Fermi, Bose and Vicious walk configurations on the directed square lattice.

F M Bhatti[1] **and J W Essam**[2]

[1]Department of Mathematics, Lahore University of Management Sciences, Sector U, DHA, Lahore, Pakistan.

[2]Department of Mathematics and Statistics, Royal Holloway College, University of London, Egham, Surrey TW20 0EX, England.

E-mail: `fmbhatti@lums.edu.pk`,`j.essam@rhul.ac.uk`

Abstract. Inui and Katori introduced Fermi walk configurations which are non-crossing subsets of the directed random walks between opposite corners of a rectangular $\ell \times w$ grid. They related them to Bose configurations which are similarly defined except that they include multisets. Bose configurations biject to vicious walker watermelon configurations. It is found that the maximum number of walks in a Fermi configuration is $\ell w + 1$ and the number of configurations corresponding to this number of walks is a $w-$dimensional Catalan number $C_{\ell,w}$. Product formulae for the numbers of Fermi configurations with ℓw and $\ell w - 1$ walks are derived. We also consider generating functions for the numbers of $n-$walk configurations as a function of ℓ and w. The Bose generating function is rational with denominator $(1-z)^{\ell w+1}$. The Fermi generating function is a polynomial which is found to have a factor $(1+z)^{\ell+w}$. The complementary factor $Q^{Fermi}_{\ell,w}(z)$ is related to the numerator of the Bose generating function which is a generalised Naryana polynomial introduced by Sulanke. Recurrence relations for the numbers of Fermi walks and for the coefficients of the polynomial $Q^{Fermi}_{\ell,w}(z)$ are obtained. Fermi configurations are such that only one walker can follow a given path; extension to configurations in which the number of walkers on any path is limited to $m \geq 1$ is discussed.

Key words: lattice paths, enumerative combinatorics, Fermi walks, interacting random walks, flows, d-dimensional Naryana and Catalan numbers, Pfaff-Saalschütz identity, Dyck paths, Young tableaux.

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1. Background and definitions.

Suppose that each of n walkers start at points of the square lattice such that $x+y=0$ and simultaneously make t steps with step vectors $(1,0)$ or $(0,1)$. Let $\{x_i(\tau), y_i(\tau)\}$ be the position of the i^{th} walker after the first τ steps and $Y_i(\tau)=y_i(\tau)-x_i(\tau)$. The walks are said to be *non-crossing* if for each $\tau \leq t$

$$Y_{i+1}(\tau) \geq Y_i(\tau) \qquad \text{for each } i \in \{1,2,\ldots,n-1\}. \tag{1.1}$$

Inui and Katori [1] considered two sets of $n-$walk non-crossing configurations, Bose and Fermi, in which all of the walks start at the lattice origin and end at the point (ℓ, w). The walks are thus confined to an $\ell \times w$ rectangular grid. No further conditions are imposed on *Bose* configurations but *Fermi* configurations are subject to the additional constraint that each of $\binom{\ell+w}{w}$ directed lattice paths between the corners of the rectangle may be used by at most one walker. Five of the Fermi configurations with $\ell=2, w=3$ are shown in figure 1.

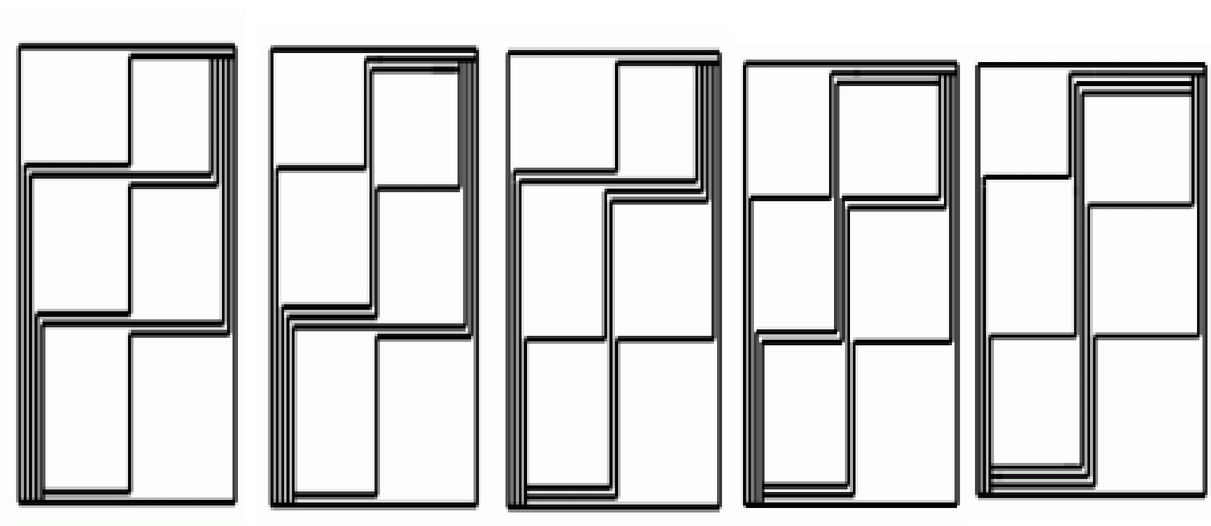

Figure 1. *The five maximal Fermi walk configurations on a* 2×3 *grid.*

The numbers of Bose and Fermi configurations will be denoted by $f^{Bose}_{\ell,w,n}$ and $f^{Fermi}_{\ell,w,n}$ respectively.

Bose configurations biject [2] to directed integer flows with a source of strength n at the origin and a sink at (ℓ, w) with the flow on a given lattice bond being equal to the number of walkers traversing that bond. The number of flows, and hence the number of Bose configurations, was conjectured by Arrowsmith et al [2] to be

$$f^{Bose}_{\ell,w,n} = \prod_{j=1}^{w} \frac{(\ell+w-j+1)_n}{(j)_n} = \prod_{j=1}^{w} \frac{(\ell+j)_n}{(j)_n} \tag{1.2}$$

where $(a)_k \equiv a(a+1)\ldots(a+k-1)$. With the usual convention $(a)_0=1$ it follows from (1.2) that $f^{Bose}_{\ell,w,0}=1$. The first equality in (1.2) was subsequently derived [3] by enumerating the vicious walker configurations [4] using a Lindstrom-Gessel-Viennot determinant [5],[6],

[7]. It is clear from this result that for fixed w and n, $f^{Bose}_{\ell,w,n}$ is a polynomial of degree nw in ℓ.

