Lattice Paths and the Constant Term

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Abstract. We firstly review the constant term method (CTM), illustrating its combinatorial connections and show how it can be used to solve a certain class of lattice path problems. We show the connection between the CTM, the transfer matrix method (eigenvectors and eigenvalues), partial difference equations, the Bethe Ansatz and orthogonal polynomials. Secondly, we solve a lattice path problem first posed in 1971. The model stated in 1971 was only solved for a special case – we solve the full model.

1. Introduction

This paper is in two parts. The first part is mostly review of the Constant Term Method (CTM) and the second part of the paper is a new application of the method where is used to solve a model which has remained unsolved since 1971.

The Constant Term Method has been used to provide elegant solutions to a number of lattice path problems, which have arisen variously in the study of several polymer models [4, 8, 9, 6, 11] and in the calculation of the stationary state of the asymmetric exclusion model (ASEP) [7, 5]. Most problems may be solved by alternative means, but the form of the solution obtained by the CTM is of interest in that it is suggestive of a purely combinatorial proof.

The CTM is a good example of a bridge between Statistical Mechanics and Pure Combinatorics. Often, in Statistical Mechanics, the desire to understand some physical system will motivate a mathematical model, the solution of which requires new combinatorial methods. The techniques thereby developed open up new possibilities in Combinatorics, as well as often suggesting new, tractable, physical models. The history of the CTM illustrates this interplay, having been originally developed to solve certain polymer models, but having lead to new combinatorics with wider application. We conclude by using the CTM to solve a previously open problem. The full details of the CTM and its generalisation may be found in Brak and Osborn [10].

2. Simple polymer models and definitions

We will concentrate on two models. The first is adsorption of a polymer on to a surface – modelled by a weighted lattice path in the half plane. The second is the problem of stabilisation of colloidal dispersions by adsorbed polymers (steric stabilization) and the destabilisation when the polymer can adsorb on surfaces of different colloidal particles (sensitized flocculation) – modelled by a weighted lattice path in a strip.

A **binomial path** of length n is a sequence of vertices $v_0 \cdots v_n$ where $v_i = (x_i, y_i) \in$, $v_i - v_{i-1} \in \{(1,1),(1,-1)\}$ and $v_0 = (0,0)$. A **Dyck path** is a binomial path for which $y_i \geq 0, i \in \{0,1,2,\ldots,n\}$. A lattice path in a **strip** of width L is a Dyck path with the additional constraint $y_i \leq L, i \in \{1,2,\ldots,n\}$. To differentiate Dyck paths in strip from Dyck paths we will occasionally refer to the latter as **half plane**

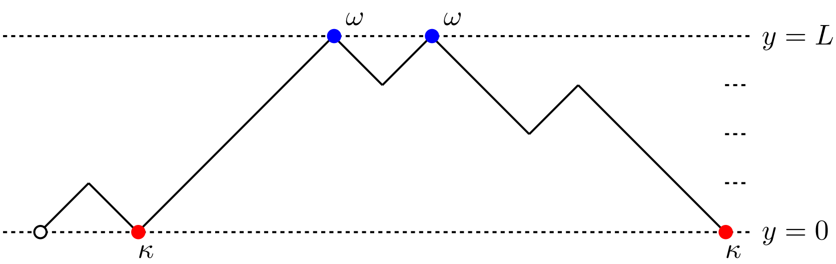

Figure 1. A path in a strip of width $L = 4$ showing the returns (lower circles) on $y = 0$ and on $y = 4$ (upper circles).

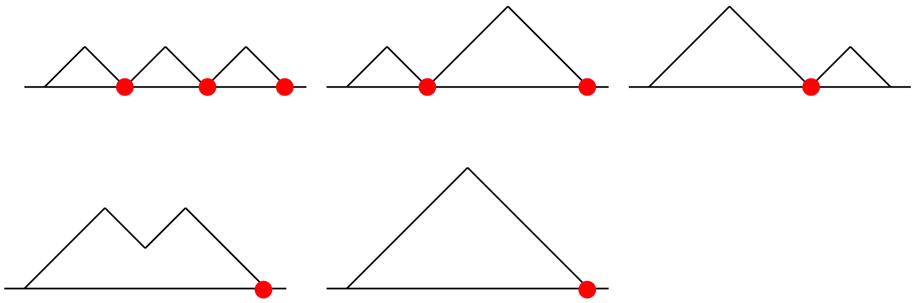

Figure 2. All Dyck paths of length 6 showing the returns (circles).

Dyck paths. The polymer models require paths weighted by "contacts" or "returns". A **contact** is any vertex of the path for which $y_i = 0$ and, for a strip $y_i = L$. A **return** is any contact except the one associated with $v_0 = (0, 0)$. Returns for which $y_i = 0$ will have weight κ and for a path in a strip returns on $y_i = L$ are given a weight ω – see figure 2.

