, G) = \sum_{G' \subseteq G} (-1)^{|E \setminus E'|} \lambda^{c(G')} \gamma_{\mathbb{P}}(G'). \tag{2.10}$$

In general we have been unable to interpret the rooted flow polynomial in terms of mod-λ flows, although in PF2 it was shown to have most of the properties of $F(\lambda, G)$. However in the case when there are only two roots of the same type (i and j, say) it is related to $F(\lambda, G)$ by

$$F_{ij}(\lambda, G) = F(\lambda, G \cup e)/(\lambda - 1) \tag{2.11}$$

where e is an additional edge connecting i and j. In this case it can be interpreted as the number of proper integer mod-λ flows when a fixed non-zero external flow is introduced at i and removed at j. The simplest example of a partitioned flow polynomial with $\mathbb{P}$ having more than one block is $F_{i,j}(\lambda, G)$. It follows immediately from the definitions (2.8) and (2.10), using $\gamma_{i,j}(G') + \gamma_{ij}(G') = 1$, that

$$F_{i,j}(\lambda, G) = F(\lambda, G) - F_{ij}(\lambda, G). \tag{2.12}$$

2.2. *The partition function and correlation functions*

The formulae in this section summarise the expressions developed in PF1 and PF2 relating the partition function and correlation functions to the transmissivity functions defined above. They are included for easy reference and may be skipped without loss of continuity.

In PF1 the partition function $Z(t, G)$ of a Potts cluster was related to the denominator of the transmissivity functions by:

$$Z(t, G) = \lambda^{|V| - |E|} \left(\prod_{e \in E} \{\exp[(\lambda - 1)K_e] + (\lambda - 1)\exp(-K_e)\} \right) D(t, G) \tag{2.13}$$

where the spin vectors have length s such that $s^2 = \lambda - 1$.

In PF2 the correlation function $\Gamma_{12\ldots m}(t, G)$ among the components $s_{11}, s_{21}, \ldots, s_{m1}$ of the m spins $s_1, s_2, \ldots, s_m$ along one of the λ special directions, say e_1, was related to the partitioned equivalent transmissivities $t_{\mathbb{P}}^{\mathrm{eq}}(t, G)$ through

$$\begin{aligned} \Gamma_{12\ldots m}(t, G) &\equiv \langle s_{11} s_{21} \ldots s_{m1} \rangle_G^{\mathrm{T}} \\ &= (\lambda - 1)^{-m/2} \sum_{\mathbb{P} \in \bar{\mathscr{P}}(M)} t_{\mathbb{P}}^{\mathrm{eq}}(t, G) F(\lambda, I_{\mathbb{P}}) \end{aligned} \tag{2.14}$$

where

$$s_{i1} = \mathbf{s}_i \cdot \mathbf{e}_1/|\mathbf{e}_1| \tag{2.15}$$

and $\langle \ldots \rangle_G^T$ means a thermal average. In (2.14) the sum is over the set $\tilde{\mathscr{P}}(M)$ of all partitions $\mathbb{P}$ of the set $M = \{1, 2, \ldots, m\}$ of roots of G into blocks which contain at least two roots. $F(\lambda, I_{\mathbb{P}})$ is the flow polynomial of the 'interface graph' $I_{\mathbb{P}}$ constructed from the partition $\mathbb{P}$ as follows: for $i = 1, \ldots, b$ associate with the block B_i a vertex v_i and connect it to an 'external' vertex u by an edge of multiplicity l_i (hence $I_{\mathbb{P}}$ has $b+1$ vertices and m edges). $F(\lambda, I_{\mathbb{P}})$ is given by

$$F(\lambda, I_{\mathbb{P}}) = \prod_{B_i \in \mathbb{P}} \frac{(\lambda - 1)}{\lambda} [(\lambda - 1)^{l_i - 1} + (-1)^{l_i}]. \tag{2.16}$$

We shall develop in the subsequent sections, SBCM for $D(t, G)$, $N_{\mathbb{P}}(t, G)$ and $t_{\mathbb{P}}^{\mathrm{eq}}(t, G)$ which provide a powerful technique for evaluating the above expressions for $Z(t, G)$ and $\Gamma_{12\ldots m}(t, G)$. We shall also discuss the BCM for $t_{\mathbb{P}}^{\mathrm{eq}}(t, G)$.

3. Properties of $F(\lambda, G)$ and $F_{\mathbb{P}}(\lambda, G)$

We now extend the deletion-contraction rules for $F_{\mathbb{P}}(\lambda, G)$ (see equation (4.14) of PF2) and $F(\lambda, G)$ (see equation (4.8) of PF1) to the cases where, instead of a single edge e, we consider a subgraph L of G which has only two vertices in common with the remainder of G. As we will see in the next section, these extensions form the heart of the proof of the SBCM for $N_{\mathbb{P}}(t, G)$ and $D(t, G)$. We also consider the combination of graphs in parallel and in series and the factorisation of partitioned flow polynomials for articulated graphs. The formulae obtained will be used in the next section to obtain the corresponding results for equivalent transmissivities and, in § 5, as the basis of an algorithm for flow polynomials. In addition to the graph theoretic interest in flow polynomials, they are also used in deriving the coefficients in series expansions for lattice problems.

In this section we shall assume, unless otherwise stated to the contrary, that G is a two-reducible m-rooted graph (Essam 1970) which is the union of two subgraphs L and H subject to the following conditions: (i) they intersect only at the vertices i and j (there are no edges in common); (ii) one of them, say H, contains all of the m roots of G. The vertices i and/or j may be rooted or not (see figure 1).

3.1. *The subgraph break-collapse equation for $F(\lambda, G)$*

Before embarking on the general analysis of the partitioned flow polynomials we present a simple intuitive derivation of the subgraph break-collapse equation (SBCE) in the unrooted case which is illustrated schematically in figure 2. Let Φ_{net} be the value of the net flow from L to H at the intersection vertex i (which, by conservation of 'fluid' mod-λ, must be equal to the value of the net flow mod-λ from H to L at j). The proper mod-λ flows in $G = H \cup L$ may be partitioned into two sets: (a) proper mod-λ flows in which $\Phi_{\mathrm{net}} = 0$ and (b) proper mod-λ flows in which $\Phi_{\mathrm{net}} \neq 0$. The proper flows on G which satisfy condition (a) may be counted by combining any proper flow in H with any proper flow in L and hence the total number of such flows will simply be $F(\lambda, H)F(\lambda, L)$ (this is illustrated by the pair of graphs just after the equality sign in figure 2). The proper flows in G subject to condition (b) may be

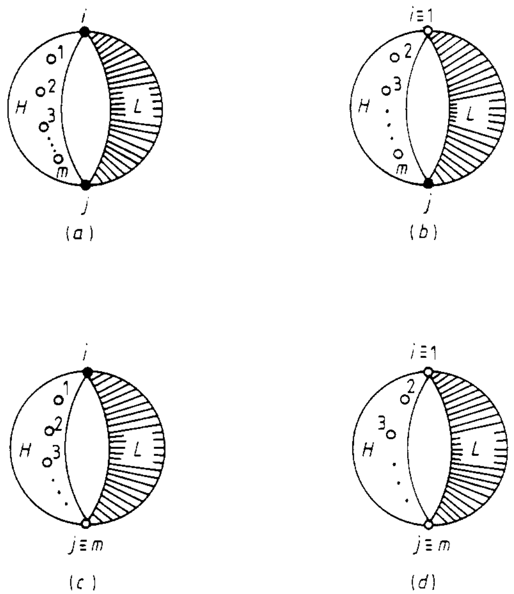

Figure 1. Pictorial representations of two-reducible m-rooted graphs $G = L \cup H$, where the intersection vertices i and j can be rooted or not. The roots $1, 2, \ldots, m$ are represented by small circles and unrooted vertices by full dots; each subgraph is represented by a half-moon shape.

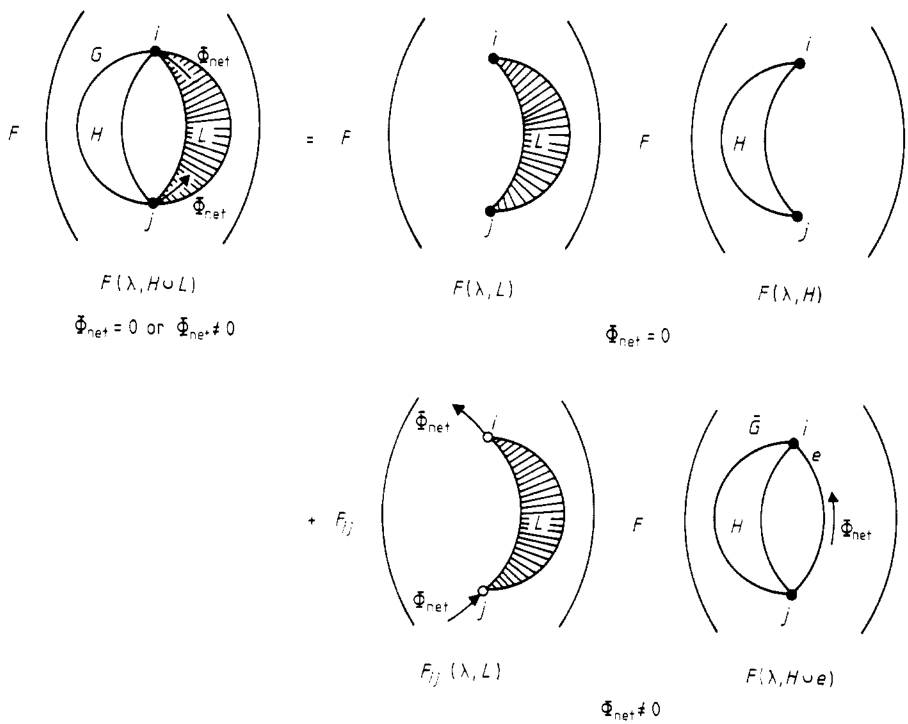

Figure 2. Pictorial representation of equation (3.1). Φ_{net} is the value of the net mod-λ flow from L to H at i which is equal to the net mod-λ flow from H to L at j.

counted by considering the proper flows on the graph $\bar{G}$ obtained from G by replacing L by a single edge e. Any proper mod-λ flow on $\bar{G}$ can be combined with any proper mod-λ flow on L, which is subject to a non-zero external flow in at j and out at i, thereby giving a proper mod-λ flow on G. In § 2.1 it was mentioned that $F_{ij}(\lambda, L)$ could be interpreted as the number of proper mod-λ flows on L with a fixed external flow. Consequently the total number of proper mod-λ flows on G with $\Phi_{\text{net}} \neq 0$ is given by $F(\lambda, \bar{G})F_{ij}(\lambda, L)$ (see the last pair of graphs of figure 2). The total number of flows is obtained by summing the contributions from flows of type (a) and (b) and hence, writing $\bar{G} = H \cup e$:

$$F(\lambda, G) = F(\lambda, L)F(\lambda, H) + F_{ij}(\lambda, L)F(\lambda, H \cup e). \tag{3.1}$$

From the deletion-contraction rule for a single edge (see equation (4.8) of PF1 with $G = H \cup e$):

$$F(\lambda, H \cup e) = F(\lambda, H_{i=j}) - F(\lambda, H) \tag{3.2}$$

and therefore

$$F(\lambda, G) = [F(\lambda, L) - F_{ij}(\lambda, L)]F(\lambda, H) + F_{ij}(\lambda, L)F(\lambda, H_{i=j}) \tag{3.3}$$

where $H_{i=j}$ is the *bicollapsed graph* obtained from H by identifying vertices i and j. This is the 'subgraph break-collapse equation' (SBCE) for $F(\lambda, G)$.

