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Directed compact percolation near a wall: II. Cluster length and size

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Abstract. The mean cluster size and length for the unbiassed growth of compact clusters near a dry wall are considered. In the case of the cluster size below p_c an exact expression is obtained for a seed of arbitrary width and distance from the surface. It is found that the critical exponent $\gamma = 1$ for any finite distance from the surface. Crossover to the bulk value $\gamma = 2$ as the distance from the surface tends to infinity is observed. This extends an existing result for the exponent β of the percolation probability which changes from a value of 2 in the presence of a surface to 1 in the bulk limit. The value $\Delta = 3$ of the scaling size exponent is unchanged by the introduction of the surface.

The cluster size above p_c and the mean cluster length are investigated using differential approximants from which we conjecture that these functions satisfy second-order differential equations. Accepting this conjecture gives a mean size exponent the same as below p_c and a logarithmic divergence of the mean length from both sides of the critical point. The latter result together with scaling theory predicts that the exponent $\nu_\parallel$ has the value 2, the same as for the bulk problem.

1. Introduction

The compact percolation model [2–4] was introduced by Domany and Kinzel [2]. The model is a simplification of the standard percolation model on the directed square lattice in that the clusters grow in such a way that no holes are formed. This allows the cluster shape to be parametrized by two random walk variables and, in the absence of boundaries, all of the usual properties of percolation clusters can be calculated analytically. The introduction of a wall which restricts the lateral growth of the cluster makes the analytical calculations far more complicated but they can still be carried through in the case of the percolation probability [7]. Here we consider the mean cluster size and length in the presence of a dry wall and find an analytic solution for the mean size below the percolation threshold. Above the threshold and on both sides of the threshold for the mean length we find that the functions appear to satisfy second-order linear differential equations with polynomial coefficients.

Domany and Kinzel [2] calculated the percolation probability and connectedness length for the unrestricted directed square lattice. Other properties such as the mean cluster length and size considered here were calculated in the absence of boundaries by Essam [3].

The introduction of a dry wall parallel to the symmetry axis of the directed square lattice, thereby restricting the lateral cluster growth, was found [1, 7] to change the critical exponent of the percolation probability from its bulk value $\beta = 1$ to $\beta = 2$. In an earlier

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paper [5], herein referred to as I, it was shown that this new value of β was dependent on the bulk growth direction being parallel to the wall. Introduction of a bias in the growth direction, by allowing different probabilities p_u and p_d for expansion of the cluster away from and towards the wall respectively, caused β to revert to its bulk value. The details of the phase diagram are given in I, where it was also shown that extreme bias towards the wall was equivalent to having a wet wall boundary condition.

In our calculations of the mean cluster size and length for the dry-wall problem we have found it necessary to restrict attention to the unbiassed case $p_u = p_d = p$. This is where new exponents are expected to be found and introduction of a bias will almost certainly cause reversion either to the bulk or to the wet wall exponents (which happen to be the same for this model).

The mean cluster size and length satisfy linear recurrence relations similar to those for the percolation probability except for the occurrence of additional inhomogeneous terms. For the percolation probabilty it was sufficient to impose the boundary condition that it remain finite on moving the seed infinitely far from the wall. Here the known bulk values for the cluster size and length must result in this limit. In the case of the mean size an exact solution of these relations has been found for $p < p_c$ resulting in an exponent $\gamma = 2$ compared with $\gamma = 1$ for the bulk. The scaling relation $\Delta = \beta + \gamma$ for the scaling size exponent gives $\Delta = 3$, the same as for the bulk. The exponent is the same for all seed widths and distances from the wall.

In the case of the mean cluster size for $p > p_c$ and mean cluster length on both sides of p_c no closed-form expressions have been obtained. The corresponding properties in the bulk are simple rational expressions but introduction of the wall appears to make these functions far less trivial. We have therefore used recurrence relations to derive high- and low-density series expansions to high order for the case of a seed of width one adjacent to the wall. Using the method of differential approximants, recently reviewed by one of us [6], we have discovered linear recurrence relations satisfied by the series coefficients which appear to be exact. The order of these relations depends on the function, but the coefficients are always polynomials of at most degree two. The corresponding second-order differential equations have singular points, one of which is the critical point and the corresponding indices determine the critical exponents. It is found that the mean size diverges as p_c is approached from above with exponent $\gamma = 1$, the same as below p_c, but there is a confluent analytic term which vanishes with exponent 3. The mean length diverges logarithmically from both sides of p_c and therefore its critical exponent $\tau = 0$.