To study the phase behaviour of the polymer models we need to calculate the partition function, or **return polynomial**

$$Z_t(\kappa, \omega) = \sum_{\pi \in \mathcal{P}_t} \kappa^{\mathrm{return}_0(\pi)} \omega^{\mathrm{return}_L(\pi)} \tag{2.1}$$

where $\mathcal{P}_t$ is the set of all paths of length t and $\mathrm{return}_y(\pi)$ is the number of returns at height y in the path π. For example, for Dyck paths of length 6, $Z_6(\kappa) = 2\kappa + 2\kappa^2 + \kappa^3$ – see figure 2 – and for strip paths with $t = 6$ and $L = 2$, $Z_6(\kappa, \omega) = \kappa\omega^2 + 2\kappa^2\omega + \kappa^3$.

There are many methods for solving lattice path and hence polymer problems. In this paper we consider three and the connections between them. The three methods are illustrated by considering the return polynomial.

3. Solving – by partial difference equations

The first method we consider uses partial difference equations. We illustrate the method by finding the return polynomial for half plane Dyck paths. The polynomial satisfies the set of partial difference equations

$$Z_{n+1}(y) = Z_n(y+1) + Z_n(y-1), \qquad y > 0 \tag{3.1}$$

$$Z_{n+1}(0) = \kappa\, Z_n(1) \tag{3.2}$$

$$Z_0(y) = \delta_{y,0} \tag{3.3}$$

which are obtained by considering the action of adding an additional step to the path – see figure 3. The partial difference equations are readily solved by separation of variables, $Z_t \to \mu^t Q_y$ which upon

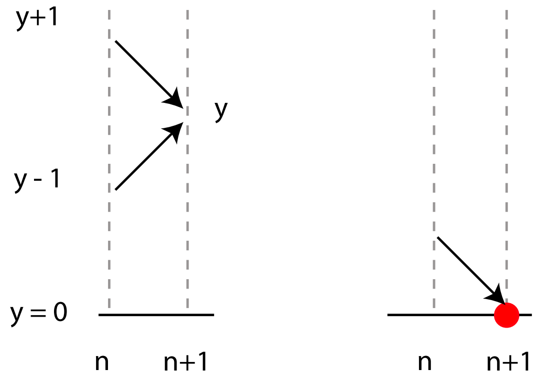

Figure 3. This figure illustrates the possible ways an extra step can be added to the path hence giving the partial difference equations (3.1) and (3.2).

substitution into (3.1) and (3.2) requires

$$\kappa Q_1 = \mu Q_0 \tag{3.4}$$

$$Q_{y+1} = \mu Q_y - Q_{y-1}, \qquad y > 0 \tag{3.5}$$

If we try $Q_y \to \rho^y$ or $Q_y \to \rho^{-y}$ in (3.5) either satisfies the equation so long as $\mu = \rho + 1/\rho$. To satisfy (3.4) try the "Bethe Anstaz" $Q_y = A\rho^y + B\rho^{-y}$ which requires $A/B = (\rho^2 - (1-\kappa))/(1-(1-\kappa)\rho^2)$. Note, the trial solution is called a Bethe Ansatz as it is a linear combination of independent "simple" solutions ie. solutions which only satisfy (3.5). Finally, to satisfy the initial condition (3.3), we make the "constant term"(CT) Ansatz[1]

$$Z_t(0) = \mathrm{CT}\left[\left(\rho + \frac{1}{\rho}\right)^t \frac{1-\rho^2}{1-(\kappa-1)\rho^2}\right] \tag{3.6}$$

where the **constant term** operator, CT is defined by

$$\mathrm{CT}[f(z)] = a_0, \qquad f(z) = \sum_{n \ge n_0} a_n z^n, \qquad n_0 \in \tag{3.7}$$

and $\sum_{n \ge n_0} a_n z^n$ is the Laurent expansion of $f(z)$ convergent in an annulus $0 < |z| < R$ where R is less than the distance from the origin to the singularity closest to the origin. The CT Ansatz is readily shown to satisfy (3.3) using induction. Note, in the case of a single variable the CT operator is just the residue of $f(z)/z$ at $z = 0$ – this is not obviously the case for the multivariable extension [4].

Evaluating the CT in (3.6) we get

$$Z_{2n} = \sum_{m \ge 0} B_{2n-m-1,m-1}\kappa^m, \qquad B_{e+d,e-d} = \frac{e-d+1}{e+1}\binom{e+d}{e} \tag{3.8}$$

where $B_{r,s}$ is a Ballot number and $C_n = B_{2n,0}$ is a Catalan number.