Sulanke [8] defined the $d-$dimensional Naryana number $N(d,n,m)$ to be the number of configurations of a d-dimensional lattice path of length dn from the origin to the point $\mathbf{n} = (n, n, \ldots, n)$ lying in the region $\{(x_1, x_2, \ldots, x_d) : 0 \le x_1 \le x_2 \le \cdots \le x_d\}$ and having m ascents (an *ascent* is a move into a higher dimension). Proposition 1 of [8] states that these numbers are given by the formula

$$N(d,n,m) = \sum_{j=0}^{m} (-1)^{m-j} \binom{dn+1}{m-j} \prod_{i=0}^{d-1} \binom{n+i+j}{n} \binom{n+i}{n}^{-1}. \tag{1.3}$$

and they have the following properties

$$N(d,n,m) = 0 \qquad \text{for } m > (d-1)(n-1) \tag{1.4}$$

$$N(d,n,(d-1)(n-1)-m) = N(d,n,m) \qquad \text{for } 0 \le m \le (d-1)(n-1) \tag{1.5}$$

$$\binom{n+d-1}{d} N(d,n,m) = \sum_{h=0}^{d} \binom{(d-1)(n-1)-m+h}{h} \binom{n+m+d-h-1}{d-h} N(d,n-1,m-h). \tag{1.6}$$

Equations (1.4) and (1.5) are from Corollary 1 of [8] and the recurrence relation is from proposition 9 of [8]. The $d-$dimensional Naryana polynomial is the generating function

$$N_{d,n}(z) \equiv \sum_{m=0}^{(d-1)(n-1)} N(d,n,m) z^m \tag{1.7}$$

and the sum of the Naryana numbers is the $d-$dimensional Catalan number

$$N_{d,n}(1) = C_{d,n} \equiv (dn)! \prod_{i=0}^{d-1} \frac{i!}{(n+i)!} \tag{1.8}$$

which determines the number of $d-$dimensional lattice paths from the origin to $\mathbf{n}$ with no constraint on the number of ascents. Sulanke attributes (1.8) to MacMahon (see [10], art. 93-103). $C_{d,2}$ is the ordinary Catalan number giving the number of Dyck paths [9].

$$N(2,n,m) = N(n,m+1) \tag{1.9}$$

where

$$N(n,m) \equiv \frac{1}{n} \binom{n}{m-1} \binom{n}{m} = \frac{(n-m+1)_m (n-m+2)_m}{(m-1)!m!} = N_{n,n-m+1} \tag{1.10}$$

is an ordinary Naryana number [11],[12],[13] introduced by Naryana in 1955 as counting the number of parallelogram polyominoes of perimeter $2(n+1)$ with m columns.

Fermi configurations are the main subject of this paper. Inui and Katori [1] gave the following relation between the numbers of Fermi and Bose configurations.

$$f^{Bose}_{\ell,w,n} = \sum_{k=1}^{n} \binom{n-1}{n-k} f^{Fermi}_{\ell,w,k} \tag{1.11}$$

The factor $\binom{n-1}{n-k}$ arises from the number of ways to assign a further $n-k$ walks to the paths used by the k Fermi walks.

For a given rectangular grid there is clearly an upper limit to the number of walks in a Fermi configuration. On the 2×3 grid of figure 1 there can be be no more than 7 walks and the figure shows the five maximal configurations. Proposition 4 below states that given ℓ and w the maximum number of walks in a Fermi configuration is $\ell w+1$. The following generating function for the numbers of Fermi walks is therefore a polynomial.

$$G^{Fermi}_{\ell,w}(z)\equiv\sum_{n=0}^{\ell w+1} z^n f^{Fermi}_{\ell,w,n} \tag{1.12}$$

For Bose walks the generating function $G^{Bose}_{\ell,w}(z)$ is defined in a similar manner except that the sum extends to infinity. Defining $f^{Fermi}_{\ell,w,0}=1$, equation (1.11) implies the relation

$$G^{Bose}_{\ell,w}(z)=G^{Fermi}_{\ell,w}\left(\frac{z}{1-z}\right). \tag{1.13}$$

In a companion paper [14] this relation together with (1.2) was used to prove the following proposition for Bose configurations which will be used here to obtain results for Fermi configurations.

Proposition 1 (Bose).

(a) $G^{Bose}_{\ell,w}(z)=\frac{Q^{Bose}_{\ell,w}(z)}{(1-z)^{\ell w+1}}$ *where* $Q^{Bose}_{\ell,w}(z)$ *is a polynomial of degree* $(\ell-1)(w-1)$

(b) $q^{Bose}_{\ell,w,n}\equiv[z^n]Q^{Bose}_{\ell,w}(z)=\sum_{k=0}^{n}(-1)^{n-k}\binom{\ell w+1}{n-k}f^{Bose}_{\ell,w,k}$

(c) $q^{Bose}_{\ell,w,n}=N(w,\ell,n)$ *and* $Q^{Bose}_{\ell,w}(z)=N_{w,\ell}(z)$

(d) $Q^{Bose}_{\ell,w}(1)=f^{Fermi}_{\ell,w,\ell w+1}=C_{w,\ell}$

(e) *For* $n=0,1,\ldots,(\ell-1)(w-1)$, $q^{Bose}_{\ell,w,(\ell-1)(w-1)-n}=q^{Bose}_{\ell,w,n}$.

(f) $z^{(\ell-1)(w-1)}Q^{Bose}_{\ell,w}(1/z)=Q^{Bose}_{\ell,w}(z)$

Notes:

- Part (b) shows that the obvious invariance of $f^{Bose}_{\ell,w,n}$ under interchange of ℓ and w is inherited by $q^{Bose}_{\ell,w,n}$. Consequently, by part (c), the Naryana numbers are invariant under interchange of their first two parameters which is not obvious from their walk definition. Also $C_{\ell,w}=C_{w,\ell}$ and for given w, $C_{\ell,w},\ell=0,1,2,\ldots$ is a sequence of $w-$dimensional Catalan numbers.
- The proposition may be stated as: the Bose generating function is a rational function with denominator $(1-z)^{\ell w+1}$ whose numerator is a Narayana polynomial and as a result the coefficients are symmetric and sum to a $w-$dimensional Catalan number.
- The second equality of (d) is proved independently here as part of proposition 4.

2. Results

In section 3, proposition 4 also gives product formulae for the numbers of Fermi configurations having one and two less walks than the maximum. A formula for the number with three less walks is also conjectured. These formulae become increasingly complicated as the number of walks is reduced.

In addition to proposition 4 we obtain further results for Fermi configurations. The following proposition is proved in section 4.

Proposition 2 (Fermi).

(a) $G^{Fermi}_{\ell,w}(z) = (1+z)^{\ell+w} Q^{Fermi}_{\ell,w}(z)$ *where* $Q^{Fermi}_{\ell,w}(z)$ *is a polynomial of degree* $(\ell-1)(w-1)$

(b) $Q^{Fermi}_{\ell,w}(-z) = (-1)^{(\ell-1)(w-1)} Q^{Fermi}_{\ell,w}(z-1)$

(c) $f^{Fermi}_{\ell,w,n} = \sum_{k=0}^{n} \binom{\ell w-k+1}{n-k} q^{Bose}_{\ell,w,k} = \sum_{k=1}^{n} (-1)^{n-k} \binom{n-1}{n-k} f^{Bose}_{\ell,w,k}$.