3.2. *The subgraph break-collapse equation for rooted flow polynomials*

Since we have no interpretation for $F_{\mathbb{P}}(\lambda, G)$ in terms of counting flows, the argument of the previous section cannot be extended to the general partitioned flow polynomial. Instead we work directly from the definition (2.10). We shall examine how the quantities $\gamma_{\mathbb{P}}(G')$, $c(G')$ and $|E'|$ relate to the corresponding ones in the partial graphs L' of L and H' of H. We shall suppose for the moment that i and j are not rooted, as shown in figure 1(a).

First let us see what is the relationship between $\gamma_{\mathbb{P}}(G')$ and $\gamma_{\mathbb{P}}(H')$. If *there is no path from i to j on L'* (i.e. $\gamma_{i,j}(L') = 1$) then in order that the m roots are $\mathbb{P}$ partitioned by G' (i.e. $\gamma_{\mathbb{P}}(G') = 1$) they must also be $\mathbb{P}$ partitioned by H' (i.e. $\gamma_{\mathbb{P}}(H') = 1$) since no root of H' can be connected to another root of H' via a path on L' (see the first parenthesis on the right-hand side of the equality sign of figure 3). If *there is a path between i and j on L'* (i.e. $\gamma_{ij}(L') = 1$; see the last parenthesis in figure 3) then the condition $\gamma_{\mathbb{P}}(G') = 1$ can be satisfied in the following cases: (i) $\gamma_{\mathbb{P}}(H') = 0$ or (ii) $\gamma_{\mathbb{P}}(H') = 1$. In case (i) some roots of H' must be connected to other roots of H' via paths on L' (see the penultimate graph of figure 3). In both cases (i) and (ii) the m roots are $\mathbb{P}$ partitioned by the bicollapsed graph $H'_{i=j}$. Hence, for every partial graph G' of G the following identity holds:

$$\gamma_{\mathbb{P}}(G') = \gamma_{i,j}(L')\gamma_{\mathbb{P}}(H') + \gamma_{ij}(L')\gamma_{\mathbb{P}}(H'_{i=j}) \qquad (\forall G' = H' \cup L' \subseteq G). \tag{3.4}$$

Now let us examine how $c(G')$ relates to $c(H')$ and $c(L')$. If $\gamma_{i,j}(L') = 1$ then no new cycle can be formed when we consider the union of L' with H', i.e. $c(G')$ is just the sum of $c(L')$ and $c(H')$ (see the first parenthesis on the right-hand side of the equality sign of figure 3). If $\gamma_{ij}(L') = 1$ then we have to consider two cases, namely (a) $\gamma_{i,j}(H') = 1$ and (b) $\gamma_{ij}(H') = 1$. In case (b), the union of paths on L' and H' between i and j gives rise to an extra cycle in $G' = L' \cup H'$, i.e. $c(G') = c(L') + c(H') + 1$ (see the last graph of figure 3). In case (a) no new cycles appear and $c(G')$ is just the sum of $c(L')$ and

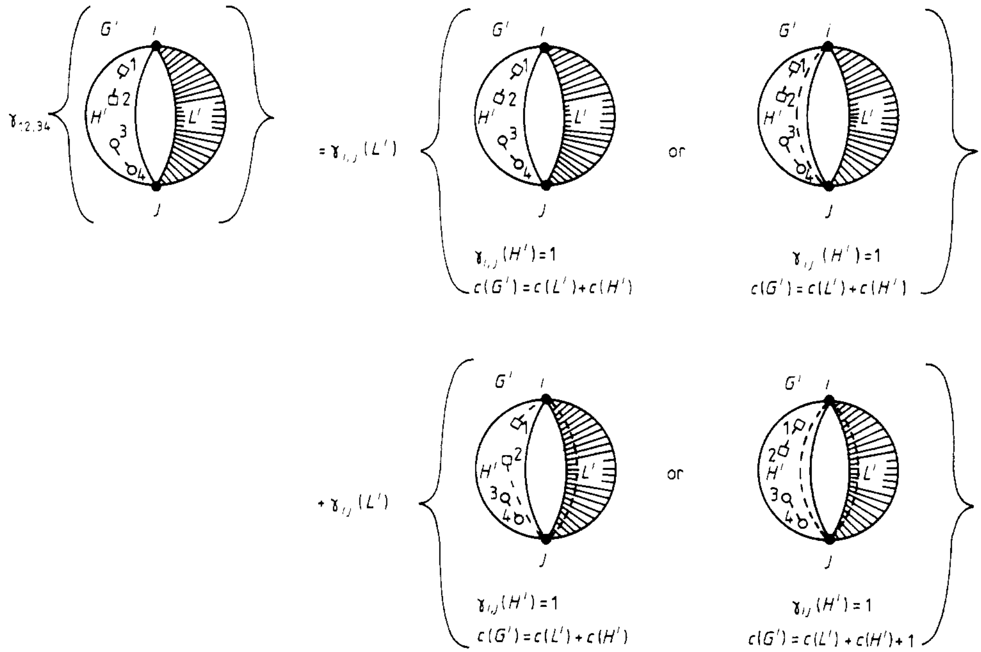

Figure 3. Illustration of equation (3.4) in the case where 1 must be connected to 2, and 3 must be connected to 4 on G', i.e. $\mathbb{P}=\{\{1,2\};\ \{3,4\}\}$. The roots of type 1 and 2 are respectively represented by small squares and circles. A broken line between any pair of vertices indicates that there is a path between these vertices.

$c(H')$ (see the penultimate graph of figure 3). It is easy to verify that in both cases (a) and (b), $c(G')$ is equal to the sum of $c(L')$ and $c(H'_{i=j})$. It follows therefore that

$$c(G')=\gamma_{i,j}(L')(c(H')+c(L'))+\gamma_{ij}(L')(c(H'_{i=j})+c(L')) \qquad (\forall G'=H'\cup L'\subseteq G). \tag{3.5}$$

Concerning the number of edges $|E'|$, since H' and L' have no edges in common and since by definition $|E(H'_{i=j})|=|E(H')|$ it follows trivially that

$$|E'|=|E(H')|+|E(L')|=|E(H'_{i=j})|+|E(L')| \qquad (\forall G'=H'\cup L'\subseteq G). \tag{3.6}$$

Combining (3.4)-(3.6) we get that

$$\begin{aligned}\gamma_{\mathbb{P}}(G')\lambda^{c(G')}(-1)^{|E\setminus E'|} &= \gamma_{i,j}(L')\lambda^{c(L')}(-1)^{|E(L)\setminus E(L')|}\gamma_{\mathbb{P}}(H')\lambda^{c(H')}(-1)^{|E(H)\setminus E(H')|}\\ &\quad+\gamma_{ij}(L')\lambda^{c(L')}(-1)^{|E(L)\setminus E(L')|}\gamma_{\mathbb{P}}(H'_{i=j})\lambda^{c(H'_{i=j})}(-1)^{|E(H_{i=j})\setminus E(H'_{i=j})|}\\ &\quad(\forall G'=H'\cup L'\subseteq G).\end{aligned} \tag{3.7}$$

Summing over all partial graphs G' of G and using (2.10) and (2.12) we obtain

$$F_{\mathbb{P}}(\lambda, G)=F_{i,j}(\lambda, L)F_{\mathbb{P}}(\lambda, H)+F_{ij}(\lambda, L)F_{\mathbb{P}}(\lambda, H_{i=j}) \tag{3.8}$$

$$=[F(\lambda, L)-F_{ij}(\lambda, L)]F_{\mathbb{P}}(\lambda, H)+F_{ij}(\lambda, L)F_{\mathbb{P}}(\lambda, H_{i=j}) \tag{3.9}$$

which is the required SBCE for $F_{\mathbb{P}}(\lambda, G)$.

Notice that the arguments used to derive the above SBCE apply also to the cases where i and/or j are roots except if i and j are roots of different types. In this latter situation, if $\gamma_{ij}(L')=1$ the condition $\gamma_{\mathbb{P}}(G')=1$ cannot be satisfied. Nevertheless (3.4) continues to be valid since

$$\gamma_{\mathbb{P}}(H'_{i=j})=0 \qquad \text{if } i \text{ and } j \text{ are roots of different types} \tag{3.10}$$

and the right-hand side of (3.4) correctly reduces to the first product. Furthermore, from (3.10) and (2.10) it follows that

$$F_{\mathbb{P}}(\lambda, H_{i=j})=0 \qquad \text{if } i \text{ and } j \text{ are roots of different types} \tag{3.11}$$

and the right-hand side of (3.8) correctly reduces to the first product. Consequently the SBCE (3.9) applies to all the four situations in figure 1.

The SBCE for $F(\lambda, G)$, which we obtained heuristically in § 3.1, can be deduced from (3.7) by imposing that $\gamma_{\mathbb{P}}(G')=1$ (and consequently $\gamma_{\mathbb{P}}(H')=\gamma_{\mathbb{P}}(H'_{i=j})=1$) for every graph $G'\subseteq G$ and using (2.8).

In the construction of $H_{i=j}$, the collapse of i and j leads to: (i) a root if at least one of them is a root, (ii) an unrooted vertex if both i and j are unrooted. Observe that when i and j are the only roots of G, and P contains a single block, then

$$F_{ij}(\lambda, H_{i=j})=F(\lambda, H_{i=j}). \tag{3.12}$$

When L is a single edge e between i and j (3.9) becomes:

$$F_{\mathbb{P}}(\lambda, H\cup e)=F_{\mathbb{P}}(\lambda, H_{i=j})-F_{\mathbb{P}}(\lambda, H) \tag{3.13}$$

which can be written equivalently in terms of $G=H\cup e$ as

$$F_{\mathbb{P}}(\lambda, G)=F_{\mathbb{P}}(\lambda, G_e^{\gamma})-F_{\mathbb{P}}(\lambda, G_e^{\delta}) \tag{3.14}$$

where G_e^{γ} and G_e^{δ} are the respective graphs obtained from G by contracting and deleting the edge e of G. Equation (3.14) is the deletion-contraction rule obtained previously (see equation (4.14) of PF2).

Using (3.13), we can write (3.9) in the form

$$F_{\mathbb{P}}(\lambda, G)=F(\lambda, L)F_{\mathbb{P}}(\lambda, H)+F_{ij}(\lambda, L)F_{\mathbb{P}}(\lambda, H\cup e) \tag{3.15}$$

which is the extension of (3.1) to partitioned flow polynomials.

3.3. *Parallel combination of graphs*

In the particular case when $\mathbb{P}=\{\{i,j\}\}$ (i.e. the only roots are the intersection vertices, see figure 4(e)) we say that H and L are in parallel. From SBCE (3.9) applied to $F_{ij}(\lambda, H\cup L)$ we obtain the *parallel equation*

$$F_{ij}(\lambda, H\cup L)=F(\lambda, L)F_{ij}(\lambda, H)+F_{ij}(\lambda, L)F(\lambda, H)+(\lambda-2)F_{ij}(\lambda, L)F_{ij}(\lambda, H) \tag{3.16}$$

where we have used (3.12), (3.2) and (2.11). This equation may be obtained intuitively by an argument similar to that leading to (3.1). The first two terms correspond to the case when there is no flow between L and H (case(a)). In the first term the external flow passes through H, and in the second term it passes through L. In the third term there is a non-zero flow between H and L. This flow can take on only $\lambda-2$ values since if it were equal to the external flow the resulting flow pattern would correspond to case (a).