[1] To simplify the comparison with other methods in this paper we only consider the case $y = 0$, the general case may be found in [6].

4. Combinatorics of the Constant Term Ansatz

Possibly the most significant aspect of the CT form is its strong combinatorial structure. Suitably interpreting the CT operator in many cases leads to a pure combinatorial solution of the problem. In fact, one can frequently obtain several different combinatorial solutions to the same problem by using different interpretations. The combinatorial structure associated with the CT form is an "involution". Examples of involutions are the method of inclusion and exclusion, the method of images (familiar in the solving of electrostatic problems) and (indirectly) the Bethe Ansatz.

In combinatorics an **involution** is a mapping $\psi : \Omega \to \Omega$ satisfying the following conditions

(i) $\psi^2 =$

(ii) If $\Omega = \Omega^+ \cup \Omega^-$, $\Omega^+ \cap \Omega^- = \phi$ then

  (a) If $x \in \Omega^+$ then $\psi(x) \in \Omega^-$ or $\psi(x) = x$

  (b) If $x \in \Omega^-$ then $\psi(x) \in \Omega^+$ or $\psi(x) = x$

The set Ω is called the **signed set**, Ω^+ the **positive set** and Ω^- the **negative set**. The set $\{x \,|\, \psi(x) = x, x \in \Omega\}$ is called the **fixed point** set. The combinatorial significance of the CT form is that it *gives the signed set of some involution*. Once you have the signed set you then have to "guess" ψ. Thus the CT form is a partial algebraic method for obtaining combinatorial solutions. We illustrate the combinatorial interpretation with an example. There are two ingredients to the interpretation. The first are binomial paths. The number of binomial paths from $(0, 0)$ to (t, y) is the binomial coefficient $\binom{t}{(t+y)/2}$; here $(t + y)/2$ is the number of up steps. This may be obtained using the CT form and the partial difference equations (3.1) and (3.3) – omit the "boundary" equation (3.2) – which are solved by the CT Ansatz

$$|\text{Binomial paths}| = \text{CT}\left[\left(\rho + \frac{1}{\rho}\right)^t \rho^{-y}\right] = \binom{t}{\frac{t+y}{2}} \tag{4.1}$$

Thus the ρ^{-y} factor in the CT argument, can be thought of as a *source of binomial paths that end at* (t, y). Now, consider the case of Dyck paths of length $2n$ with $\kappa = 1$. From (3.6) we get

$$|\text{Dyck paths}| = \text{CT}\left[\left(\rho + \frac{1}{\rho}\right)^{2n} \left(1 - \rho^2\right)\right] \tag{4.2}$$

$$= \text{CT}\left[\left(\rho + \frac{1}{\rho}\right)^{2n} 1\right] - \text{CT}\left[\left(\rho + \frac{1}{\rho}\right)^{2n} \rho^2\right] \tag{4.3}$$

$$= \binom{2n}{n} - \binom{2n}{n-1} \tag{4.4}$$

$$= \frac{1}{n+1}\binom{2n}{n} \tag{4.5}$$

Combinatorially we can interpret the first CT term $\text{CT}\left[(\rho + 1/\rho)^{2n}\, 1\right]$ as defining the elements of Ω^+ and the second term $\text{CT}\left[(\rho + 1/\rho)^{2n}\, \rho^2\right]$ as defining the elements of Ω^- ie. Ω^+ is the set of all binomial paths from $(0, 0)$ to $(2n, 0)$ and Ω^- is the set of all binomial paths from $(0, 0)$ to $(2n, -2)$ – see figure 4. The difference $1 - \rho^2$ in (4.2) represents a pairing off of paths in the sets Ω^+ and Ω^-. The paths left unpaired in Ω^+, the fixed point set, are the Dyck paths. In this case the involution ψ is defined by the well known method of images or Andrè principle [2, 3]: if the steps of the path intersect the line $y = -1$ then the steps after the rightmost vertex in common with $y = -1$ are reflected in the line $y = -1$ to obtain a new path – the dashed (blue) steps in figure 4. The reflected path ends at $(2n, -2)$. This takes a path from Ω^+ (respec. Ω^-) and gives a unique path in Ω^- (respec. Ω^+), unless the path does not intersect $y = -1$ (ie. all the Dyck paths) in which case the path maps to itself.