(d) $q^{Fermi}_{\ell,w,n} \equiv [z^n] Q^{Fermi}_{\ell,w}(z) = \sum_{k=0}^{n} \binom{(\ell-1)(w-1)-k}{n-k} q^{Bose}_{\ell,w,k} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n+\ell+w-1}{n-k} f^{Bose}_{\ell,w,k}$

(e) $q^{Fermi}_{\ell,w,(\ell-1)(w-1)-n} = \sum_{k=n}^{(\ell-1)(w-1)} \binom{k}{n} N(w,\ell,k) = \frac{1}{n!} \sum_{m=0}^{n} s(n,m) \mu_{\ell,w}(m)$ *where* $\mu_{\ell,w}(m)$*is the* m^{th} *moment of the w-dimensional Naryana distribution*
$\mu_{\ell,w}(m) \equiv \sum_{k=0}^{(\ell-1)(w-1)} k^m N(w,\ell,k)$
and $s(v,m)$ *is a Stirling number of the first kind [15]*

(f) $q^{Fermi}_{\ell,w,(\ell-1)(w-1)} = C_{\ell,w}$

(g) $z^{(\ell-1)(w-1)} Q^{Fermi}_{\ell,w}\left(\frac{1}{z}\right) = Q^{Bose}_{\ell,w}(1+z)$ *and hence* $Q^{Fermi}_{\ell,w}(1) = Q^{Bose}_{\ell,w}(2)$

Notes:

- From part (d) $q^{Fermi}_{\ell,w,n}$ inherits the symmetry of $f^{Bose}_{\ell,w,n}$ under interchange of ℓ and w.
- The sequences $Q^{Bose}_{\ell,w}(2), \ell = 0,1,2,\ldots$ are the little $w-$Schröder numbers [8] which generalise the little Schröder numbers [13] [16] obtained by setting $w = 2$.

Examples of proposition 2(a) are

$$G^{Fermi}_{2,4}(z) = (1+z)^6(1+2z)(1+7z+7z^2)$$
$$G^{Fermi}_{3,4}(z) = (1+z)^7(1+28z+238z^2+882z^3+1596z^4+1386z^5+462z^6)$$
$$G^{Fermi}_{4,4}(z) = (1+z)^8(1+2z)(1+60z+1050z^2+7986z^3+31020z^4+66066z^5+78078z^6+48048z^7+12012z^8)$$

The additional factor $(1+2z)$ always appears when both indices are even.

In section 5 we consider recurrence relations and prove the following proposition.

Proposition 3 (Recurrence relations).

(a) $\binom{\ell+w-1}{w} f^{Bose}_{\ell,w,n} = \binom{\ell+w+n-1}{w} f^{Bose}_{\ell-1,w,n}$

(b) $\binom{\ell+w-1}{w} q^{Bose}_{\ell,w,n} = \sum_{h=0}^{w} \binom{\ell+w+n-h-1}{w-h} \binom{(\ell-1)(w-1)-n+h}{h} q^{Bose}_{\ell-1,w,n-h}$

(c) $\binom{\ell+w-1}{w} q^{Fermi}_{\ell,w,n} = \binom{\ell+w+n-1}{w} \sum_{h=0}^{w-1} \binom{w-1}{h} q^{Fermi}_{\ell-1,w,n-h}$

(d) $\binom{\ell+w-1}{w} f^{Fermi}_{\ell,w,n} = \sum_{h=0}^{w} \binom{\ell+w+n-h-1}{w-h} \binom{n-1}{h} f^{Fermi}_{\ell-1,w,n-h}$.

By proposition 1(c), recurrence relation (b) is the Naryana recurrence (1.6). It is reported in [8] that this was obtained using the Mathematica package MULTISUM written by Wegschaider [17]. The package yielded a recurrence relation for the sum in proposition 1(b) in the cases $w = 3, 4$ and 5 and the general result was then conjectured and proved by substitution of the sum. Here we obtain the relations (c) and (d) in the same way. The second binomial coefficient in the sum of part (b) is zero for $n > (\ell-1)(w-1)$ in agreement with proposition 1(a).

The case w =2

Equation (1.2) shows that $f^{Bose}_{\ell,w,n}$ is hypergeometric which together with proposition 1(b) and proposition 2, parts (c) and (d) means that $q^{Bose}_{\ell,w,n}$, $f^{fermi}_{\ell,w,n}$ and $q^{Fermi}_{\ell,w,n}$ are sums of hypergeometric terms. By Zeilberger's algorithm [18], [19] these sequences therefore satisfy linear recurrence relations with polynomial coefficients. An excellent account of the algorithm is given in the lecture notes of Wilf [20]. Using Paule and Schorn's Mathematica implementation [23] of the algorithm the recurrence relations found for $q^{Bose}_{\ell,2,n}$ and $q^{Fermi}_{\ell,2,n}$ are first order leading to the product forms

$$q^{Bose}_{\ell,2,n} = \frac{(\ell-n)_n(\ell-n+1)_n}{(1)_n(2)_n} = \frac{1}{\ell}\binom{\ell}{n}\binom{\ell}{n+1} = N(\ell, n+1) \tag{2.1}$$

and

$$q^{Fermi}_{\ell,2,n} = \frac{(\ell-n)_n(\ell+2)_n}{(1)_n(2)_n} = \frac{1}{\ell}\binom{\ell+n+1}{n}\binom{\ell}{n+1} = c_{\ell,\ell-n-1}. \tag{2.2}$$

where the $c_{\ell,n}$ are Kirkman numbers [24]

$$c_{\ell,n} = \frac{1}{\ell}\binom{\ell}{n}\binom{2\ell-n}{\ell+1}. \tag{2.3}$$

The sum of these coefficients is a *small Schröder number* [16]

$$Q^{Fermi}_{\ell,2}(1) = Q^{Bose}_{\ell,2}(2) = \sum_{n=0}^{\ell-1} N_{\ell,n+1}2^n = s_n$$

A number of other occurrences of the small Schröder numbers are listed in [9] on page 239.

The recurrence relation found in the case of $f^{fermi}_{\ell,2,n}$ was second order

$$n(n+1)f^{Fermi}_{\ell,2,n} = (\ell(\ell+1) - 2n(n-2) + 2\ell n)f^{Fermi}_{\ell,2,n-1} - (n-2)(n-2\ell-3)f^{Fermi}_{\ell,2,n-2}. \tag{2.4}$$

Equation (2.1) also follows from proposition 1(c) and (1.9). Setting $w = 2$ in proposition 2(e)

$$q^{Fermi}_{\ell,2,\ell-n-1} = \sum_{k=n}^{\ell-1}\binom{k}{n}N(\ell, k+1) = c_{\ell,n} \tag{2.5}$$

in agreement with (2.2). This sum was performed [13] by applying a Vandermonde convolution formula.