The corresponding equation for unrooted graphs may be obtained from (3.1), (3.2) and (2.11)

$$F(\lambda, G) = F(\lambda, L)F(\lambda, H) + (\lambda - 1)F_{ij}(\lambda, L)F_{ij}(\lambda, H) \tag{3.17}$$

which will be required in calculating the denominator of the transmissivity for parallel combinations. Again a simple interpretation is possible. The first and second terms correspond to no flow and positive flow between H and L respectively but this time there are $\lambda - 1$ values for the internal circulation.

3.4. *Factorisation rules for articulated graphs*

When two graphs G_1 and G_2 intersect at only one vertex i, then (property (ii) of $F(\lambda, G)$ in PF1):

$$F(\lambda, G_1 \cup G_2) = F(\lambda, G_1)F(\lambda, G_2). \tag{3.18}$$

The factorisation rule for $F_{\mathbb{P}}(\lambda, G_1 \cup G_2)$ depends on the distribution of the roots of $G_1 \cup G_2$. There are four cases to consider which are pictorially illustrated in figure 4.

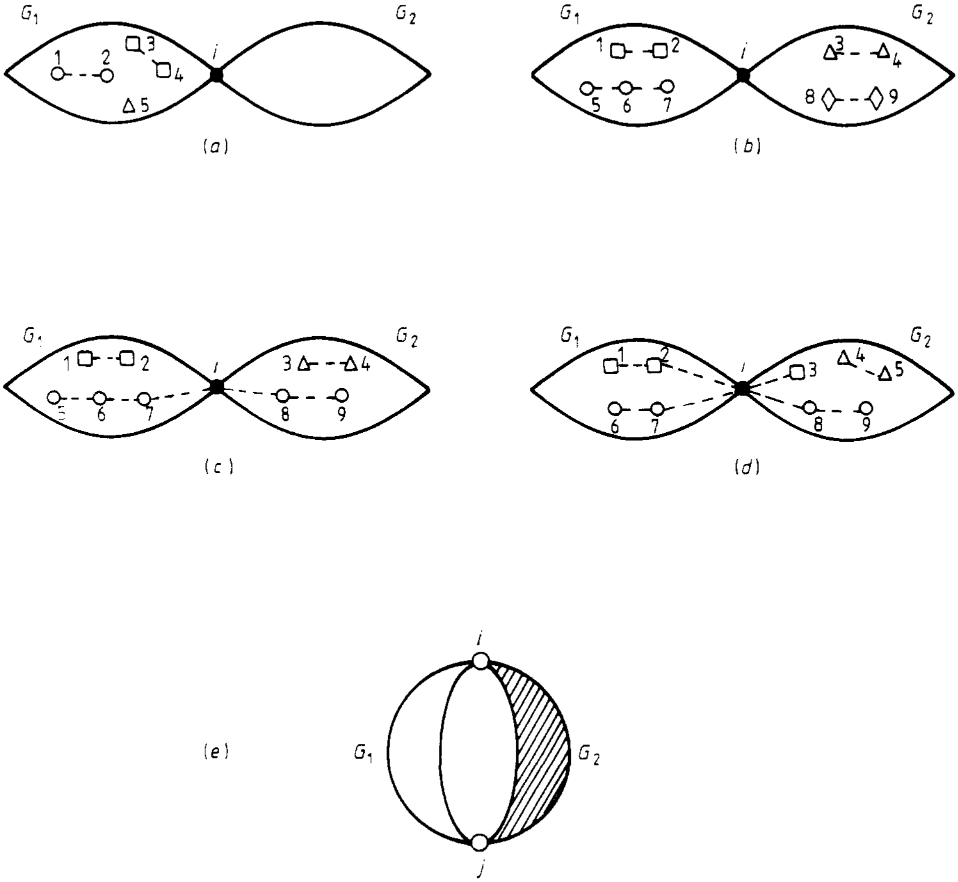

Figure 4. Pictorial representations of two graphs G_1 and G_2 which share an articulation vertex ((*a*), (*b*), (*c*) and (*d*)) and which are in parallel (*e*). Roots of the same type are represented by the same symbol (○ or △ or □ or ◇). In case (*e*), the only roots of G_1 and G_2 are i and j. Roots of the same type are connected by a broken line.

Figure 4(*a*). There are no roots in G_2 except possibly *i*. This case is covered by property (v) of PF2 and

$$F_{\mathbb{P}}(\lambda, G_1 \cup G_2) = F_{\mathbb{P}}(\lambda, G_1) F(\lambda, G_2). \tag{3.18a}$$

Figure 4(*b*). There are roots in both G_1 and G_2 but each block of $\mathbb{P}$ is contained within either G_1 or G_2. If *i* is a root it is again possible to factorise $F_{\mathbb{P}}$ and

$$F_{\mathbb{P}}(\lambda, G_1 \cup G_2) = F_{\mathbb{P}'}(\lambda, G_1) F_{\mathbb{P}''}(\lambda, G_2) \tag{3.18b}$$

where $\mathbb{P}'$ and $\mathbb{P}''$ are the restrictions of $\mathbb{P}$ to G_1 and G_2 respectively. If *i* is not a root the combination rule is not a simple factorisation.

Figure 4(*c*). Exactly one block of $\mathbb{P}$ contains roots in both G_1 and G_2. The equation is the same as (3.18*b*) except that if *i* is not a root then it must be converted into a root of the same type as the common block before calculating $\mathbb{P}'$ and $\mathbb{P}''$.

Figure 4(*d*). If more than one block of $\mathbb{P}$ contains roots in both G_1 and G_2 then $\gamma_{\mathbb{P}}(G')$ is zero for all partial graphs of $G_1 \cup G_2$ and so $F_{\mathbb{P}}(\lambda, G_1 \cup G_2) = 0$.

3.5. Series combination of graphs

A two-rooted graph with roots 1 and 2 is said to be a series combination if it is the union of two graphs G_1 and G_2 which have only one *non-rooted vertex i* in common and roots 1 and 2 belong to G_1 and G_2 respectively. Since this is a special case of figure 4(*c*) above we have

$$F_{12}(\lambda, G_1 \cup G_2) = F_{1i}(\lambda, G_1) F_{i2}(\lambda, G_2). \tag{3.19}$$

In terms of counting flows the result follows since the same external flow passes through G_1 and G_2 and the two factors count the number of internal flows for each graph consistent with the given external flow.

4. Properties of $N_{\mathbb{P}}(t, G)$, $D(t, G)$ and $t_{\mathbb{P}}^{\mathrm{eq}}(t, G)$

The equations derived in § 3 for $F(\lambda, G')$ and $F_{\mathbb{P}}(\lambda, G')$ will now be used to deduce the corresponding equations for their respective generating functions, i.e. the denominator $D(t, G)$ and the numerator $N_{\mathbb{P}}(t, G)$ of $t_{\mathbb{P}}^{\mathrm{eq}}(t, G)$. The general strategy is to apply the results of § 3 to the subgraphs in (2.7) and (2.9). The validity of applying the formulae derived for G to its subgraphs G' will only be discussed at its first occurrence. The subgraph break-collapse equation for $t_{\mathbb{P}}^{\mathrm{eq}}(t, G)$ is used to introduce the concept of an effective edge. Series/parallel equations are formulated for effective edges as well as an 'effective break-collapse equation'. These will be used in the algorithms of the following section.

4.1. The subgraph break-collapse equation

We begin by deducing the SBCE for $N_{\mathbb{P}}(t, G)$ from the result (3.9) for $F_{\mathbb{P}}(\lambda, G)$. Since a subgraph $G'' = L'' \cup H''$ of $G' = H' \cup L'$ is also a subgraph of $G = H \cup L$, it follows that (3.7) holds also for all $G'' \subseteq G'$. Consequently (3.9) remains true for any subgraph G' of G, namely

$$F_{\mathbb{P}}(\lambda, G') = [F(\lambda, L') - F_{ij}(\lambda, L')] F_{\mathbb{P}}(\lambda, H') + F_{ij}(\lambda, L') F_{\mathbb{P}}(\lambda, H'_{i=j}). \tag{4.1}$$

Combining (2.9) and (4.1) we get the following SBCE for $N_{\mathbb{P}}(t, G)$:

$$N_{\mathbb{P}}(t, G) = [D(t, L) - N_{ij}(t, L)] N_{\mathbb{P}}(t, H) + N_{ij}(t, L) N_{\mathbb{P}}(t, H_{i=j}). \tag{4.2}$$

Similarly (2.7) combined with (3.3) applied to G' gives

$$D(t, G) = [D(t, L) - N_{ij}(t, L)] D(t, H) + N_{ij}(t, L) D(t, H_{i=j}). \tag{4.3}$$

Notice that when i and j are roots of different types then (3.11) holds for all $H'_{i=j} \subseteq H_{i=j}$. Therefore using (2.9)

$$N_{\mathbb{P}}(t, H_{i=j}) = 0 \qquad \text{if } i \text{ and } j \text{ are roots of different types.} \tag{4.4}$$

If i and j are roots of the same type and G has no other roots, then (3.12) applies for all $H'_{i=j} \subseteq H_{i=j}$ and so

$$N_{ij}(t, H_{i=j}) = D(t, H_{i=j}). \tag{4.5}$$

From (4.2), (4.3) and the definition (2.4) of $t_{\mathbb{P}}^{\text{eq}}(t, G)$ we obtain the following 'subgraph break-collapse equation' for $t_{\mathbb{P}}^{\text{eq}}(t, G)$

$$t_{\mathbb{P}}^{\text{eq}}(t, G) = \frac{[D(t, L) - N_{ij}(t, L)] N_{\mathbb{P}}(t, H) + N_{ij}(t, L) N_{\mathbb{P}}(t, H_{i=j})}{[D(t, L) - N_{ij}(t, L)] D(t, H) + N_{ij}(t, L) D(t, H_{i=j})} \tag{4.6}$$

or equivalently

$$t_{\mathbb{P}}^{\text{eq}}(t, G) = \frac{[1 - t_{ij}^{\text{eq}}(t, L)] N_{\mathbb{P}}(t, H) + t_{ij}^{\text{eq}}(t, L) N_{\mathbb{P}}(t, H_{i=j})}{[(1 - t_{ij}^{\text{eq}}(t, L)] D(t, H) + t_{ij}^{\text{eq}}(t, L) D(t, H_{i=j})} \tag{4.6'}$$

When L is a single edge e, (4.6′) reduces to

$$t_{\mathbb{P}}^{\text{eq}}(t, H \cup e) = \frac{(1 - t_e) N_{\mathbb{P}}(t, H) + t_e N_{\mathbb{P}}(t, H_{i=j})}{(1 - t_e) D(t, H) + t_e D(t, H_{i=j})} \tag{4.7}$$

or, in terms of $G = H \cup e$,

$$t_{\mathbb{P}}^{\text{eq}}(t, G) = \frac{(1 - t_e) N_{\mathbb{P}}(t, G_e^{\delta}) + t_e N_{\mathbb{P}}(t, G_e^{\gamma})}{(1 - t_e) D(t, G_e^{\delta}) + t_e D(t, G_e^{\gamma})} \tag{4.7'}$$

which recovers our previous result (equation (5.2) of PF2) and extends the BCE of Tsallis and Levy (1981) from $t_{12\ldots m}^{\text{eq}}(t, G)$ to $t_{\mathbb{P}}^{\text{eq}}(t, G)$.