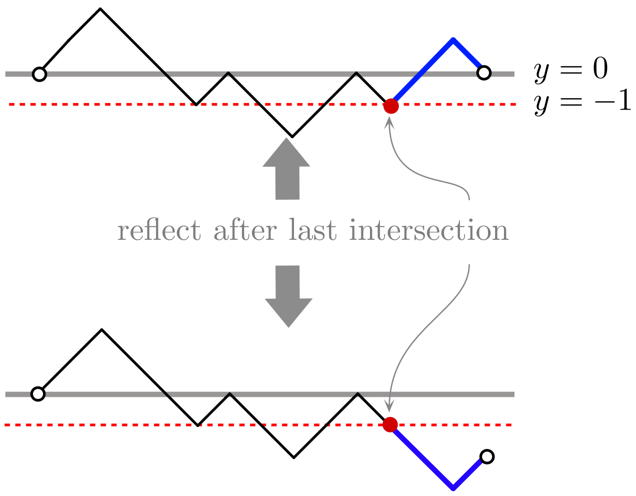

Figure 4. Paths stepping below $y = 0$ are cancelled by paths starting at $y = -2$. The canceling pairs are obtained by reflecting the thicker (blue) steps in the line $y = -1$.

Applying these ideas to the general κ case, (3.6) gives, either

$$Z_{2n} = \mathrm{CT}\left[\left(\rho+\frac{1}{\rho}\right)^{2n}\frac{1-\rho^2}{1-(\kappa-1)\rho^2}\right] \tag{4.6}$$

$$= \sum_{m\geq 0}(\kappa-1)^m\,\mathrm{CT}\left[\left(\rho+\frac{1}{\rho}\right)^{2n}(1-\rho^2)\rho^{2m}\right] \tag{4.7}$$

$$= \sum_{m\geq 0}(\kappa-1)^m\left[\binom{2n}{n}-\binom{2n}{n-m-1}\right] \tag{4.8}$$

or, rearranging the denominator before expanding, gives an alternative expression

$$Z_{2n} = \mathrm{CT}\left[\left(\rho+\frac{1}{\rho}\right)^{2n}\frac{1-\rho^2}{1-(\kappa-1)\rho^2}\right] \tag{4.9}$$

$$= \mathrm{CT}\left[\left(\rho+\frac{1}{\rho}\right)^{2n}\frac{\rho^{-1}-\rho}{\rho+\rho^{-1}}\,\frac{1}{1-\kappa\rho/(\rho+\rho^{-1})}\right] \tag{4.10}$$

$$= \sum_{m\geq 0}\kappa^m\,\mathrm{CT}\left[\left(\rho+\frac{1}{\rho}\right)^{2n-m-1}(1-\rho^2)\rho^{m-1}\right] \tag{4.11}$$

$$= \sum_{m\geq 0}\kappa^m\left[\binom{2n-m-1}{n-1}-\binom{2n-m-1}{n}\right] \tag{4.12}$$

The coefficient of $(\kappa-1)^m$ is the number of paths with m marked returns. Thus, from (4.8), we can construct an involution on a signed set of "terraced" paths of length $2n$. The terraced paths biject to paths with m marked returns. The positive set contains binomial paths counted by $\binom{2n}{n}$ and the negative set contains paths counted by $\binom{2n}{n-m-1}$ – see [6] for details of this involution.

Similarly, from (4.12) the coefficient of κ^m, which is the number of paths with *exactly* m returns, we can define a *different* involution (based on paths of length $2n-m-1$) – see [6] for details.

It should be clear that by expanding the argument of the CT operator in different ways we are lead to different combinatorial solutions to the problem.

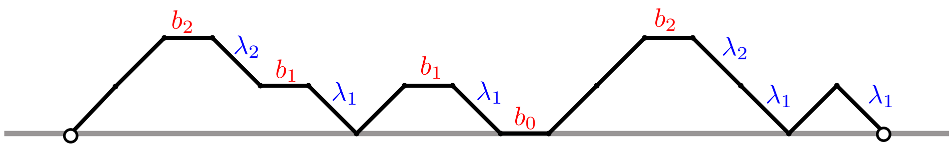

Figure 5. An example of a Motzkin path with weight $b_0 b_1^2 b_2^2 \lambda_1^4 \lambda_2^2$.

5. Solving – by transfer matrix

In this section we discuss a second method of solving certain lattice path problems and show the connection with the CT solution of the partial difference equations. Transfer matrices can be used to enumerate lattice paths. Thus, in the case of return polynomials, the transfer matrix[2] is

$$T = \begin{pmatrix} 0 & 1 & & & & \\ \kappa & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & 1 & 0 & 1 & \\ & & & & \ddots & \\ & & & & \omega & 0 \end{pmatrix} \tag{5.1}$$