In section 6 the number of configurations in which each path is used by at most m walks is considered and will be denoted by $f^{(m)}_{\ell,w,n}$. This interpolates between the Fermi and Bose configurations, $m = 1$ corresponds to Fermi configurations and replacing m by ∞ gives Bose configurations. Each path in a Fermi configuration may be replaced by between 1 and m paths so the generating function for these *multi-walks* is

$$G^{(m)}_{\ell,w}(z) \equiv \sum_{n=0}^{m(\ell w+1)} z^n f^{(m)}_{\ell,w,n} = G^{Fermi}_{\ell,w}(z + z^2 + \cdots + z^m) \tag{2.6}$$

which generalises (1.13). Expanding in powers of z relates the numbers of configurations to the numbers of Fermi configurations.

$$f^{(m)}_{\ell,w,n} = \sum_{k=\lceil \frac{n}{m} \rceil}^{n} C^{(m)}_{k,n-k} f^{(Fermi)}_{\ell,w,k} \tag{2.7}$$

where $C^{(m)}_{k,i}$ is the coefficient of z^i in the expansion of $(1 + z + z^2 + \cdots + z^{m-1})^k$. The $C^{(2)}_{k,i}$ are clearly binomial coefficients. For higher values of m the expansion was considered by Leonhard Euler [21] who called the $m = 3, 4, 5, \ldots$ coefficients trinomial, quadrinomial, quintinomial, ... and obtained the recurrence relation

$$C^{(m)}_{k,i} = \sum_{\ell=\lceil \frac{i}{m-1} \rceil}^{i} \binom{k}{\ell} C^{(m-1)}_{\ell,i-\ell} \tag{2.8}$$

and the symmetry property $C^{(m)}_{k,i} = C^{(m)}_{k,(m-1)k-i}$. The recurrence may be initialised by $C^{(1)}_{k,i} = \delta_{i,0}$. Clearly the coefficients vanish outside the range $[0, (m-1)k]$ and $C^{(m)}_{k,0} = C^{(m)}_{k,(m-1)k} = 1$. By proposition 2(a), $G^{(m)}_{\ell,w}(z)$ is a polynomial with factor $(1 + z + z^2 + \cdots + z^m)^{\ell+w}$. It will be shown that the alternative generating function

$$H^{(m)}_{w,n}(z) \equiv \sum_{\ell=0}^{\infty} z^\ell f^{(m)}_{\ell,w,n} \tag{2.9}$$

is a rational function with denominator $(1-z)^{wn+1}$ and numerator of degree at most $(n-1)w$ the coefficients of which sum to $C_{w,n}$. In the case of Bose configurations (infinite m) the rationality of this generating follows from proposition 1(a) since it was shown in [14] that $f^{Bose}_{\ell,w,n}$ is invariant under any permutation of its subscripts and hence $H^{Bose}_{w,n}(z) = G^{Bose}_{w,n}(z)$. The surprisng symmetry with respect to interchange of ℓ and n was shown to follow from duality. Similar rational generating functions were found by Guttmann and Vöge [22] for $f-$friendly walks which include vicious walkers and osculating walkers as the cases $f = 0$ and $f = 1$. As noted earlier, vicious walk watermelon configurations biject to Bose configurations.

3. Maximal and near maximal Fermi walk configurations.

For $w = 1$ the Fermi walk configurations are easily enumerated. The possible paths are determined by the $\ell + 1$ positions at which a single left step can be made. When there are

n walks the possible configurations are determined by choosing subsets of n of these paths leading to n walks which use different paths and are non-crossing as required.

$$f^{Fermi}_{\ell,1,n} = \binom{\ell+1}{n} \qquad \text{for } n \le \ell+1. \tag{3.1}$$

For more than $\ell+1$ walks at least one path must be used more than once so no Fermi configurations are possible.

Sample data for $w = 2, 3$ and 4 are given in appendix Appendix A. Notice that for $w = 2$ the maximum number of walks in a Fermi configuration is $2\ell + 1$ and the number of maximal Fermi walk configurations as a function of ℓ is given by the Catalan number C_ℓ. Parts (a) and (b) of proposition 4 extend these observations to general width and state that the maximum number of walks is $n^{max}_{\ell,w} = \ell w + 1$ and the number of maximal configurations is the $w-$dimensional Catalan number $C_{\ell,w}$. A direct proof of these results here is based on equations (1.2) and (1.11). A second proof will be given in [25] using a bijection to standard Young tableaux [26] which are enumerated by a product formula in terms of hook lengths [27].

Equation (1.2) may be used to write $f^{Bose}_{\ell,w,k}$ in the form

$$f^{Bose}_{\ell,w,n} = \frac{(\ell!)^w}{(\ell w)!} C_{\ell,w} \prod_{j=0}^{w-1} \binom{\ell+j+n}{\ell}. \tag{3.2}$$

Möbius inversion [28] of (1.11) gives

$$f^{Fermi}_{\ell,w,n} = \sum_{k=1}^{n} (-1)^{n-k} \binom{n-1}{n-k} f^{Bose}_{\ell,w,k}. \tag{3.3}$$

Let

$$f_{\ell,w}(x) \equiv f^{Fermi}_{\ell,w,n^{max}_{\ell,w}-x} \tag{3.4}$$

We will show that $f^{(w)}_\ell(x) = 0$ if and only if $x < 0$ which implies that $n^{max}_{\ell,w}$ is the maximum number of walks in a Fermi walk configuration. Thus x measures the distance from the maximal configuration.

Substituting (3.2) into (3.3)

$$f_{\ell,w}(x) = \frac{(\ell w - x)!}{(\ell w)!} C_{\ell,w}\, g^{(w)}_\ell(x) \tag{3.5}$$

where

$$g^{(w)}_\ell(x) = \frac{(\ell!)^w}{(\ell w - x)!} \sum_{k=0}^{\ell w - x} (-1)^{\ell w - x - k} \binom{\ell w - x}{k} \prod_{j=1}^{w} \binom{\ell+j+k}{\ell}. \tag{3.6}$$

3.1. Product formulae for $x = 0, 1, 2$ and 3.

Proposition 4. *Given $\ell \geq 0$ and $w \geq 1$*

(i) The maximum number of walks in a Fermi walk configuration is $\ell w + 1$.

(ii) The number of maximal Fermi walk configurations is given by the w-dimensional Catalan number

$$f_{\ell,w}(0) = C_{\ell,w} \tag{3.7}$$

(iii) The number of configurations with one less than the maximimum number of walks is

$$f_{\ell,w}(1) = \frac{1}{2}(w+1)(\ell+1)C_{\ell,w} \tag{3.8}$$

(iv) The number of configurations with two less than the maximum number of walks is

$$f_{\ell,w}(2) = \frac{(w+1)\,(\ell+1)\,\left(3\,\ell\,w\,(\ell+w+\ell\,w) - 5\,\ell\,w - 4\,(\ell+w-1)\right)C_{\ell,w}}{24(\ell w-1)} \tag{3.9}$$

Note: All formulae are symmetric under interchange of ℓ and w as expected.

The proposition follows from the following lemma together with equation (3.12) below by using (3.6).