4.2. Effective edges: series and parallel equations and the effective break-collapse equation

We may think of equation (4.7) as making the dependence of $t_{\mathbb{P}}^{\text{eq}}(t, G)$ on the transmissivity t_e, of edge e, explicit since the other factors on the right are independent of t_e. A comparison of (4.6′) and (4.7) shows that if G has a subgraph L which is attached at only two vertices, say i and j, and which has no internal roots, then the partitioned equivalent transmissivity of G is the same as that of the graph $H \cup e_{\text{eff}}$ obtained by replacing L by a single effective edge e_{eff} with transmissivity $t_{\text{eff}} = t_{ij}^{\text{eq}}(t, L)$. The edge e_{eff} will be said to represent the two-rooted graph L. It may be that $H \cup e_{\text{eff}}$ has a further subgraph of the same type as L in which case the replacement process may be repeated. In fact more and more effective edges may be created until one arrives at a two-irreducible graph. The actual substitution $t_{\text{eff}} = t_{ij}^{\text{eq}}(t, L)$ may only be carried out when $t_{\mathbb{P}}^{\text{eq}}(t, H \cup e_{\text{eff}})$ is known. If this is not the case, the current graph is discarded by the computer and replaced by $H \cup e_{\text{eff}}$, and the numerator N_{eff} and denominator

D_{eff} are stored along with each effective edge as part of the graph description. Usually $t_{ij}^{\text{eq}}(t, L)$ will not be known and attention is transferred to the calculation of this quantity before $H \cup e_{\text{eff}}$ is created. This latter calculation may itself be carried out by a reduction process.

In the case that L is the series or parallel combination of a pair of edges, t_{eff} may be obtained by a direct calculation for the pair of edges *in isolation* using (2.7) and (2.9) which yields (see also Domb (1974)) for series combination

$$t_{\text{eff}}^{(s)} = t_1 t_2 \tag{4.8}$$

and for parallel combination

$$t_{\text{eff}}^{(p)} = [t_1 + t_2 + (\lambda - 2) t_1 t_2]/[1 + (\lambda - 1) t_1 t_2]. \tag{4.9}$$

The replacement of a pair of edges in series or parallel by a single edge is not restricted to the original edges of G. Suppose that $e_{\text{eff}}^{(1)}$ and $e_{\text{eff}}^{(2)}$ are effective edges with thermal transmissivities $t_{\text{eff}}^{(1)} = N_1/D_1$ and $t_{\text{eff}}^{(2)} = N_2/D_2$, then another consequence of the effective edge rule is that we may make the substitutions $t_1 = N_1/D_1$ and $t_2 = N_2/D_2$ in (4.8) and (4.9), and thereby obtain the equations for combining effective edges in series or parallel. With the definitions

$$t_{\text{eff}}^{(s)} = N_{\text{eff}}^{(s)}/D_{\text{eff}}^{(s)} \qquad \text{and} \qquad t_{\text{eff}}^{(p)} = N_{\text{eff}}^{(p)}/D_{\text{eff}}^{(p)} \tag{4.10}$$

the series combination rule is

$$N_{\text{eff}}^{(s)} = N_1 N_2 \tag{4.11a}$$

$$D_{\text{eff}}^{(s)} = D_1 D_2 \tag{4.11b}$$

and the parallel combination rule is

$$N_{\text{eff}}^{(p)} = N_1 D_2 + D_1 N_2 + (\lambda - 2) N_1 N_2 \tag{4.12a}$$

$$D_{\text{eff}}^{(p)} = D_1 D_2 + (\lambda - 1) N_1 N_2. \tag{4.12b}$$

This result is at the heart of both the BCM and SBCM for $t_{\mathbb{P}}^{\text{eq}}(t, G)$, since it allows one to replace all the edges in series and/or parallel of a graph G by *effective edges*, thereby generating a graph $\tilde{G}$ which contains edges whose thermal transmissivities are the ratios of multilinear functions in t_e. When all such replacements have been made, there may still be a subgraph L of $\tilde{G}$ which is attached at only two vertices and has no internal roots. We shall call the operation of replacing such a subgraph L, *which has no edges in series and / or parallel*, by an effective edge e_{eff}, the *non-reducible subgraph replacement*. In the SBCM, the above replacements are carried out repeatedly. When $\tilde{G}$ is no longer two-reducible, one chooses an edge e_{eff} with thermal transmissivity $t_{\text{eff}} = N_{\text{eff}}/D_{\text{eff}}$, to be deleted and contracted, and applies the following *effective break-collapse equations*:

$$N_{\mathbb{P}}(t, G) = (D_{\text{eff}} - N_{\text{eff}}) N_{\mathbb{P}}(t, G^{\delta}_{e_{\text{eff}}}) + N_{\text{eff}} N_{\mathbb{P}}(t, G^{\gamma}_{e_{\text{eff}}}) \tag{4.13a}$$

$$D(t, G) = (D_{\text{eff}} - N_{\text{eff}}) D(t, G^{\delta}_{e_{\text{eff}}}) + N_{\text{eff}} D(t, G^{\gamma}_{e_{\text{eff}}}). \tag{4.13b}$$

Equations (4.13) result from (4.7′) by replacing t_e by $N_{\text{eff}}/D_{\text{eff}}$ and multiplying the numerator and the denominator by D_{eff}. When $\mathbb{P}$ has just one block, equations (4.13) reduce to the BCE conjectured by Tsallis (1987). Notice that the SBCE (4.6) is equivalent to a non-reducible subgraph replacement followed immediately by applying equations (4.13) to the effective edge created.

4.3. *Combination of graphs in parallel*

The graph we shall consider is shown in figure 4(e) and is the parallel combination of graphs G_1 and G_2. The common root points are labelled i and j. Using equations (3.16) and (3.17) applied to G' together with the relations (2.9) and (2.7) we deduce the 'parallel rule for graphs':

$$N_{ij}(t, G_1 \cup G_2) = N_{ij}(t, G_1)D(t, G_2) + N_{ij}(t, G_2)D(t, G_1) + (\lambda - 2)N_{ij}(t, G_1)N_{ij}(t, G_2) \qquad (4.14a)$$

$$D(t, G_1 \cup G_2) = D(t, G_1)D(t, G_2) + (\lambda - 1)N_{ij}(t, G_1)N_{ij}(t, G_2). \qquad (4.14b)$$

This provides an alternative derivation of equations (4.12).

From equations (4.14) it may be seen that the variables $X = D + (\lambda - 1)N$ and $Y = D - N$ satisfy a product rule. In the general case of n graphs G_α ($\alpha = 1, 2, \ldots, n$) in parallel with equivalent transmissivities N_α / D_α between i and j, repetition of this rule leads to

$$D\left(t, \bigcup_{\alpha=1}^{n} G_\alpha\right) = \lambda^{-1}[X(t) + (\lambda - 1)Y(t)] \qquad (4.15a)$$

and

$$N\left(t, \bigcup_{\alpha=1}^{n} G_\alpha\right) = D\left(t, \bigcup_{\alpha=1}^{n} G_\alpha\right) - Y(t) \qquad (4.15b)$$

where

$$X(t) = \prod_{\alpha=1}^{n} [D_\alpha + (\lambda - 1)N_\alpha] \qquad (4.15c)$$

and

$$Y(t) = \prod_{\alpha=1}^{n} [D_\alpha - N_\alpha]. \qquad (4.15d)$$

These equations are used for combining more than two effective edges in parallel.

4.4. *Factorisation rules for articulated graphs*

Here we suppose, as in § 3.4, that G_1 and G_2 have just one vertex in common. The equations of § 3.4 together with (2.7) and (2.9) give the following factorisation rules:

$$D(t, G_1 \cup G_2) = D(t, G_1)D(t, G_2) \qquad (4.16)$$

(cf property (ii) of $D(t, G)$ in PF1).

When all roots belong to G_1, as in figure 4(a), the formula for factorisation of the numerator is

$$N_{\mathbb{P}}(t, G_1 \cup G_2) = N_{\mathbb{P}}(t, G_1)D(t, G_2) \qquad (4.17a)$$

(cf property (v) of $N_{\mathbb{P}}(t, G)$ in PF2). The equations for the cases represented by figures 4(b) and 4(c) are:

$$N_{\mathbb{P}}(t, G_1 \cup G_2) = N_{\mathbb{P}'}(t, G_1)N_{\mathbb{P}''}(t, G_2) \qquad (4.17b)$$

with the condition that in figure 4(b) the articulation point must be rooted. In the case represented by figure 4(d)

$$N_{\mathbb{P}}(t, G_1 \cup G_2) = 0. \qquad (4.17c)$$

The corresponding relations for $t_{\mathbb{P}}^{\mathrm{eq}}(t, G_1 \cup G_2)$ are given by the ratios of (4.17) and (4.16).

The series combination rule for graphs is an example of the case represented by figure 4(c) and rederives (4.11) for effective edges in series. It is easily shown that these factorisation rules apply also to graphs having effective edges.

5. Computer algorithms for equivalent transmissivities and flow polynomials

The results derived in §§ 3 and 4 lead to two methods, the SBCM and the BCM, for computing partitioned flow polynomials and equivalent transmissivities. Both methods allow the calculation of multispin correlation functions (2.14) and of the partition function (2.13) without having to examine all the subgraphs G' of G which contribute to $D(t, G)$ (see (2.7)) or to $N_{\mathbb{P}}(t, G)$ (see (2.9)). Similarly these methods provide a way of calculating $F(\lambda, G)$ and $F_{\mathbb{P}}(\lambda, G)$ without the examination of the subgraphs G' appearing in (2.8) and (2.10) respectively. The algorithms will be presented with reference to the transmissivities, but the extension to flow polynomials is immediate by replacing $D(t, G)$ and $N_{\mathbb{P}}(t, G)$ by $F(\lambda, G)$ and $F_{\mathbb{P}}(\lambda, G)$ respectively.

5.1. The SBCM for $t_{\mathbb{P}}^{\mathrm{eq}}(t, G)$

The SBCM for $t_{\mathbb{P}}^{\mathrm{eq}}(t, G)$ consists essentially in applying successively a combination of: (i) the factorisation rules for articulated graphs ((4.16) and (4.17)), (ii) the relations for effective edges in series (4.11) and in parallel (4.12), (iii) the non-reducible subgraph replacement, and (iv) the effective BCE (4.13). We can compute $t_{\mathbb{P}}^{\mathrm{eq}}(t, G)$ for a given m-rooted graph G and a given partition $\mathbb{P} = \{B_1, B_2, \ldots, B_b\}$ of the roots using a recursive language (e.g. PL1 or PASCAL) and a recursive procedure $T(G, \mathbb{P}, N, D)$. In this procedure we assume that all the graphs are decorated, i.e. to each edge $e = [i, j]$ we associate a pair (N_e, D_e) where $N_e \equiv N_{ij}(t, e)$ and $D_e \equiv D(t, e)$ are the numerator and denominator of the effective thermal transmissivity $t_{\mathrm{eff}} = t_{ij}^{\mathrm{eq}}(t, e)$. In general, for a 'non-effective' edge e this pair is just $(t_e, 1)$, but for an effective edge both N_e and D_e are multilinear functions of the t_e whose coefficients are polynomials in λ. In particular applications it may be unnecessary to have a separate t variable for each edge, and a special value of λ may be substituted. In this case N_e and D_e become polynomials in one or more variables and require less storage space.

The inputs of $T(G, \mathbb{P}, N, D)$ are the above 'decorated' graph G and the partition $\mathbb{P}$, and the outputs are the numerator N and denominator D of $t_{\mathbb{P}}^{\mathrm{eq}}(t, G)$. When $\mathbb{P}$ is the null partition $\mathbb{P}_0$ (i.e. when there are no roots at all) this procedure calculates only D and makes $N = D$ for reasons which we shall see later. The main steps of the recursive procedure $T(G, \mathbb{P}, N, D)$ are the following.