This is a particular case of a tridiagonal or Jacobi matrix

$$T = \begin{pmatrix} b_0 & 1 & & & \\ \lambda_1 & b_1 & 1 & & \\ & \lambda_2 & b_2 & 1 & \\ & & & \ddots & \\ & & & \lambda_L & b_L \end{pmatrix} \tag{5.2}$$

which enumerates Motzkin paths [13, 14] in a strip of width L with weights $\{b_k, \lambda_k\}$. **Motzkin paths** are similar to Dyck paths but are allowed a horizontal step, $v_{i+1} - v_i = (1, 0)$ – see figure 5. The **weight polynomial** is defined as

$$Z_t = \sum_{\pi \in S_t} W(\pi) \tag{5.3}$$

where $W(\pi)$ is the weight of the path π and S_t the set of paths. The transfer matrix methods gives the weight polynomial for paths which start at height y and end at height y' as the y, y' matrix element (we index the matrix elements starting from zero up to L) of the t^{th} power of T,

$$Z_t(y, y') = \left(T^t\right)_{y,y'} \tag{5.4}$$

To evaluate (5.4) we diagonalise T. The eigenvectors $T\vec{P} = \mu_k \vec{P}$ with eigenvalue μ_k have components which satisfy the equations

$$\lambda_k P_{k-1} + b_k P_k + P_{k+1} = \mu_k P_k, \quad 0 \le k \le L \tag{5.5}$$

$$P_{L+1} = 0, \tag{5.6}$$

We choose a length for the vectors by setting $P_0 = 1$ and $P_{-1} = 0$. Equation (5.5) is easily recognised as the standard three term recurrence for orthogonal polynomials [1], $P_k(x)$,

$$P_{k+1}(x) = (x - b_k) P_k(x) - \lambda_k P_{k-1}(x) \tag{5.7}$$

[2] Note, in the matrix the weights are actually associated with the *edges* however, except in isolated cases the return polynomials are the same.

Thus we see that the eigenvalues and right eigenvectors are related to orthogonal polynomials by

$$\text{Eigenvalues } \mu_k: \quad P_{L+1}(\mu_k) = 0 \tag{5.8}$$

$$\text{Eigenvectors } \vec{P}_k: \quad (P_0(\mu_k), P_1(\mu_k), \cdots, P_L(\mu_k)). \tag{5.9}$$

Note, the Jacobi matrix in the form (5.2) is not symmetric and hence has different left and right eigenvectors. The left eigenvectors, Q_k satisfy similar equations

$$\lambda_k\, Q_{k+1}(x) = (x - b_k) Q_k(x) - Q_{k-1}(x) \tag{5.10}$$

$$Q_{L+1} = 0 \tag{5.11}$$

with $Q_0 = 1$ and $Q_{-1} = 0$, and are directly related to the right eigenvectors via

$$P_k = h_{1,k}\, Q_k, \qquad k \geq 1 \tag{5.12}$$

$$h_{k,k'} = \prod_{i=k}^{k'} \lambda_i. \tag{5.13}$$

Using the eigenvectors to diagonalise T in (5.4) gives the generating function in the form

$$Z_t(y, y') = \sum_{k=0}^{L} \frac{1}{N_k} P_y(\mu_k)\, \mu_k^t\, Q_{y'}(\mu_k) \tag{5.14}$$

where $N_k = \sum_{y=0}^{L} Q_y(\mu_k) P_y(\mu_k)$ is the normalisation. The problem is how to evaluate the sum (5.14)? We do this by comparing the transfer matrix result with the partial difference equation CT solution.

If we compare the equations (3.4) and (3.5) obtained in solving the partial difference equations with equation (5.10) we see that the problem of solving the partial difference equations is, after separation of variables, precisely the same as finding the left eigenvectors of a particular Jacobi matrix. The CT Ansatz used to satisfy the initial condition (3.3) is a statement of the completeness of the set of eigenvectors ie. $\sum_{k=0}^{L} P_y(\mu_k) Q_{y'}(\mu_k)/N_k = \delta_{y,y'}$. Now we have the connection between the two methods it suggests the transfer matrix form of solution (5.14) can be written in CT form. This is in fact the case.

6. Solving – by generating function

The third method we consider is the generating function method of Viennot [14] and Flajolet [13]. In their work there is a direct relationship between the Motzkin path *length* generating function and orthogonal polynomials. The path length generating function

$$G(x) = \sum_{t \geq 0} Z_t(y, y')\, x^t \tag{6.1}$$

is given by

$$G(x) = \bar{x} \frac{P_y^{(0)}(\bar{x})\, h_{y,y'}\, P_{L-y'}^{(y'+1)}(\bar{x})}{P_{L+1}^{(0)}(\bar{x})}, \qquad \bar{x} = \frac{1}{x} \tag{6.2}$$

where the **weight shifted** orthogonal polynomials, $P_k^{(s)}(x)$ satisfy the equation

$$P_{k+1}^{(s)}(x) = (x - b_{k+s}) P_k^{(s)}(x) - \lambda_{k+s} P_{k-1}^{(s)}(x) \tag{6.3}$$

and $h_{y,y'}$ is given by (5.13).