Lemma 1. *For any non-negative integers $\{r\} \equiv \{r_1, r_2, \ldots, r_w\}$ and $\{\ell\} \equiv \{\ell_1, \ell_2, \ldots, \ell_w\}$ define*

$$D(L, \{\ell\}, \{r\}) \equiv \frac{(\prod_{j=1}^{w} \ell_j!)}{L!} \sum_{k=0}^{L} (-1)^{L-k} \binom{L}{k} \prod_{j=1}^{w} \binom{r_j + k}{\ell_j}$$

then

$$D(L, \{\ell\}, \{r\}) = \begin{cases} 0, & L > \ell_1 + \ell_2 + \cdots + \ell_w \\ 1, & L = \ell_1 + \ell_2 + \cdots + \ell_w \\ \sum_{i=1}^{w} \ell_i r_i + \sum_{1 \leq i < j \leq w} \ell_i \ell_j, & L = \ell_1 + \ell_2 + \cdots + \ell_w - 1 \end{cases}$$

Proof. Now

$$\binom{r_j + k}{\ell_j} = \frac{1}{\ell_j!} \sum_{m_j=1}^{\ell_j} s(\ell_j, m_j)(k + r_j)^{m_j} = \frac{1}{\ell_j!} \sum_{m_j=1}^{\ell_j} s(\ell_j, m_j) \sum_{n_j=0}^{m_j} \binom{m_j}{n_j} k^{n_j} r_j^{m_j - n_j}$$

where $s(\ell_j, m_j)$ is a Stirling number of the first kind [15]. Substituting in the definition of D leads to

$$D(L, \{\ell\}, \{r\}) = \sum_{n_1=0}^{\ell_1} \ldots \sum_{n_w=0}^{\ell_w} S(n_1 + n_2 + \ldots n_w, L) \times \\ \sum_{m_1=n_1}^{\ell_1} \cdots \sum_{m_w=n_w}^{\ell_w} \prod_{j=1}^{w} \left(s(\ell_j, m_j) \binom{m_j}{n_j} r_j^{m_j - n_j} \right) \tag{3.10}$$

where

$$S(n, L) \equiv \frac{1}{L!} \sum_{k=0}^{L} (-1)^{L-k} \binom{L}{k} k^n$$

is a Stirling number of the second kind [15].

Notice that $n_j \le m_j \le \ell_j$ and it is a property of Stirling numbers that if $L > n$ then $S(n, L) = 0$. Hence

$$D = 0 \qquad \text{unless} \qquad \sum_{j=1}^{w} \ell_i \ge \sum_{j=1}^{w} m_i \ge \sum_{j=1}^{w} n_i \ge L \tag{3.11}$$

(i) $L > \sum_{j=1}^{w} \ell_j$. In this case $D = 0$ follows directly from (3.11).

(ii) $L = \sum_{j=1}^{w} \ell_j$. The only indices which satisfy this condition and (3.11) are given by $n_j = m_j = \ell_j$ and since $S(L, L) = s(\ell, \ell) = 1$ the multiple sum in (3.10) evaluates to unity as required.

(iii) $L = \sum_{j=1}^{w} \ell_j - 1$. There are now two possibilities.

(a) $\sum_j n_j = L + 1 = \sum_j \ell_j$. Again $n_j = m_j = \ell_j$ and this case contributes $S(L + 1, L) = \frac{1}{2}(L + 1)L$ to D.

(b) $\sum_j n_j = L = \sum_j \ell_j - 1$. In this case $n_i = \ell_i - 1$ for some i and $n_j = m_j = \ell_j$ for $j \neq i$. Either $m_i = \ell_i$ or $\ell_i - 1$ and $s(\ell, \ell - 1) = -\frac{1}{2}\ell(\ell - 1)$ so the contribution to D is

$$\sum_{i=1}^{w} \left(\ell_i r_i - \frac{1}{2} \ell_i (\ell_i - 1) \right)$$

which combined with (a) gives the stated result.

□

The formulae for higher values of L become increasingly complicated. When $L = \sum_{j=1}^{w} \ell_j - 2$. the formula for D is

$$\begin{aligned} S(L+2, L) + S(L+1, L) \sum_{i=1}^{w} (\ell_i r_i + s(\ell_i, \ell_i - 1)) \\ + \sum_{i=1}^{w} \left(\binom{\ell_i}{2} r_i^2 + (\ell_i - 1) r_i s(\ell_i, \ell_i - 1) + s(\ell_i, \ell_i - 2) \right) \\ + \sum_{1 \le i_1 < i_2 \le w} (\ell_{i_1} r_{i_1} + s(\ell_{i_1}, \ell_{i_1} - 1)) (\ell_{i_2} r_{i_2} + s(\ell_{i_2}, \ell_{i_2} - 1)) \end{aligned} \tag{3.12}$$

Proof of the proposition. Using the lemma with $L = \ell w - x$, $\ell_j = \ell$ and $r_j = \ell + j$ shows that $g_\ell^{(w)}(x) = 0$ for $x < 0$, $g_\ell^{(w)}(0) = 1$ and $g_\ell^{(w)}(1) = \frac{1}{2} w(w + 1)\ell(\ell + 1)$. The case $x = 2$ requires the results

$$S(L + 2, L) = h_2(1, 2, \dots, L) = \sum_{1 \le i \le j \le L} ij \quad = L(L + 1)(L + 2)(3L + 1)/24,$$

$$s(\ell, \ell - 2) = e_2(1, 2, \dots, \ell - 1) = \sum_{1 \le i < j \le \ell - 1} ij \quad = \ell(\ell - 1)(\ell - 2)(3\ell - 1)/24.$$

which together with (3.12) yields

$$g_\ell^{(w)}(2) = \frac{1}{24} w\,(w+1)\;\ell\,(\ell+1)\;\big(3w(w+1)\ell^2 + (3w^2-5w-4)\ell - 4(w-1)\big)\,.$$

Here $h_2(e_2)$ is the complete(elementary) symmetric function of degree 2. Extension to greater values of x would involve higher degree symmetric functions.

The proposition follows from equations (3.6) and (3.5). □

We observe that for given $x, w > 0$, $g_\ell^{(w)}(x)$ is a polynomial in ℓ of degree $2x$ which contains a factor

$$\prod_{i=0}^{\lfloor \frac{1}{w}(x-1)\rfloor} (\ell-i)(\ell-i+1) \tag{3.13}$$

and that further linear factors occur, for example when $x=3$

$$\begin{aligned} g_\ell^{(w)}(3) = &\frac{1}{48} w(w+1)\ell(\ell+1)\times \\ &(\ell w-2)(\ell w+w-2)(\ell(w+1)-2)\,(\ell(w+1)+w-1)\,. \end{aligned} \tag{3.14}$$

Notice the expected symmetry under interchange of ℓ and w. Assuming this result gives

$$f_{\ell,w}(3) = \frac{(w+1)\,(\ell+1)\;(\ell w+w-2)(\ell(w+1)-2)\,(\ell(w+1)+w-1)\,C_{\ell,w}}{48(\ell w-1)} \tag{3.15}$$

The polynomials $g_\ell^{(w)}(x)$ for $x = 4, 5$ and early values of w are listed in appendix Appendix B. Similar data was used to obtain the formula for $x=3$ and data for higher odd values of x is consistent with the occurrence of the four factors

$$(\ell w-x+1)(\ell w+w-x+1)(\ell(w+1)-x+1)(\ell(w+1)+w-x+2). \tag{3.16}$$

Finally equation (3.1) yields the $w=1$ formula for general x

$$g_\ell^{(1)}(x) = \frac{1}{x!}\prod_{i=0}^{x-1}(\ell-i)(\ell-i+1) \tag{3.17}$$

4. Proof of proposition 2.

(a) Combining (1.13) with proposition 1(a)

$$\begin{aligned} G_{\ell,w}^{Fermi}(z) &= (1+z)^{\ell w+1} Q_{\ell,w}^{Bose}\left(\frac{z}{1+z}\right) && (4.1)\\ &= (1+z)^{\ell+w} Q_{\ell,w}^{Fermi}(z) && (4.2) \end{aligned}$$

where

$$Q_{\ell,w}^{Fermi}(z) = \sum_{k=0}^{(\ell-1)w-1} q_{\ell,w,k}^{Bose} z^k (1+z)^{(\ell-1)(w-1)-k} \tag{4.3}$$

is a polynomial of degree $(\ell-1)(w-1)$.