Step I. Splitting into pieces.
The object of this part is to partition G into subgraphs $G_1, \ldots, G_r$ in such a way that the factorisation rules of § 4.4 may be used. In the case when $\mathbb{P}$ has more than one block the procedure is complicated by the possible occurrence of a situation represented by figure 4(b) with i unrooted. The procedure is as follows. Find the l articulation points $a_1, a_2, \ldots, a_l$ of G, and identify the pieces into which G is separated by these points (using, for example, the algorithm described by Tucker (1980)). Iteratively split off all pieces having no roots and exactly one articulation point of G. Label these pieces $G_1, \ldots, G_q$ and set the corresponding partitions $\mathbb{P}_1, \ldots, \mathbb{P}_q$ equal to $\mathbb{P}_0$. Con-

sidering the remainder of G, whenever an articulation point a_j belongs to any path connecting roots of a type i ($i = 1, 2, \ldots, b$) we transform a_j into a root of type i. If any a_j becomes a root of two or more types then (4.17c) holds and we need to calculate only $D(t, G)$. In this case we set KEY = 0, ignore all roots of G, label the remaining pieces $G_{q+1}, \ldots, G_r$ and make $\mathbb{P}_k = \mathbb{P}_0$ for $k = q+1, \ldots, r$. Otherwise set KEY = 1 and split the remainder of G into composite pieces $G_{q+1}, \ldots, G_r$ by separation at the *rooted* articulation points only. For $k = q+1, \ldots, r$, the partition $\mathbb{P}_k$ is defined by the blocks of roots of the same type which belong to G_k. If $\mathbb{P}_k$ has one unique block with a single root then make $\mathbb{P}_k = \mathbb{P}_0$. For example, the respective partitions $\mathbb{P}_1$ and $\mathbb{P}_2$ of the roots of G_1 and G_2 in the cases shown in figure 4 are as follows.

Figure 4(a). $\mathbb{P}_1 = \{\{1,2\}; \{3,4\}; \{5\}\}$; $\mathbb{P}_2 = \mathbb{P}_0$.

Figure 4(b). (i) i is a root represented by a circle: $\mathbb{P}_1 = \{\{1,2\}; \{5,6,7,i\}\}$; $\mathbb{P}_2 = \{\{i\}; \{3,4\}; \{8,9\}\}$.

(ii) i is not a root. In this case the graph must be treated as a composite piece with corresponding partition $\mathbb{P}$ given by $\mathbb{P} = \{\{1,2\}; \{5,6,7\}; \{3,4\}; \{8,9\}\}$.

Figure 4(c). $\mathbb{P}_1 = \{\{1,2\}; \{5,6,7,i\}\}$; $\mathbb{P}_2 = \{\{3,4\}; \{i,8,9\}\}$.

Figure 4(d). $\mathbb{P}_1 = \mathbb{P}_0$; $\mathbb{P}_2 = \mathbb{P}_0$.

Step II. Calculation of N_k and D_k.
For each composite piece G_k do the following.
While $|V(G_k)| > 2$ and (a), (b) or (c), described below, is possible, do the first one which is possible.

(a) Edges in series.
Replace two effective edges $e_{\text{eff}}^{(1)}$ and $e_{\text{eff}}^{(2)}$ in series with respective effective transmissivities N_1/D_1 and N_2/D_2 by a single edge with effective transmissivity given by equations (4.11).

(b) Edges in parallel.
Replace two effective edges $e_{\text{eff}}^{(1)}$ and $e_{\text{eff}}^{(2)}$ in parallel with respective effective transmissivities N_1/D_1 and N_2/D_2 by a single edge with effective transmissivity given by equations (4.12).

(c) Non-reducible subgraph replacement.
(c1) Look for an articulation pair $\{i,j\}$, breaking G_k into the subgraphs L_k and H_k such that $H_k \cap L_k = \{i,j\}$, $H_k \cup L_k = G_k$, all the roots belong to H_k (except possibly i and/or j) and the numbers of edges in H_k and L_k are as nearly equal as possible.
(c2) Call $T(L_k, \{\{i,j\}\}, NL_k, DL_k)$.
(c3) Construct $\bar{G}_k$ obtained from G_k by replacing L_k by a single edge e_{eff} with effective transmissivity NL_k/DL_k. Replace G_k by $\bar{G}_k$.

(d) Break-collapse.
If $|V(G_k)| > 2$ then do the following, else do (e).
(d1) Look for one edge $e = [i,j]$ of $E(G_k)$ where the sum of the number of edges incident with i and those incident with j is the maximum possible. The thermal transmissivity of e is N_e/D_e.
(d2) Construct G_k^δ from G_k by deleting the edge e and call $T(G_k^\delta, \mathbb{P}_k, N_k^\delta, D_k^\delta)$.
(d3) Construct G_k^γ and $\mathbb{P}_k^\gamma$ from G_k^δ and $\mathbb{P}_k$ respectively by identifying the vertices i and j. If $\mathbb{P}_k^\gamma$ has a single root or if i and j are roots of different types then set $\mathbb{P}_k^\gamma = \mathbb{P}_0$. Call $T(G_k^\gamma, \mathbb{P}_k^\gamma, N_k^\gamma, D_k^\gamma)$.
(d4) Check if i and j are roots of different types. If so set $N_k^\gamma = 0$ (cf (4.4)).
(d5) Set (see (4.13)) $D_k = (D_e - N_e)D_k^\delta + N_e D_k^\gamma$ and $N_k = (D_e - N_e)N_k^\delta + N_e N_k^\gamma$

if $\mathbb{P}_k \neq \mathbb{P}_0$, else $N_k = D_k$.

(e) Terminal conditions.

Check if G_k is in a terminal condition, i.e. we know N_k and D_k explicitly. This happens when G_k consists of n $(n \geq 1)$ edges e_α $(\alpha = 1, 2, \ldots, n)$ in parallel with effective thermal transmissivities N_α / D_α. Set D_k equal to the right-hand side of (4.15a) and N_k equal to the right-hand side of (4.15b) or (4.15d) according to the respective cases where the two vertices of G_k are roots of the same type or not.

Step III. Calculation of N and D.

After computing N_k and D_k for all values of k $(k = 1, 2, \ldots, r)$, set

$$D = \prod_{k=1}^{r} D_k \tag{5.1a}$$

and

$$N = \prod_{k=1}^{r} N_k \quad \text{if } \mathbb{P} \neq \mathbb{P}_0 \tag{5.1b}$$

else

$$N = 0 \qquad \text{if KEY} = 0 \tag{5.1c}$$

or

$$N = D \qquad \text{if KEY} = 1. \tag{5.1d}$$

Notice that (5.1b) is true only because the procedure sets $N_k = D_k$ when $\mathbb{P}_k = \mathbb{P}_0$; otherwise we would have (cf (4.17a)) to replace N_k by D_k whenever $\mathbb{P}_k$ was equal to the null partition.

It is worth emphasising that a similar SBCM holds for the calculation of $F_{\mathbb{P}}(\lambda, G)$ and $F(\lambda, G)$. In this case $F_{\mathbb{P}}(\lambda, G)$ and $F(\lambda, G)$ play the same role as $N_{\mathbb{P}}(t, G)$ and $D(t, G)$, and to each edge e we associate the pair $(F_{ij}(\lambda, e), F(\lambda, e))$ instead of $(N_{ij}(t, e), D(t, e))$. For a non-effective edge this pair is equal to $(1, 0)$. All the formulae remain valid provided we replace N_α and D_α by $F_{ij}^{(\alpha)}$ and $F^{(\alpha)}$, respectively.

5.2. An illustration of the SBCM for $t_{\mathbb{P}}^{\mathrm{eq}}(t, G)$

Figure 5 illustrates the SBCM described above by the calculation of $t_{12,3}^{\mathrm{eq}}(t, G)$ for a three-rooted graph G whose edges have the same thermal transmissivity $t_e = t$, $\forall e \in E$. The equivalent transmissivities which appear there are defined as follows:

$$t_r = \frac{2t^2 + (\lambda - 2)t^4}{1 + (\lambda - 1)t^4} \tag{5.2a}$$

$$t_p = \frac{2t + (\lambda - 2)t^2}{1 + (\lambda - 1)t^2} \tag{5.2b}$$

$$t_v = \frac{4t + 6(\lambda - 2)t^2 + 4(\lambda^2 - 3\lambda + 3)t^3 + (\lambda - 2)(\lambda^2 - 2\lambda + 2)t^4}{1 + 6(\lambda - 1)t^2 + 4(\lambda - 1)(\lambda - 2)t^3 + (\lambda - 1)(\lambda^2 - 3\lambda + 3)t^4} \tag{5.2c}$$

$$t_1 \equiv t_{ij}^{\mathrm{eq}}(t, L) = \frac{2t^2 + 2t^3 + 5(\lambda - 2)t^4 + (\lambda - 2)(\lambda - 3)t^5}{1 + 2(\lambda - 1)t^3 + (\lambda - 1)t^4 + (\lambda - 1)(\lambda - 2)t^5} \tag{5.2d}$$

$$t_q = \frac{3t^2 + 2t^3 + 7(\lambda - 2)t^4 + \lambda(\lambda - 1)t^5 + (5\lambda^2 - 19\lambda + 19)t^6 + (\lambda - 2)(\lambda^2 - 4\lambda + 5)t^7}{1 + 2(\lambda - 1)t^3 + 3(\lambda - 1)t^4 + \lambda(\lambda - 1)t^5 + 5(\lambda - 1)(\lambda - 2)t^6 + (\lambda - 1)(\lambda - 2)(\lambda - 3)t^7}. \tag{5.2e}$$

(*a*)