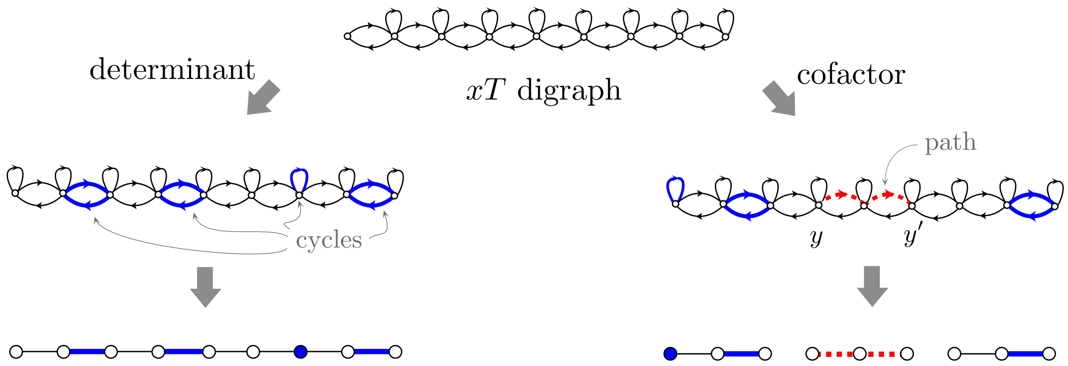

Figure 6. An example of a combinatorial representation of the determinant (left) and cofactor (right) of a Jacobi matrix T. Starting from the digraph (top) with adjacency matrix xT, the determinant is represented by configurations of non-intersecting cycles (left-middle) on the digraph and, for the cofactor (right-middle), an additional path from y to y'. These configurations then biject to a set of pavings (left bottom) or a pair of pavings and a path (right bottom).

To get (6.2) from (6.1) algebraically, use the Neumann result

$$\sum_{t\geq 0} Z_t(y,y')\, x^t = \sum_{t\geq 0} T^t_{y,y'} x^t = \left((1-xT)^{-1}\right)_{y,y'} \tag{6.4}$$

The y, y' matrix element of the inverse matrix $(1-xT)^{-1}$ is then obtained in the usual way via cofactors to give $G = \text{cof}(1-xT)_{y,y'}/\det(1-xT)$. The determinant of $1-xT$ leads to the denominator of (6.2) and the cofactor leads to the numerator.

We mention, without going into any details, that equation (6.2) can be obtained combinatorially from (6.4) – see [14]. It is a classical result that the determinant $\det(1-xT)$ can be represented combinatorially as enumerating weighted non-intersecting cycles on a digraph with (weighted) adjacency matrix xT – see figure 6. Because of the tridiagonal form of the Jacobi matrix, the digraph is rather simple – see figure 6 – in this case the non-intersecting cycles on the digraph biject to "pavings" (by monomers and dimers) of a line graph. Enumerating these pavings leads to precisely the three term recurrence (5.7) and hence to orthogonal polynomials. Similarly, the cofactor is represented by non-intersecting cycles *and a path* on the digraph between vertices y to y' – see figure 6. The cofactor configurations biject to a pair of pavings (hence the two polynomials in the numerator) and a weighted path (the $h_{y,y'}$ factor in the numerator).

Equation (6.2) reduces the problem of finding the generating function to that of finding explicit forms for the three orthogonal polynomials. If we want the return polynomial, Z_t, we still have to then find the coefficient of x^t. We shall do this by connecting this problem to that of the transfer matrix.

We can see the connection to the transfer matrix method by using a partial fraction expansion of (6.2). Since $P^{(0)}_{L+1}$ is an orthogonal polynomial its zeros, μ_k, are all real and simple, thus we have

$$G(x) = \bar{x}\sum_{k=0}^{L}\frac{A_k}{\bar{x}-\mu_k} = \sum_{t\geq 0} x^t \sum_{k=0}^{L} A_k \mu_k^t \tag{6.5}$$

with $A_k = P^{(0)}_y(\mu_k)h_{y,y'}P^{(y'+1)}_{L-1}(\mu_k)/P^{(0)'}_{L+1}(\mu_k)$ where $P^{(0)'}$ is the derivative. Beginning with the Christoffel-Darboux formula and using a lemma in [10], it can be shown that $A_k = P_y(\mu_k)Q_{y'}(\mu_k)/N_k$ where N_k is the same normalisation as (5.14). Thus,

$$Z_t(y,y') = \sum_{k=0}^{L}\mu_k^t A_k = \sum_{k=0}^{L}\mu_k^t \frac{P_y(\mu_k)Q_{y'}(\mu_k)}{N_k} \tag{6.6}$$

which is manifestly the same form at the transfer matrix solution (5.14). Thus again we are required to sum over the zeros of P_{L+1}.