(b) Replace z by $-z$ in (4.3)

(c) The first equality is obtained by expanding (4.1) in powers of z and the second is (3.3).

(d) The first equation follows by expanding the factor $(1+z)^{(\ell-1)(w-1)-k}$ in (4.3). Combining (4.2) and (1.13) gives

$$Q^{Fermi}_{\ell,w}(z) = (1+z)^{-(\ell+w)} G^{Bose}_{\ell,w}\left(\frac{z}{1+z}\right) \tag{4.4}$$

and the second equation follows by expanding in powers of z.

(e) The first equality is obtained on replacing n by $(\ell-1)(w-1)-n$ in part (d) and using proposition 1, parts (e) and (c). The second is obtained by expanding the binomial coefficient in terms of Stirling numbers using

$$k(k-1)\dots(k-n+1) = \sum_{m=0}^{n} s(n,m)k^m. \tag{4.5}$$

(f) Set $n = 0$ in part (e) and use (1.8).

(g) Replace z by $1/z$ in (4.3) and use proposition 1(e).

5. Recurrence Relations.

5.1. Proof of proposition 3.

(a) In [14] it was shown that by duality that $f^{Bose}_{\ell,w,n}$ is invariant under any permutation of its indices. Equation (1.2) can therefore be rewritten in the form

$$f^{Bose}_{\ell,w,n} = \prod_{j=1}^{\ell} \frac{(j+k)_w}{(j)_w} \tag{5.1}$$

and hence

$$f^{Bose}_{\ell,w,k} = \frac{(\ell+k)_w}{(\ell)_w} f^{Bose}_{\ell-1,w,k}. \tag{5.2}$$

from which the result follows.

(b) The proof of this part may be found in [8] and uses the Pfaff-Saalschütz identity [29]

$$\binom{\ell+w+k-1}{w}\binom{\ell w+1}{n-k} = \sum_{h=0}^{w}(-1)^h \binom{(\ell-1)(w-1)-n-h}{h}\binom{\ell+w+n-h-1}{w-h}\binom{(\ell-1)w+1}{n-h-k}. \tag{5.3}$$

Our proof of parts (c) and (d) uses a similar technique.

(c) The proposition may be written in the form

$$q^{Fermi}_{\ell,w,n} = \frac{(\ell+n)_w}{(\ell)_w}\sum_{h=0}^{w-1}\binom{w-1}{h} q^{Fermi}_{\ell-1,w,n-h} \tag{5.4}$$

A variant of the Vandermonde convolution ([30] section 1.3, equation (5)) gives

$$\binom{\ell+n-1}{n-k} = \sum_{h=0}^{w-1} (-1)^h \binom{w-1}{h} \binom{n-h+\ell+w-2}{n-h-k} \tag{5.5}$$

Combining this with (5.2) gives

$$\begin{aligned}\binom{n+\ell+w-1}{n-k} f^{Bose}_{\ell,w,k} &= \frac{(\ell+n)_w}{(\ell)_w} \binom{\ell+n-1}{n-k} f^{Bose}_{\ell-1,w,k} \\ &= \frac{(\ell+n)_w}{(\ell)_w} \sum_{h=0}^{w-1} \binom{w-1}{h} (-1)^h \binom{n-h+\ell+w-2}{n-h-k} f^{Bose}_{\ell-1,w,k}\end{aligned} \tag{5.6}$$

Multiplying by $(-1)^{n-k}$, summing over k and using proposition 2(d) completes the proof.

(d) The following identity has been verified using Paule and Schorn's Mathematica implementation [23] of Zeilberger's algorithm [18].

$$\begin{aligned}&\binom{\ell+w+k-1}{w} \binom{n-1}{n-k} = \\ &\qquad \sum_{h=0}^{w} (-1)^h \binom{n-1}{h} \binom{\ell+w+n-h-1}{w-h} \binom{n-h-1}{n-h-k}.\end{aligned} \tag{5.7}$$

Using part (a)

$$\begin{aligned}&\binom{\ell+w-1}{w} \binom{n-1}{n-k} f^{Bose}_{\ell,w,k} = \\ &\qquad \sum_{h=0}^{w} (-1)^h \binom{n-1}{h} \binom{\ell+w+n-h-1}{w-h} \binom{n-h-1}{n-h-k} f^{Bose}_{\ell-1,w,k}.\end{aligned} \tag{5.8}$$

Multiplying by $(-1)^{n-k}$, summing over k and using proposition 2(c) completes the proof.

5.2. Recurrence relations for generating functions.

Equation 9 of [8] is a recurrence relation for the Naryana polynomials which translates to

$$(\ell+1)Q^{Bose}_{\ell,2}(z) = (2\ell-1)(1+z)Q^{Bose}_{\ell-1,2}(z) - (\ell-2)(1-z)^2 Q^{Bose}_{\ell-2,2}(z) \tag{5.9}$$

Now using (4.1) and (4.2)

$$Q^{Bose}_{\ell,w}\left(\frac{z}{1+z}\right) = \frac{Q^{Fermi}_{\ell,w}(z)}{(1-z)^{(w-1)(\ell-1)}}. \tag{5.10}$$

Replacing z by $z/(1+z)$ in the recurrence relation and using (5.10) with $w = 2$ leads to the corresponding Fermi recurrence relation

$$(\ell+1)Q^{Fermi}_{\ell,2}(z) = (2\ell-1)(1+2z)Q^{Fermi}_{\ell-1,2}(z) - (\ell-2)Q^{Fermi}_{\ell-2,2}(z). \tag{5.11}$$

For $w = 3$, Proposition 7 of [8] gives

$$\begin{aligned}
&(3\ell-4)(\ell+1)^2(\ell+2)Q^{Bose}_{\ell,3}(z) = \\
&\qquad (\ell+1)\,(3\ell-2)\left[4\,\left(1+z+z^2\right)+\ell\,(3\ell-5)\,\left(1+7z+z^2\right)\right]Q^{Bose}_{\ell-1,3}(z) \\
&\qquad +(\ell-2)\left(12-29\,\ell+30\,\ell^2-9\,\ell^3\right)(1-z)^4 Q^{Bose}_{\ell-2,3}(z) \\
&\qquad +(\ell-2)\,(\ell-3)\,(\ell-4)\,(3\ell-1)\,(1-z)^6 Q^{Bose}_{\ell-3,3}(z)
\end{aligned} \tag{5.12}$$

and the corresponding Fermi recurrence is

$$\begin{aligned}
&(3\ell-4)(\ell+1)^2(\ell+2)Q^{Fermi}_{\ell,3}(z) = \\
&\qquad (\ell+1)\,(3\ell-2)\left[4\,\left(1+3z+3z^2\right)+\ell\,(3\ell-5)\,\left(1+9z+9z^2\right)\right]Q^{Fermi}_{\ell-1,3}(z) \\
&\qquad +(\ell-2)\left(12-29\,\ell+30\,\ell^2-9\,\ell^3\right)Q^{Fermi}_{\ell-2,3}(z) \\
&\qquad +(\ell-2)\,(\ell-3)\,(\ell-4)\,(3\ell-1)\,Q^{Fermi}_{\ell-3,3}(z)
\end{aligned} \tag{5.13}$$

These relations are valid for $\ell \geq 2$ and may be initialised by $Q^{Bose}_{1,w}(z) = Q^{Fermi}_{1,w}(z) = 1$ and arbitrary values for $\ell \leq 0$. Notice that in the Fermi case only the coefficient of $Q^{Fermi}_{\ell-1,w}(z)$ depends on z.