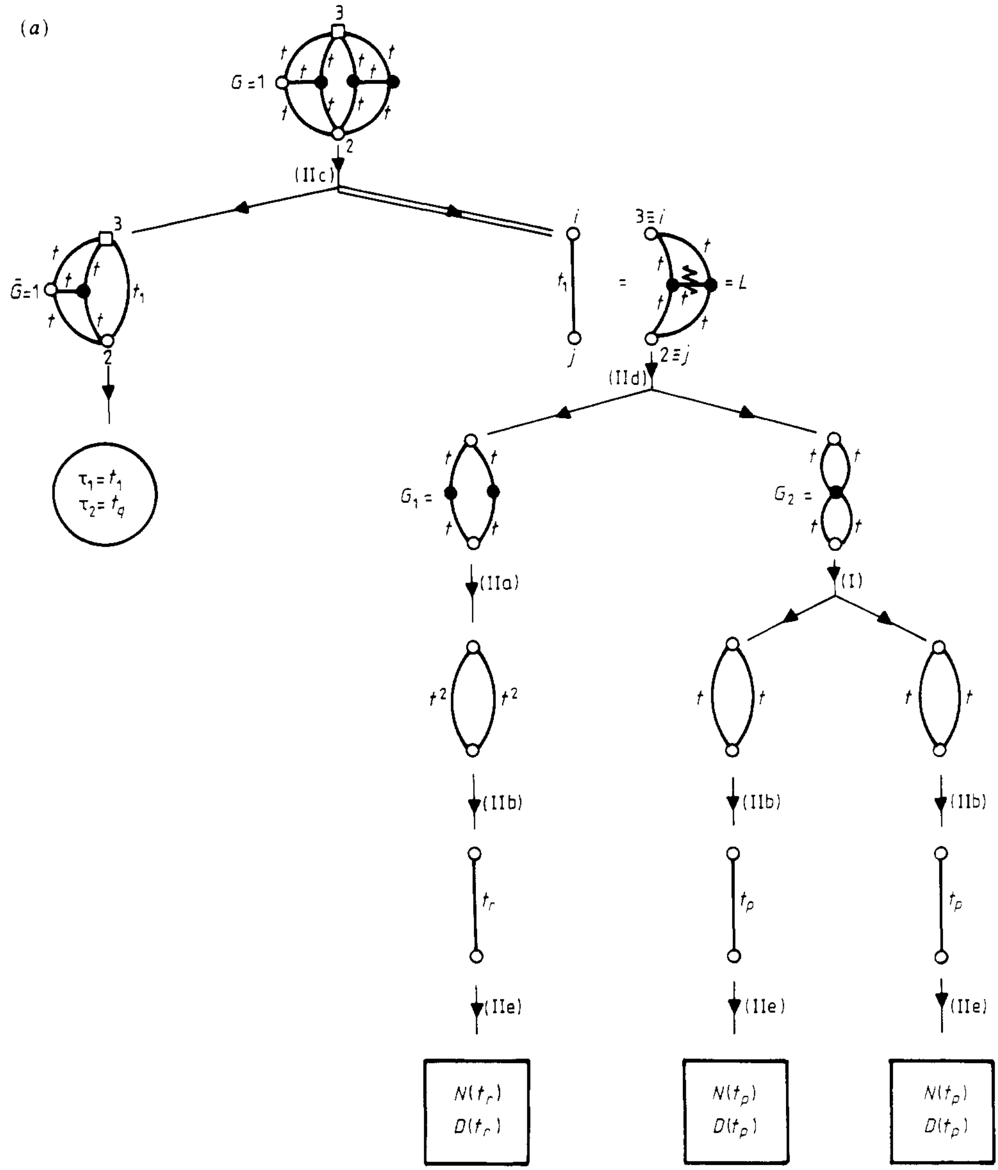

Figure 5. A schematic representation of the calculation of $t^{eq}_{12,3}(t, G)$ for the drawn graph G 'decorated' with equal thermal transmissivities t through the SBCM using the algorithm described in § 5.1. The steps used from one graph to the other are indicated in parentheses. The effective thermal transmissivities associated with their respective effective edges are also indicated. A zigzag bar indicates the edge t to be deleted and contracted. The respective polynomials at the top and bottom of a rectangle represent the numerator N and denominator D of the equivalent transmissivity of the preceding graph. Transmissivities t_r, t_p, t_v, t_1 and t_q are defined in equations (5.2). The double line (⇉) points to the subgraph replacement. The calculation of $t^{eq}_{12,3}(t, \bar{G})$, with $\bar{G}$ drawn in (*a*), is represented in (*b*) (overleaf) where $\tau_1 = t_1$ and $\tau_2 = t_q$.

(b)

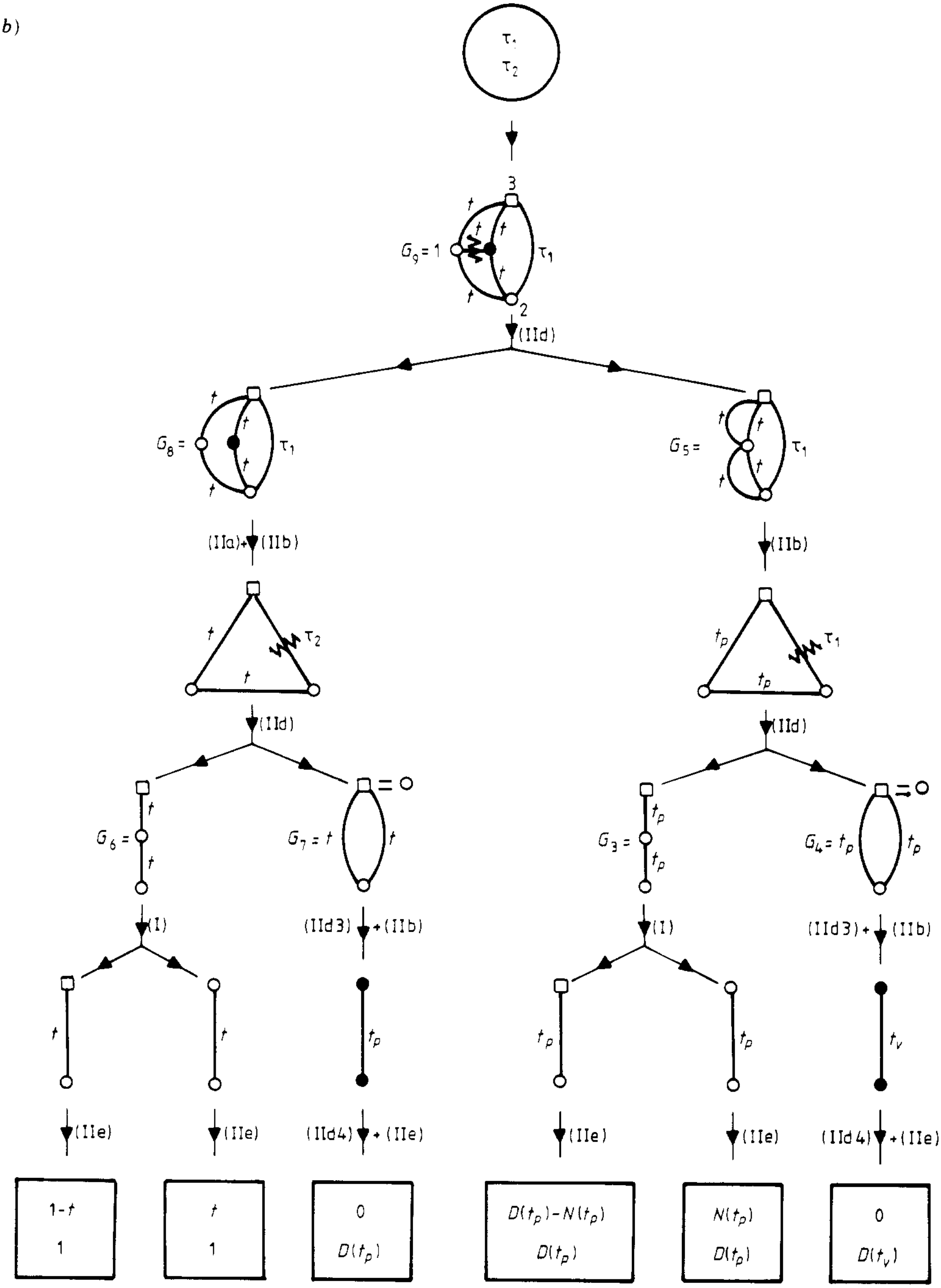

Figure 5. (continued)

The effective thermal transmissivity t_1 was calculated first in order to replace L by a single edge, thus generating the graph $\bar{G}$ which appears on the left-hand side of figure 5(a).

Combining the results of figure 5(b) (where $\tau_1 = t_1$ and $\tau_2 = t_q$) with the effective BCE (4.13) we get the following equivalent transmissivities for the graphs generated by the application of the SBCM to $t^{\mathrm{eq}}_{12,3}(t, \bar{G})$:

$$t^{\mathrm{eq}}_{12,3}(t, G_8) = \frac{\begin{array}{l} t - t^2 - 3t^3 + (2\lambda - 1)t^4 - (6\lambda - 15)t^5 \\ \quad + (4\lambda - 11)t^6 + (4\lambda - 9)t^7 - (6\lambda - 13)t^8 + 2(\lambda - 2)t^9 \end{array}}{\begin{array}{l} 1 + 2(\lambda - 1)t^3 + 6(\lambda - 1)t^4 + (\lambda - 1)(\lambda + 2)t^5 \\ \quad + 12(\lambda - 1)(\lambda - 2)t^6 + 2(\lambda - 1)(\lambda^2 - 3\lambda + 3)t^7 \\ \quad + (\lambda - 1)(5\lambda^2 - 19\lambda + 19)t^8 + (\lambda - 1)(\lambda - 2)(\lambda^2 - 4\lambda + 5)t^9 \end{array}} \tag{5.3}$$

$$t^{\mathrm{eq}}_{12,3}(t, G_5) = \frac{\begin{array}{l} 2t + (\lambda - 6)t^2 - 2(\lambda - 1)t^3 + (3\lambda + 2)t^4 \\ \quad + 2(\lambda^2 - 10\lambda + 15)t^5 + (-8\lambda^2 + 55\lambda - 82)t^6 \\ \quad + 2(6\lambda^2 - 33\lambda + 43)t^7 + (\lambda - 2)(-8\lambda + 21)t^8 + 2(\lambda - 2)^2 t^9 \end{array}}{\begin{array}{l} 1 + 2(\lambda - 1)t^2 + 2(\lambda - 1)t^3 + (\lambda^2 + 7\lambda - 8)t^4 + (\lambda - 1)(13\lambda - 14)t^5 \\ \quad + 2(\lambda - 1)(\lambda^2 + 11\lambda - 25)t^6 + 2(\lambda - 1)(15\lambda^2 - 59\lambda + 59)t^7 \\ \quad + (\lambda - 1)(9\lambda^3 - 57\lambda^2 + 122\lambda - 87)t^8 \\ \quad + (\lambda - 1)(\lambda - 2)(\lambda^3 - 6\lambda^2 + 14\lambda - 11)t^9 \end{array}}. \tag{5.4}$$

Combining (5.3) and (5.4) with (4.13) we finally get that:

$$\begin{aligned} N_{12,3}(t, G) = {} & t + (\lambda - 8)t^3 + 4t^4 - (5\lambda - 18)t^5 + 2(\lambda^2 - 5\lambda + 2)t^6 - (8\lambda^2 - 55\lambda + 80)t^7 \\ & + 4(3\lambda^2 - 19\lambda + 27)t^3 - (8\lambda^2 - 45\lambda + 59)t^9 + 2(\lambda - 2)(\lambda - 3)t^{10} \end{aligned} \tag{5.5a}$$

and

$$\begin{aligned} D(t, G) = {} & 1 + 4(\lambda - 1)t^3 + 6(\lambda - 1)t^4 + 2(\lambda^2 + \lambda - 2)t^5 + 8(\lambda - 1)(3\lambda - 5)t^6 \\ & + 4(\lambda - 1)(\lambda^2 + \lambda - 5)t^7 + (\lambda - 1)(33\lambda^2 - 131\lambda + 131)t^8 \\ & + (\lambda - 1)(10\lambda^3 - 68\lambda^2 + 154\lambda - 116)t^9 + (\lambda - 1)(\lambda - 2)^2(\lambda^2 - 5\lambda + 8)t^{10}. \end{aligned} \tag{5.5b}$$

Notice that the coefficient of $t^{|E'|}$ in $D(t, G)$ (see (2.7)) is given by the sum of the $F(\lambda, G')$ corresponding to all the subgraphs G' with $|E'|$ edges, each of which belongs to a cycle (see § 6 of PF2). For example, the coefficient of t^3 in (5.5b) is the sum of the flow polynomials corresponding to the four subgraphs shown in figure 6. In the

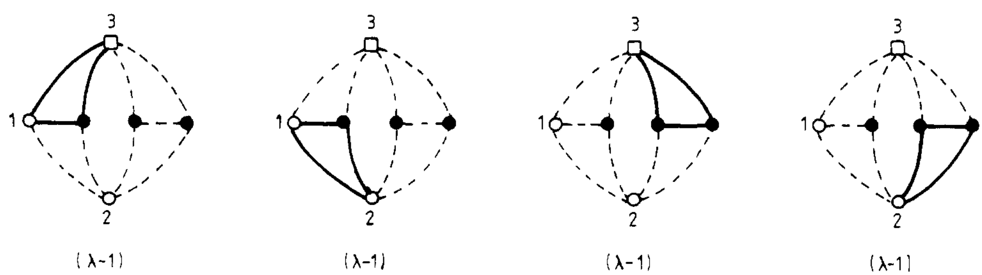

Figure 6. The partial graphs G' of G (drawn at the top of figure 5(a)) with their respective flow polynomials $F(\lambda, G')$ which contribute to the coefficient of t^3 in $D(t, G)$ (equation (5.5b)). The broken lines indicate missing edges.

case of $N_{12,3}(t, G)$, the coefficient of $t^{|E'|}$ (see (2.9)) is the sum of the $F_{12,3}(\lambda, G')$ corresponding to all the subgraphs G' with $|E'|$ edges which have no 'dangling end' and in which 1 and 2 are connected but not via 3 (otherwise $\gamma_{12,3}(G'')$ would vanish for all $G'' \subseteq G'$). For example, there are only seven subgraphs G' (see figure 7) which contribute to the coefficient of t^3 in (5.5*a*).