We can see some of the workings of the various techniques by considering (unweighted) Dyck paths in a strip. In this case $b_k = 0, \forall k$ and $\lambda_k = 1, \forall k$ and the orthogonal polynomials are Chebyshev polynomials, F_k (a particular case of Chebychev polynomials),

$$F_k(x) = \frac{\rho^{k+1} - \rho^{-k-1}}{\rho - \bar{\rho}}, \qquad x = \rho + \bar{\rho}, \qquad \bar{\rho} = \frac{1}{\rho} \tag{6.7}$$

The zeros, $F_{L+1}(x) = 0$, in the ρ variable, are all roots of unity, $\rho^{2L+4} = 1$ (some are cancelled by the $\rho - \bar{\rho}$ factor in the denominator of (6.7) and hence there are only $L + 1$ zeros) and the sum over the zeros and normalisation needed in (5.14) or (6.4) become simple geometric series. Interestingly, the result obtained is exactly the same as the following

$$Z_t(y, y') = \mathrm{CT}_{\bmod 2L+4} \left[(\rho + \bar{\rho})^t \, (1 - \rho^2) \, \rho^L \, F_y(\rho + \bar{\rho}) \, F_{L-y'}(\rho + \bar{\rho}) \right] \tag{6.8}$$

where $\mathrm{CT}_{\bmod 2L+4}$ is a generalisation of (3.7)

$$\mathrm{CT}_{\bmod 2L+4}[f(z)] = \sum_{m\in} a_{(2L+4)m}. \tag{6.9}$$

Since we have a CT result we can use the same combinatorial interpretation stated earlier. This leads to interpreting the factor $(1 - \rho^2) \, \rho^L \, F_y(\rho + \bar{\rho}) \, F_{L-y'}(\rho + \bar{\rho})$ as giving rise to the "images" of the method of images. In this case the images are the same as are used in the method of inclusion-exclusion to enumerate Dyck paths in a strip – see figure 6. If we consider the half plane limit (and, for simplicity, $y = 0$), $L \to \infty$, (6.8) becomes

$$Z_t(0) = \mathrm{CT} \left[(\rho + \bar{\rho})^t (1 - \rho^2) \right] \tag{6.10}$$

comparing this with (4.2) shows we get the same result obtained from the partial difference equations. This is no coincidence, we have shown the generating function method is equivalent to the transfer matrix method which is in turn equivalent to the partial difference equation method and hence the CT connection.

7. Strip paths

We now consider the solution of a long standing problem in statistical mechanics [12] – that of a Dyck path in strip with return weights on the lines $y = 0$ and $y = L$ – see figure 2. This problem has remained unsolved since 1971. The Jacobi transfer matrix of this problem has $\lambda_1 = \kappa$, $\lambda_L = \omega$ (otherwise $\lambda_k = 1$) and $b_k = 0, \forall k$.

To solve the problem we can apply any of the three methods discussed above. All require the calculation of the same set of orthogonal polynomials. One way of obtaining the polynomials, P_k, is to solve the recurrence relation (5.5). This would normally give the polynomials as a function of x. However, in this form you do *not* get a simple CT expression. In order to achieve a useful CT expression it is necessary to write the polynomials P_k in terms of the Chebyshev polynomials (6.7). It is possible to do this algebraically (using Greens functions) or combinatorially - see [10] for details. One then finds that

$$\begin{aligned} P_k = F_k + (1 - \kappa) F_{k-2} + (1 - \omega) F_{k-L-1} F_{L-1} \\ + (1 - \kappa)(1 - \omega) F_{k-L-1} F_{L-3} \end{aligned} \tag{7.1}$$