6. Configurations in which each path is used at most m times

In this section we examine the generating function $H^{(m)}_{w,n}(z)$ defined by equation (2.9). It was shown in [14] that

$$H^{Fermi}_{w,n}(z) \equiv H^{(1)}_{w,n}(z) = \frac{P^{Fermi}_{w,n}(z)}{(1-z)^{wn+1}} \tag{6.1}$$

where

$$P^{Fermi}_{w,n}(z) = \sum_{k=1}^{n}(-1)^{n-k}\binom{n-1}{k-1}(1-z)^{(n-k)w}Q^{Bose}_{w,k}(z). \tag{6.2}$$

The proof used the previously mentioned symmetry of $f^{Bose}_{\ell,w,n}$ under interchange of ℓ and n. The degree $(n-1)w$ of this polynomial is determined by the $k = 1$ term and the coefficient of $z^{(n-1)w}$ is $(-1)^{(n-1)(w+1)}$. Setting $z = 1$ gives $P^{Fermi}_{w,n}(1) = Q^{Bose}_{w,n}(1) = C_{w,n}$. Tables of $Q^{Bose}_{w,n}(z)$ and $P^{Fermi}_{w,n}(z)$ for low values of n and w are given in an appendix to [14].

The extension to general m follows from (6.1) by using (2.7) to show that

$$H^{(m)}_{w,n}(z) = \sum_{k=\lceil \frac{n}{m}\rceil}^{n} C^{(m)}_{k,n-k}H^{Fermi}_{w,k}(z) \tag{6.3}$$

$$= \frac{P^{(m)}_{w,n}(z)}{(1-z)^{wn+1}} \tag{6.4}$$

where

$$P^{(m)}_{w,n}(z) = \sum_{k=\lceil \frac{n}{m}\rceil}^{n} C^{(m)}_{k,n-k}(1-z)^{(n-k)w}P^{Fermi}_{w,k}(z). \tag{6.5}$$

w	1	2	3	4
n				
1	1	1	1	1
2	1	$1+z$	$1+3z+z^2$	$1+6z+6z^2+z^3$
3	$2z-z^2$	$7z-5z^2+4z^3-z^4$	$16z+5z^2+30z^3-14z^4+6z^5-z^6$	$30z+85z^2+246z^3+43z^4+78z^5-27z^6+8z^7-$
4	$z+z^2-z^3$	$6z+17z^2-23z^3$ $+21z^4-8z^5+z^6$	$19z+153z^2+68z^3+299z^4-142z^5$ $+89z^6-30z^7+7z^8-z^9$	$45z+804z^2+3198z^3+8407z^4+6796z^5+4360z$ $+130z^7+418z^8-191z^9+70z^{10}-14z^{11}+z^{12}$

Table 1. The polynomials $P^{(2)}_{w,n}(z)$

Setting $z=1$ shows that the sum of the numerator coefficients is an $n-$dimensional Catalan number, $P^{(m)}_{w,n}(1) = P^{Fermi}_{w,n}(1) = C_{w,n}$.

Now the degree of each term in (6.5) is $(n-1)w$ but expanding the factor $(1-z)^{w(n-k)}$ gives positive and negative terms so that the degree of $P^{(m)}_{w,n}(1)$ is at most $(n-1)w$. The coefficient of $z^{(n-1)w}$ in $P^{(m)}_{w,n}(z)$ is

$$[z^{(n-1)w}]P^{(m)}_{w,n}(z) = (-1)^{(n-1)w-1} \sum_{k=\lceil \frac{n}{m} \rceil}^{m} (-1)^k C^{(m)}_{k,n-k} \tag{6.6}$$

$$= (-1)^{(n-1)w-1} \begin{cases} 1 & n=0 \mod m+1 \\ -1 & n=1 \mod m+1 \\ 0 & \text{otherwise} \end{cases} \tag{6.7}$$

The maximum degree is therefore only achieved when $n=0$ or $1 \mod m+1$. Table 6 shows a sample of the numerator polynomials in the case $m=2$.

Now $C^{(m)}_{k,i}$ becomes independent of m for $m \geq k+i$. From (6.5) it follows that for $m \geq n$, $P^{(m)}_{w,n}(z)$ is independent of m as expected since there are not enough walks to exceed the upper limit. In this case the configurations enumerated by $H^{(m)}_{w,n}(z)$ using (6.3) are just the Bose configurations so that for $m \geq n$, $P^{(m)}_{w,n}(z) = Q^{Bose}_{w,n}(z)$ and the degree is therefore $(n-1)(w-1)$.

7. Summary and conclusion

The number of configurations of n fully directed walks between the corners of an ℓ by w rectangular grid and subject to the Fermi condition of Inui and Katori [1] has been investigated. It has been shown that the maximum number of walks in such a configuration is $\ell w+1$ and the number of maximal configurations is the generalised Catalan number $C_{\ell,w}$. Product formulae have been found for fixed numbers of walks close to the maximum but complete factorisation eventually fails as the number of walks is reduced (see Appendix B). As a consequence of the upper limit on the number of walks in a Fermi configuration the generating function $G^{Fermi}_{\ell,w}(z)$ is a polynomial whereas the Bose generating function is an infinite series. In a companion paper [14] this led to proposition 1 which shows that $G^{Bose}_{\ell,w}(z)$ is a rational function whose numerator $Q^{Bose}_{\ell,w}(z)$ is a generalised Naryana polynomial [8]. Here we use this to show (proposition 2) that $G^{Fermi}_{\ell,w}(z)$ may be factorised as $(1+z)^{\ell+w}Q^{Fermi}_{\ell,w}(z)$ where the second factor is related to $Q^{Bose}_{\ell,w}(z)$. The number of Fermi configurations $f^{Fermi}_{\ell,w,n}$ is found to satisfy a recurrence relation (proposition 3) similar to that obtained by Sulanke for the generalised Narayana numbers. The coefficients of the polynomial $Q^{Fermi}_{\ell,w}(z)$ satisfy a recurrence relation simpler than that for $f^{Fermi}_{\ell,w,n}$. The generating function $H^{(m)}_{w,n}(z)$ for the number of configurations of non-crossing walkers, with fixed w and n, in which the number of walkers on a given path is at most m is also considered. It is shown to be a rational function similar to $G^{Bose}_{\ell,w}(z)$. It is clear from the results of this paper that there is considerable scope for bijective combinatorics involving Fermi configurations. This will be investigated in a subsequent study [25].