The value $\tau = 0$, although not rigorously obtained, is also supported by scaling theory. Applying this theory to compact percolation [3] it was shown that the parallel scaling length exponent is given by $\nu_{\|} = \tau + \beta$. Thus with $\tau = 0$ and $\beta = 2$ we obtain $\nu_{\|} = 2$ which is the same as for the bulk lattice [2]. This result is similar in nature to the result that the scaling size exponent does not change on introducing the wall. We have established the latter result rigorously as stated above.

2. The mean cluster size near a dry wall

The general compact percolation model in the presence of a wall was described in I, where a diagram of a typical cluster near a dry wall may be found. We consider a restricted directed square lattice whose sites are the points of the $t - x$ plane such that $t \geqslant 0, x \geqslant 0$ and $t + x$ is even. The cluster grows from a seed of m atoms which occupy contiguous sites in the column $t = 0$. The position of the seed will be specified by its centre-of-mass co-ordinate y. Thus the occupied sites have $x = n, n + 2, n + 4, \ldots, n + 2(m - 1)$ where

$n = y - m + 1 \geqslant 0$ and hence the least value of y is $m - 1$ which corresponds to the seed being adjacent to the wall. The random growth takes place by one column at each time step. A site in column t is occupied with probability 1 if both its predecessors in column $t - 1$ are occupied and with probability p if just one of its two predecessors is occupied; otherwise the site will be unoccupied. If there is only one atom in column $t - 1$ the cluster terminates with probability $(1-p)^2$. For p below the percolation threshold p_c, termination at some stage will occur with probability 1. The size of a cluster which terminates will be defined as the total number of atoms it contains including the seed atoms.

2.1. General difference equations

The mean size of finite clusters with seed of width m and centre of mass y will be denoted by $S_{m,y}(p)$. The functional dependence on p will be dropped in situations where a fixed value of p is considered. A recurrence relation for the mean size may be obtained by a simple extension of the argument in section 3 of I for the termination probability. By translational invariance in t, any finite cluster C, with at least one growth stage, may be constructed by concatenating its seed together with a cluster C' having one less growth stage. The seed of C' is in column $t = 1$ and its width and centre of mass will be denoted by m' and y' respectively. If $y \geqslant m$, so that the seed is not adjacent to the wall, then the possibilities for (m', y') are $(m, y+1)$, $(m, y-1)$, $(m+1, y)$ and $(m-1, y)$ with respective probabilities $a = b = p(1-p), c = p^2$, and $d = (1-p)^2$. In all cases C' has m more atoms than C. This leads to the recurrence relation

$$\bar{S}_{m,y} = a\bar{S}_{m,y+1} + b\bar{S}_{m,y-1} + c\bar{S}_{m+1,y} + d\bar{S}_{m-1,y} + mQ_{m,y} \qquad \text{for } y \geqslant m \geqslant 1 \tag{1}$$

where $Q_{m,y}(p) = 1 - P_{m,y}(p)$ is the probability that C is finite, i.e. the complement of the percolation probability $P_{m,y}(p)$ determined in I. The factor $Q_{m,y}$ is necessary above p_c since only finite clusters are counted. Also for $p > p_c$ it is the unnormalized mean size

$$\bar{S}_{m,y} = S_{m,y}Q_{m,y} \tag{2}$$

which appears in the relation. Below p_c, $Q_{m,y} = 1$ and the bar may be dropped. To make (1) valid for $m = 1$ we have imposed the boundary condition

$$\bar{S}_{0,y} = 0 \qquad \text{for } y \geqslant m - 1. \tag{3}$$

If the seed is adjacent to the wall, i.e. $y = m - 1$, only two of the above possibilities occur and hence

$$\bar{S}_{m,m-1} = p\bar{S}_{m,m} + (1-p)\bar{S}_{m-1,m-1} + mQ_{m,m-1} \qquad m \geqslant 1. \tag{4}$$