5.3. *The* BCM *for* $t_{\mathbb{P}}^{\mathrm{eq}}(t, G)$

The BCM for $t_{\mathbb{P}}^{\mathrm{eq}}(t, G)$ consists in combining: (i) the factorisation rules for articulated graphs; (ii) the equations for effective edges in series and in parallel; (iii) the effective BCE (4.13). Unlike the SBCM, it does not search for the mentioned pair of vertices $\{ij\}$ which appears in the 'non-reducible subgraph replacement'. Figure 8 shows schematically the application of the BCM to the calculation of $t_{12,3}^{\mathrm{eq}}(t, G)$ corresponding to the same graph used in the illustration of the SBCM. In this figure, t_r and t_p are defined by (5.2*a*) and (5.2*b*) while t_w and t_s are respectively

$$t_w = \frac{3t^2 + 3(\lambda - 2)t^4 + (\lambda^2 - 3\lambda + 3)t^6}{1 + 3(\lambda - 1)t^4 + (\lambda - 1)(\lambda - 2)t^6} \tag{5.6a}$$

and

$$t_s = \frac{5t^2 + 4(\lambda - 2)t^3 + (\lambda^2 + 2\lambda - 6)t^4 + 4(\lambda - 2)^2 t^5 + (\lambda^3 - 5\lambda^2 + 10\lambda - 7)t^6}{1 + 2(\lambda - 1)t^2 + (\lambda - 1)(\lambda + 3)t^4 + 4(\lambda - 1)(\lambda - 2)t^5 + (\lambda - 1)(\lambda - 2)^2 t^6}. \tag{5.6b}$$

Combining the results of figure 8 with the effective BCE we obtain the expected expressions (5.5) for $N_{12,3}(t, G)$ and $D(t, G)$.

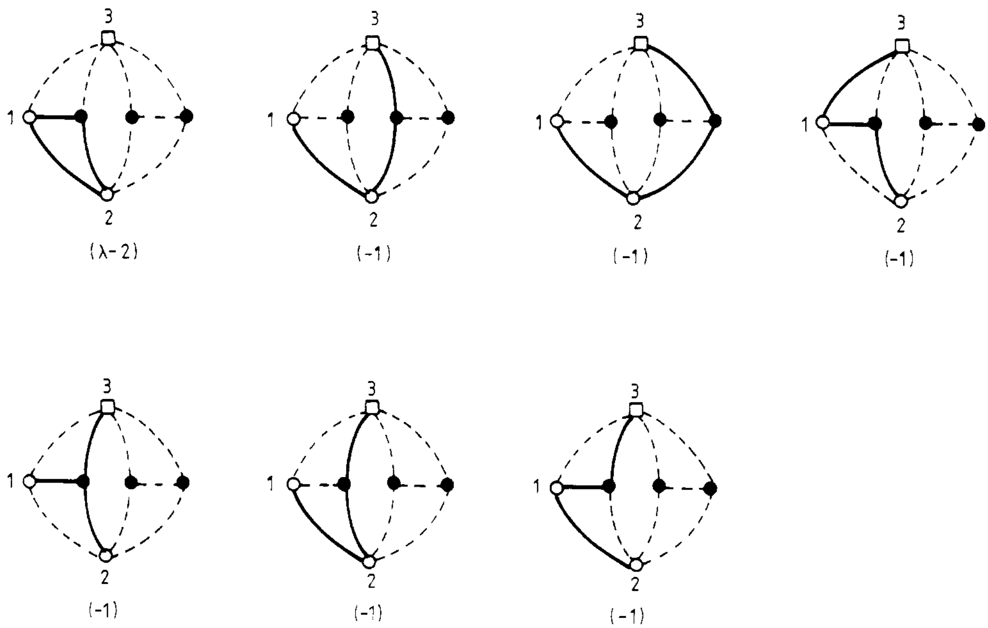

Figure 7. The partial graphs G' of G (drawn in figure 5(*a*)) with their respective $F_{12,3}(\lambda, G')$ which contribute to the coefficient of t^3 in $N_{12,3}(t, G)$ (equation (5.5*a*)). The broken lines indicate missing edges.

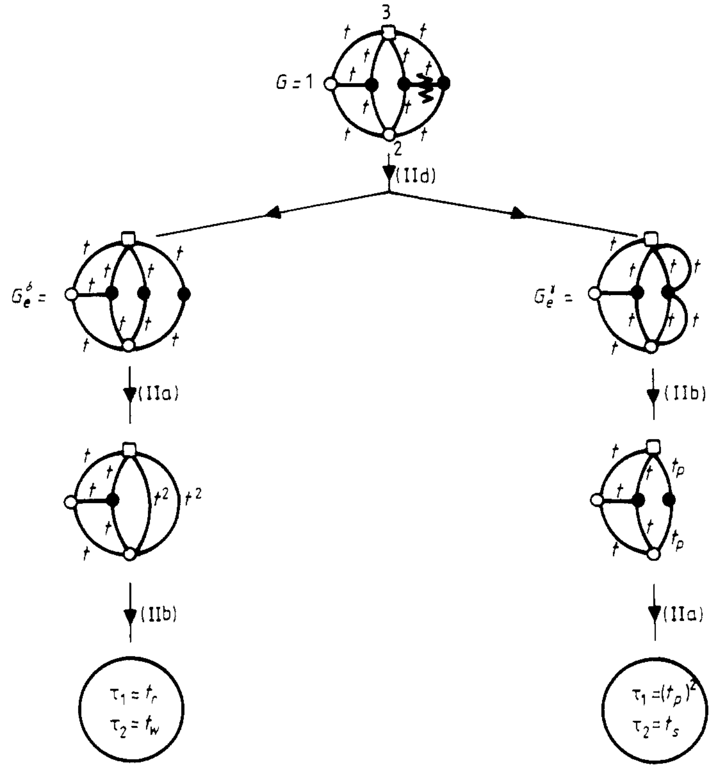

Figure 8. A schematic representation of the calculation of $t^{\rm eq}_{12,3}(t, G)$ for the same graph G given in figure 5 through the BCM described in § 5.3. The calculation of $t^{\rm eq}_{12,3}(t, G_e^\delta)$ and $t^{\rm eq}_{12,3}(t, G_e^\gamma)$ require the calculation of $t^{\rm eq}_{12,3}(t, G_9)$ (see figure 5(*b*)) where ($\tau_1 = t_r$, $\tau_2 = t_w$) and ($\tau_1 = t_p^2$, $\tau_2 = t_s$) respectively. The effective transmissivities t_r, t_p, t_w and t_s can be found in equations (5.2*a*), (5.2*b*), (5.6*a*) and (5.6*b*) respectively.

Comparing figures 5 and 8 we see that for this two-reducible graph the BCM necessitates the calculation of the equivalent transmissivities of more graphs (23) than does the SBCM (17 graphs). Figure 8 shows the application of the BCE seven times while in figure 5 one 'non-reducible subgraph replacement' is made and the BCE is used four times.

When the partition $\mathbb{P}$ of the m roots has only one block, it is possible to construct a BCM for $t^{\rm eq}_{12\ldots m}(t, G)$ without using the split procedure described in step 1 of § 5.1 (Tsallis 1987).

6. The SBCM for bond percolation

In this section we consider the $\lambda \to 1$ limit of our formulae in order to obtain results for the connectedness functions of bond percolation theory. As we have seen in § 7 of PF2, the flow polynomial vanishes for $\lambda = 1$ (except for the null graph) and $F_{\mathbb{P}}(1, G)$ is the partitioned d weight $d_{\mathbb{P}}(G)$ (see (7.6) of PF2) which is a generalisation of the ordinary d weight which occurs in the expansion of the pair connectedness (see, e.g., Essam 1971b). In this limit, the t and p variables become equal, and $t^{\rm eq}_{\mathbb{P}}(t, G)$ reduces to the partitioned m-rooted connectedness $C_{\mathbb{P}}(p, G)$ which generalises the pair connectedness $C_{12}(p, G)$ which appears in bond percolation.

6.1. *Main formulae for $d_{\mathbb{P}}(G)$*

From (3.15) we obtain the following SBCE for $d_{\mathbb{P}}(G)$:

$$d_{\mathbb{P}}(G) = d_{ij}(L)d_{\mathbb{P}}(H \cup e). \tag{6.1}$$

The factorisation equations corresponding to the cases illustrated in figure 4 are (see equations (3.18)):

figure 4(a) $\quad d_{\mathbb{P}}(G_1 \cup G_2) = 0 \quad$ (6.2a)

figure 4(b, c) $\quad d_{\mathbb{P}}(G_1 \cup G_2) = d_{\mathbb{P}'}(G_1)d_{\mathbb{P}''}(G_2) \quad$ (6.2b)

figure 4(d) $\quad d_{\mathbb{P}}(G_1 \cup G_2) = 0. \quad$ (6.2c)

From (3.16) we obtain the following equation for parallel combination:

$$d_{ij}(G_1 \cup G_2) = -d_{ij}(G_1)d_{ij}(G_2). \tag{6.3}$$

Equations (6.1), (6.2c) and (6.3) agree, when $m = 2$ and $\mathbb{P}$ has a single block, with known results (Essam 1971b).

6.2. *The SBCM and BCM for $C_{\mathbb{P}}(p, G)$*

Considering now the probability $C_{\mathbb{P}}(p, G)$ that the m roots of G are connected in blocks according to the partition $\mathbb{P}$, we can see from equations (4.2) and (7.2) and equation (7.3) of PF2 that it satisfies the following SBCE:

$$C_{\mathbb{P}}(p, G) = [1 - C_{ij}(p, L)]C_{\mathbb{P}}(p, H) + C_{ij}(p, L)C_{\mathbb{P}}(p, H_{i=j}) \tag{6.4}$$

which, for $\mathbb{P} = \{\{1, 2\}\}$, recovers equation (3.14) of Essam (1971b), referred to as the edge substitution equation for the pair connectedness.

Equation (6.4) can be interpreted as follows. $C_{\mathbb{P}}(p, G)$ can be written as the sum of the probabilities of two disjoint events: (a) the probability P_a that the roots of G are $\mathbb{P}$ partitioned and that i and j are not connected in L; (b) the probability P_b that the roots of G are $\mathbb{P}$ partitioned and that i and j are connected in L.