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Appendix A. The numbers of Fermi walk configurations for $w = 2, 3$ and 4.

(a) $w = 2$

ℓ / n	0	1	2	3	4	5	6	7	8
1	1	3	6	10	15	21	28	36	45
2	0	3	14	40	90	175	308	504	780
3	0	1	16	85	295	805	1876	3906	7470
4	0	0	9	105	594	2331	7280	19404	45990
5	0	0	2	76	771	4529	19348	66780	197484
6	0	0	0	30	650	6083	36644	166608	621180
7	0	0	0	5	345	5685	50464	309537	1476135
8	0	0	0	0	105	3640	50813	434493	2701610
9	0	0	0	0	14	1526	37100	462952	3849715
10	0	0	0	0	0	378	19152	372708	4288140
11	0	0	0	0	0	42	6636	223272	3724140
12	0	0	0	0	0	0	1386	96558	2497110
13	0	0	0	0	0	0	132	28512	1268190
14	0	0	0	0	0	0	0	5148	471900
15	0	0	0	0	0	0	0	429	121407
16	0	0	0	0	0	0	0	0	19305
17	0	0	0	0	0	0	0	0	1430

(b) $w = 3$

ℓ	0	1	2	3	4	5	6	7	8
n									
1	1	4	10	20	35	56	84	120	165
2	0	6	40	155	455	1120	2436	4830	8910
3	0	4	85	650	3171	11816	36624	99120	241560
4	0	1	105	1681	13783	77560	340116	1245300	3972144
5	0	0	76	2848	40411	346136	2147412	10600248	44034606
6	0	0	30	3235	83475	1107352	9776688	65119605	351728685
7	0	0	5	2450	124265	2624240	33351627	301118850	2117140795
8	0	0	0	1190	134288	4698883	87461913	1079098615	9912173425
9	0	0	0	336	104608	6421968	179322472	3059222880	36934857883
10	0	0	0	42	57330	6711852	290434872	6960555504	111397726440
11	0	0	0	0	21000	5329632	373396758	12834078180	275329223610
12	0	0	0	0	4620	3162390	380973582	19287937350	562596057870
13	0	0	0	0	462	1359072	306746088	23681211840	956041192260
14	0	0	0	0	0	399828	192564372	23726383395	1355652938211
15	0	0	0	0	0	72072	92303211	19305532818	1605558804135
16	0	0	0	0	0	6006	32636604	12640617990	1585882849980
17	0	0	0	0	0	0	8024016	6561755200	1301064338860
18	0	0	0	0	0	0	1225224	2638656020	880193482740
19	0	0	0	0	0	0	87516	792603240	485591190084

(c) $w = 4$

ℓ	0	1	2	3	4	5	6	7
n								
1	1	5	15	35	70	126	210	330
2	0	10	90	455	1694	5166	13650	32340
3	0	10	295	3171	21238	105966	429870	1492260
4	0	5	594	13783	163982	1313046	8012850	39963792
5	0	1	771	40411	856366	10909746	98928336	699551061
6	0	0	650	83475	3201050	64848960	868345125	8632370175
7	0	0	345	124265	8877688	287867997	5684065080	79065603265
8	0	0	105	134288	18689069	982887633	28699196855	557695903325
9	0	0	14	104608	30269162	2634822946	114539208070	3113010521852
10	0	0	0	57330	37940910	5623892190	367901819460	14036581816950
11	0	0	0	21000	36765750	9644235300	963715352250	51935620238250
12	0	0	0	4620	27306048	13350415452	2078439922680	159600327635430
13	0	0	0	462	15263226	14929494822	3714545258820	411116322541815
14	0	0	0	0	6216210	13442303160	5522111069475	893801821197285
15	0	0	0	0	1741740	9667362705	6837708147270	1648096898999265
16	0	0	0	0	300300	5476390920	7043966829900	2585470479091500
17	0	0	0	0	24024	2389583196	6013865497640	3455749990733040
18	0	0	0	0	0	774954180	4225562012100	3934689718087700
19	0	0	0	0	0	175907160	2416869569400	3809080075647132
20	0	0	0	0	0	24942060	1107021739824	3123364634717184
21	0	0	0	0	0	1662804	396290534640	2156256587104620

Appendix B. The polynomials $g_\ell^{(w)}(x)$

$$g_\ell^{(w)}(x) = \frac{1}{x!} w(w+1)\ell(\ell+1) f_\ell^{(w)}(x)$$

(a) $x = 4$

w	$\mathbf{f_\ell^{(w)}(4)}$
1	$\frac{1}{2}(-3+\ell)(-2+\ell)^2(-1+\ell)^2\ell$
2	$\frac{1}{2}(-1+\ell)^2\ell(4+4\ell-45\ell^2+27\ell^3)$
3	$2(-1+\ell)\ell(4+9\ell-22\ell^2-36\ell^3+54\ell^4)$
4	$\frac{2}{5}(-18-57\ell+212\ell^2+393\ell^3-850\ell^4-750\ell^5+1250\ell^6)$
5	$\frac{1}{10}(-288+28\ell+3512\ell^2-577\ell^3-13475\ell^4+1125\ell^5+16875\ell^6)$
6	$\frac{1}{2}(-144+292\ell+1556\ell^2-2407\ell^3-6209\ell^4+4851\ell^5+9261\ell^6)$

(b) $x = 5$

w	$\mathbf{f_\ell^{(w)}(5)}$
1	$\frac{1}{2}(-4+\ell)(-3+\ell)^2(-2+\ell)^2(-1+\ell)^2\ell$
2	$\frac{1}{2}(-2+\ell)(-1+\ell)^2\ell(-4+3\ell)(-1+3\ell)(-2-13\ell+9\ell^2)$
3	$4(-1+\ell)^2\ell(-4+3\ell)(-1+3\ell)(-2-7\ell-2\ell^2+18\ell^3)$
4	$4(-1+\ell)\ell(-4+5\ell)(1+5\ell)(6+5\ell-29\ell^2-20\ell^3+50\ell^4)$
5	$\frac{1}{2}(-2+3\ell)(1+3\ell)(-4+5\ell)(1+5\ell)(24-6\ell-133\ell^2+10\ell^3+225\ell^4)$
6	$\frac{1}{2}(-2+3\ell)(1+3\ell)(-4+7\ell)(3+7\ell)(40-46\ell-229\ell^2+154\ell^3+441\ell^4)$
7	$(-2+4\ell)(2+4\ell)(-4+7\ell)(3+7\ell)(30-59\ell-167\ell^2+224\ell^3+392\ell^4)$
8	$(-2+4\ell)(2+4\ell)(-4+9\ell)(5+9\ell)(42-115\ell-215\ell^2+480\ell^3+648\ell^4)$
9	$\frac{1}{2}(-2+5\ell)(3+5\ell)(-4+9\ell)(5+9\ell)(112-390\ell-493\ell^2+1770\ell^3+2025\ell^4)$
10	$\frac{1}{2}(-2+5\ell)(3+5\ell)(-4+11\ell)(7+11\ell)(144-606\ell-493\ell^2+2970\ell^3+3025\ell^4)$

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