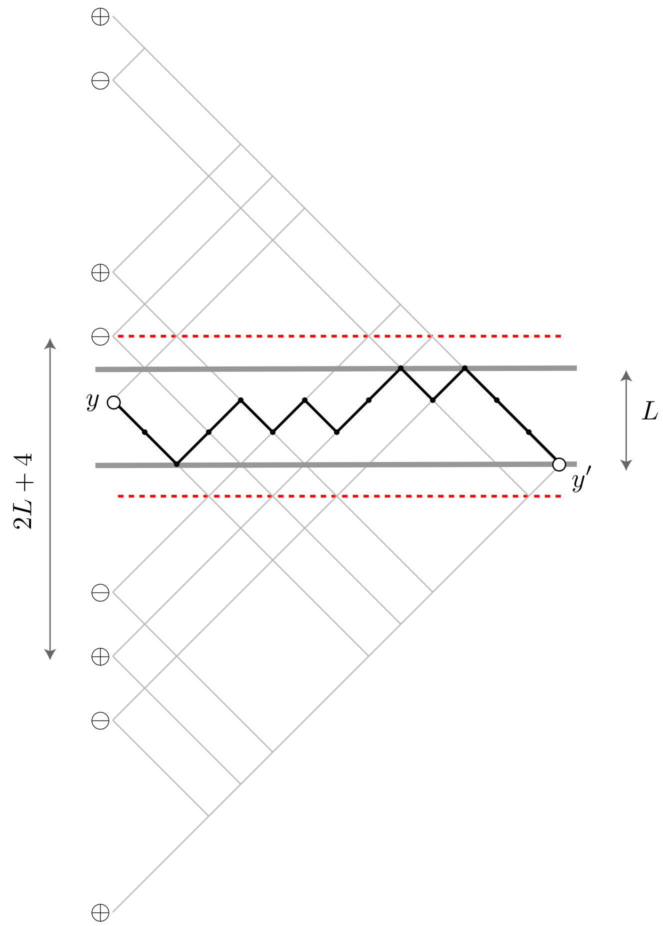

Figure 7. To enumerate unweighted Dyck paths in a strip of width L (dark path), "image" paths starting at the $\oplus$ (respec. the $\ominus$) symbol are added (resp. subtracted). The grey lines show the constructions used to locate the images.

and then, after some calculation one finds,

$$Z_{2n}(\kappa,\omega) = \mathrm{CT}\left[\left(\rho+\frac{1}{\rho}\right)^{2n}\frac{1-\rho^2}{\rho^2-\sum_{s=0}^{L}\beta_s\rho^{-2s}}\sum_{r=0}^{L}\alpha_r\rho^{-2r}\right] \tag{7.2}$$

where

$$\alpha_r = \begin{cases} 2-\omega & r\in\{1,\dots,L-1\},\\ 1 & r\in\{0,L\},\end{cases} \tag{7.3}$$

$$\beta_s = \begin{cases} 2\kappa+2\omega-\kappa\omega-4 & s\in\{1,\dots,L-2\},\\ \kappa+\omega-3 & s\in\{0,L-1\}\ ,\\ -1 & s=L\ .\end{cases} \tag{7.4}$$

and hence

$$Z_{2n}(0,0) = \sum_{r=0}^{L}\sum_{s=0}^{n}\sum_{s_0,\dots,s_L}{}' C_{2n,n^*}\binom{s}{s_0,\dots,s_L}\alpha_r\prod_{m=0}^{L}\beta_m^{s_m} \tag{7.5}$$

where the multinomial sum has the constraint $s_0 + \cdots + s_L = s$, $s_i \in \{0, \dots, s\}$, $C_{n,m}$ is the generalised Catalan number,

$$C_{n,m} = \binom{n}{m} - \binom{n}{m-1} \tag{7.6}$$

and $n^* = \ell - s - r - \sum_{m=0}^{L} m s_m$. Note, by using the recurrence relation for the Chebyshev polynomials you can change the form of (7.2) and hence the form of the solution (7.5).

It is of interest to note that, in the above case the zeros of P_{L+1} (ie. the eigenvalues) are no longer roots of unity (in the ρ variable), although numerical evidence suggests most still lie on the unit circle. There are however, several special cases where the zeros are still roots of unity, these are $(\kappa, \omega) \in \{(1,1), (1,2), (2,1), (2,2)\}$ and the case $\kappa + \omega = \kappa\omega$. This can be seen by substituting these special cases into (7.1) , resulting in P_{L+1} reducing back down to a single Chebyshev polynomial eg. for $(1,1)$ $P_{L+1} \to F_{L+1}$, and hence roots of unity are obtained.

The case $\kappa+\omega = \kappa\omega$, although highlighted for mathematical reasons, does have physical significance. It turns out (see [11]) to be the line of zero force – the physical effect of the polymer in the strip is to produce an attractive or repulsive force between the top and bottom of the strip. The line $\kappa + \omega = \kappa\omega$ separates the repulsive regime from the attractive regime.

8. Conclusion

We have illustrated how the three methods for enumerating a class of lattice paths – the partial difference equation method, the transfer matrix method and the length generating function method are all very closely interconnected. The constant term Ansatz is used to solve partial difference equations. The CT form of the solution has the feature of having a strong combinatorial connection, giving the signed set of an involution and hence suggesting a direct combinatorial solution. The transfer matrix method is a traditional method used in statistical mechanics to solve physical models - the connection with the constant term solution and the generating function method illustrates the strong interconnectedness between statistical mechanics and pure combinatorics.

We have also used the constant term solution to solve a new problem. This problem is of physical interest (modelling a polymer interacting with two boundaries) and of combinatorial interest as it is a combinatorial problem that appears not to be solvable by using a "simple" set of images – in statistical mechanics it represents a problem that can't be solved by a "Bethe Ansatz".

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