According to probability theory, the probability $P(\alpha_1 \cap \alpha_2)$ that two events α_1 and α_2 occur simultaneously is given by

$$P(\alpha_2 \cap \alpha_1) = P(\alpha_2|\alpha_1)P(\alpha_1) \tag{6.5}$$

where $P(\alpha_1)$ is the probability that event α_1 occurs and $P(\alpha_2|\alpha_1)$ is the conditional probability that event α_2 occurs given that α_1 occurs. In case (a), $P_a(\alpha_1)$ represents the probability that i and j are not connected in L (hence $P_a(\alpha_1) = 1 - C_{ij}(p, L)$) and $P_a(\alpha_2|\alpha_1)$ is the probability that the roots of G are $\mathbb{P}$ partitioned given that i and j are not connected on L ($P_a(\alpha_2|\alpha_1) = C_{\mathbb{P}}(p, H)$ since in this case the roots of G must be $\mathbb{P}$ partitioned on H itself). In case (b), $P_b(\alpha_1)$ is the probability that i and j are connected on L (hence $P_b(\alpha_1) = C_{ij}(p, L)$) and $P_b(\alpha_2|\alpha_1)$ is the probability that the roots of G are $\mathbb{P}$ partitioned given that i and j are connected on L. This conditional probability is equal to $C_{\mathbb{P}}(p, H_{i=j})$ since, as we have seen in § 3.2, when $\gamma_{ij}(L') = 1$ we need to consider the connections among the roots of $H'_{i=j}$.

The factorisation rules for articulated graphs corresponding to figure 4 are given by (cf equation (4.17) and equations (7.2) and (7.3) of PF2):

figure 4(*a*) $$C_{\mathbb{P}}(p, G_1 \cup G_2) = C_{\mathbb{P}}(p, G_1) \qquad (6.6a)$$

figure 4(*b, c*) $$C_{\mathbb{P}}(p, G_1 \cup G_2) = C_{\mathbb{P}'}(p, G_1) C_{\mathbb{P}''}(p, G_2) \qquad (6.6b)$$

figure 4(*d*) $$C_{\mathbb{P}}(p, G_1 \cup G_2) = 0. \qquad (6.6c)$$

The parallel equation for $C_{ij}(p, G)$ can be written as (see equation (4.12*a*) and equations (7.2) and (7.3) of PF2):

$$1 - C_{ij}(p, G_1 \cup G_2) = (1 - C_{ij}(p, G_1))(1 - C_{ij}(p, G_2)). \qquad (6.7)$$

Equation (6.6*b*) particularised for $P = \{\{i, j\}\}$ and equation (6.7) are respectively the same as equations (3.3) and (3.1) of Essam (1971b).

The above formulae may be used, instead of the corresponding transmissivity equations, in the algorithms of § 5 to define the SBCM and BCM for $C_{\mathbb{P}}(p, G)$. The recursive procedure $T(G, \mathbb{P}, N, D)$ is replaced by $CO(G, \mathbb{P}, C)$ which has only one output C, the partitioned connectedness, instead of the pair N, D. Also only one multilinear function, the equivalent pair connectedness, is associated with each edge of G.

7. Summary

We have proved, through graph theory, the formulae which appear in the BCM for the partitioned equivalent transmissivities which, for a one-block partition, has been so extensively used in real space renormalisation group calculations for the Potts model. We have also developed a more refined method (the SBCM) based on the same formulae, but which is considerably *more efficient* than the BCM since it allows for the replacement of *any* subgraph L, which is attached at only two vertices and which has no internal roots, by a single effective edge at any stage of the calculation. In the BCM the *only* subgraphs L considered were edges in series and/or parallel which were replaced by effective edges with effective thermal transmissivities given by equations (4.11) and (4.12) respectively. These equations, together with the factorisation rules for articulated graphs (equations (4.16) and (4.17)) and the effective break-collapse equation (4.13), constitute the main expressions used in the two above methods. We derive them from similar relations which we have proved to be true for the flow polynomials and the partitioned flow polynomials. Both the SBCM and the BCM provide a way of computing the *exact* expressions for the partitioned equivalent transmissivities, and hence for the multispin correlation function (see equation (2.14)) and for the partition function (see equation (2.13)), for finite graphs. They avoid the time-consuming summation over spin states and may be used either in the extension (or derivation) of series expansions or of real space renormalisation group calculations on crystal and hierarchical lattices.

Acknowledgments

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Appendix. The SBCM for other quantities

Let us first quote very briefly previous results (see PF2) expressed in the p variable of Kasteleyn and Fortuin (1969).

The partition function $Z(p, G)$ is:

$$Z(p, G) = \left(\prod_{e \in E} \exp[(\lambda - 1)K_e] \right) D(p, G) \tag{A1a}$$

with

$$D(p, G) \equiv \langle \lambda^{\omega} \rangle_{G,p} \tag{A1b}$$

where $\omega(G')$ is the number of components of G' and p_e is given by

$$p_e = 1 - \exp(-\lambda K_e). \tag{A1c}$$

The multilinear form of $D(p, G)$ is

$$D(p, G) = \sum_{G' \subseteq G} (-1)^{|E'|} P(\lambda, G') \prod_{e \in E'} p_e \tag{A2a}$$

where $P(\lambda, G)$ is the chromatic polynomial of G with λ colours given by

$$P(\lambda, G) = \sum_{G' \subseteq G} (-1)^{|E'|} \lambda^{\omega(G')} \tag{A2b}$$

$\Gamma_{12\ldots m}(p, G)$ is given by (2.14) where now

$$t_{\mathbb{P}}^{\text{eq}}(p, G) = N_{\mathbb{P}}(p, G)/D(p, G) \tag{A3a}$$

with

$$N_{\mathbb{P}}(p, G) \equiv \langle \lambda^{\omega} \gamma_{\mathbb{P}} \rangle_{G,p}. \tag{A3b}$$

The multinear form of $N_{\mathbb{P}}(p, G)$ is

$$N_{\mathbb{P}}(p, G) = \sum_{G' \subseteq G} (-1)^{|E'|} P_{\mathbb{P}}(\lambda, G') \prod_{e \in E'} p_e \tag{A4a}$$

where the partitioned m-rooted chromatic polynomial is

$$P_{\mathbb{P}}(\lambda, G) = \sum_{G' \subseteq G} (-1)^{|E'|} \lambda^{\omega(G')} \gamma_{\mathbb{P}}(G'). \tag{A4b}$$

The partitioned m-rooted rank function $W_{\mathbb{P}}(x, y, G)$, which extends the Whitney rank function $W(x, y, G)$ (see Essam 1971a) becomes, for $y = \lambda x$,

$$W_{\mathbb{P}}(x, \lambda x, G) \equiv W_{\mathbb{P}}(G) = \sum_{G' \subseteq G} x^{|E'|} \lambda^{c(G')} \gamma_{\mathbb{P}}(G'). \tag{A5}$$

$P_{\mathbb{P}}(\lambda, G)$, $N_{\mathbb{P}}(p, G)$ and $W_{\mathbb{P}}(x, y, G)$ are related to $F_{\mathbb{P}}(\lambda, G')$ through equations (4.5), (4.6) and (4.13) of PF2 respectively. Using these relations and equations (3.8), (3.16), (3.17) and (3.18) we can define the SBCM for these functions using the algorithm of § 5.

A1. SBCM for $P_{\mathbb{P}}(\lambda, G)$

SBCE:

$$P_{\mathbb{P}}(\lambda, G) = \lambda^{-2}\{P_{i,j}(\lambda, L) P_{\mathbb{P}}(\lambda, H) + \lambda P_{ij}(\lambda, L) P_{\mathbb{P}}(\lambda, H_{i=j})\} \tag{A6a}$$

$$P(\lambda, G) = \lambda^{-2}\{P_{i,j}(\lambda, L) P(\lambda, H) + \lambda P_{ij}(\lambda, L) P(\lambda, H_{i=j})\}. \tag{A6b}$$

Factorisation equations:

$$P_{\mathbb{P}}(\lambda, G_1 \cup G_2) = \lambda^{-1} P_{\mathbb{P}}(\lambda, G_1) P(\lambda, G_2) \tag{A7a}$$

$$P_{\mathbb{P}}(\lambda, G_1 \cup G_2) = \lambda^{-1} P_{\mathbb{P}'}(\lambda, G_1) P_{\mathbb{P}''}(\lambda, G_2) \tag{A7b}$$

$$P_{\mathbb{P}}(\lambda, G_1 \cup G_2) = 0. \tag{A7c}$$

Parallel equations:

$$P(\lambda, G_1 \cup G_2) = \lambda^{-2}\{P(\lambda, G_1) P(\lambda, G_2) + (\lambda - 1) P_{ij}(\lambda, G_1) P_{ij}(\lambda, G_2)\} \tag{A8}$$

$$P_{ij}(\lambda, G_1 \cup G_2) = \lambda^{-2}\{P_{ij}(\lambda, G_1) P(\lambda, G_2) + P_{ij}(\lambda, G_2) P(\lambda, G_1) + (\lambda - 2) P_{ij}(\lambda, G_1) P_{ij}(\lambda, G_2)\}. \tag{A9}$$

A2. SBCM *for* $N_{\mathbb{P}}(p, G)$

SBCE:

$$N_{\mathbb{P}}(p, G) = \lambda^{-2}\{N_{i,j}(p, L) N_{\mathbb{P}}(p, H) + \lambda N_{ij}(p, L) N_{\mathbb{P}}(p, H_{i=j})\} \tag{A10a}$$

$$D(p, G) = \lambda^{-2}\{N_{i,j}(p, L) D(p, H) + \lambda N_{ij}(p, L) D(p, H_{i=j})\}. \tag{A10b}$$

Factorisation equations:

$$N_{\mathbb{P}}(p, G_1 \cup G_2) = \lambda^{-1} N_{\mathbb{P}}(p, G_1) D(p, G_2) \tag{A11a}$$

$$N_{\mathbb{P}}(p, G_1 \cup G_2) = \lambda^{-1} N_{\mathbb{P}'}(p, G_1) N_{\mathbb{P}''}(p, G_2) \tag{A11b}$$

$$N_{\mathbb{P}}(p, G_1 \cup G_2) = 0. \tag{A11c}$$

Parallel equations:

$$D(p, G_1 \cup G_2) = \lambda^{-2}\{D(p, G_1) D(p, G_2) + (\lambda - 1) N_{ij}(p, G_1) N_{ij}(p, G_2)\} \tag{A12}$$

$$N_{ij}(p, G_1 \cup G_2) = \lambda^{-2}\{N_{ij}(p, G_1) D(p, G_2) + N_{ij}(p, G_2) D(p, G_1) + (\lambda - 2) N_{ij}(p, G_1) N_{ij}(p, G_2)\}. \tag{A13}$$

A3. SBCM *for* $W_{\mathbb{P}}(G)$

SBCE:

$$W_{\mathbb{P}}(G) = W_{i,j}(L) W_{\mathbb{P}}(H) + W_{ij}(L) W_{\mathbb{P}}(H_{i=j}) \tag{A14a}$$

$$W(G) = W_{i,j}(L) W(H) + W_{ij}(L) W(H_{i=j}). \tag{A14b}$$

Factorisation equations:

$$W_{\mathbb{P}}(G_1 \cup G_2) = W_{\mathbb{P}}(G_1) W(G_2) \tag{A15a}$$

$$W_{\mathbb{P}}(G_1 \cup G_2) = W_{\mathbb{P}'}(G_1) W_{\mathbb{P}''}(G_2) \tag{A15b}$$

$$W_{\mathbb{P}}(G_1 \cup G_2) = 0. \tag{A15c}$$

Parallel equations:

$$W(G_1 \cup G_2) = W(G_1) W(G_2) + (\lambda - 1) W_{ij}(G_1) W_{ij}(G_2) \tag{A16}$$

$$W_{ij}(G_1 \cup G_2) = W_{ij}(G_1) W(G_2) + W_{ij}(G_2) W(G_1) + (\lambda - 2) W_{ij}(G_1) W_{ij}(G_2). \tag{A17}$$

Using the above formulae we can construct further SBCM and BCM along similar lines to the algorithm of § 5.

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