Finally in the limit $y \to \infty$ the bulk mean size, which has been calculated in [4], must result as

$$\lim_{y\to\infty} S_{m,y} = S_m(\text{bulk}) = \frac{m}{|1-2p|}\left(\frac{1}{1-u} + \frac{m-1}{2}\right) \tag{5}$$

where

$$u = \begin{cases} \left(\dfrac{p}{1-p}\right)^2 & \text{for } p < p_c \\ \left(\dfrac{1-p}{p}\right)^2 & \text{for } p > p_c. \end{cases} \tag{6}$$

Equation (4) may be included as the case $y = m - 1$ of (1) provided we impose the simpler boundary condition

$$\bar{S}_{m+1,m-1} = \bar{S}_{m,m} \qquad \text{for } m \geqslant 1. \tag{7}$$

2.2. Exact solution for the low-density region

In I a number of solutions of the homogeneous part of (1) were given including $m(p/(1-p))^{y-m}$. This solution is appropriate for $p < p_c$ since it vanishes as $y \to \infty$ and may therefore be added to S_m(bulk), which is a particular solution of the inhomogeneous equation, in an attempt to include the effect of the wall in the low-density region. Hence

$$S_{m,y} = S_m(\text{bulk}) + Am\left(\frac{p}{1-p}\right)^{y-m}. \tag{8}$$

The modified function still satisfies (3) so it remains to satisfy (7). It turns out that this equation may be satisfied for all m and y by an appropriate choice of A which results in the required solution

$$S_{m,y}(p) = S_m(\text{bulk}) - \frac{mp^2}{(1-2p)^2}\left(\frac{p}{1-p}\right)^{y-m}. \tag{9}$$

Notice that, for finite y, on collecting the terms on the right of (9) over the common denominator $(1-2p)^2$, the numerator has a simple zero at $p = p_c$ and hence the pole of multiplicity two in the bulk mean size is reduced to a simple pole. This crossover phenomenon parallels that observed by Bidaux and Privman [1] for the percolation probability. Here $\gamma = 1$ for any finite distance of the seed from the wall and $\gamma = 2$ in the bulk limit.

For the case of a source of width 1 adjacent to the wall (9) reduces to the simple result

$$S_{1,0}(p) = \frac{1-p}{1-2p}. \tag{10}$$

2.3. Differential approximant for the high-density region

In the region $p > p_c$ we have been unable to find a closed solution for the mean size. Even the case $m = 1$, $y = 0$, which had such a simple form in the low-density region, appears to be a much more complicated function but still having a simple pole at $p = p_c$ as we show below. Above the critical probability it is convenient to work with the variable $q = 1 - p$ and the unnormalized mean size in this region, $\bar{S}_{m,y}(q)$, will be considered to be a function of q.

We have used the recurrence relation (1) together with the boundary conditions (3) and (7) to obtain the coefficients S_n^- in the expansion of $\bar{S}_{1,0}(q)$, in powers of q. The first 50 of these are given in table 1.

Using the method of differential approximants [6] we have found the following eighth-order recurrence relation:

$$\begin{aligned}(n+n^2)S_n^- &+ (12 - 14n - 4n^2)S_{n-1}^- - (276 - 170n + 10n^2)S_{n-2}^- \\ &+ (1764 - 888n + 92n^2)S_{n-3}^- - (5292 - 2321n + 239n^2)S_{n-4}^- \\ &+ (8640 - 3370n + 320n^2)S_{n-5}^- - (7920 - 2780n + 240n^2)S_{n-6}^- \\ &+ (3840 - 1224n + 96n^2)S_{n-7}^- - (768 - 224n + 16n^2)S_{n-8}^- = 0\end{aligned} \tag{11}$$

Table 1. *Coefficients in the series expansions of the mean cluster length and size. Superscripts* $+$ and $-$ denote expansion in powers of p and q, respectively.

n	S_n^-	L_n^+	L_n^-
0	0	1	0
1	1	1	1
2	5	2	5
3	15	3	13
4	36	6	26
5	80	9	46
6	169	20	76
7	351	26	122
8	714	76	193
9	1454	55	309
10	2911	364	496
11	5913	−166	830
12	11676	2484	1366
13	24004	−4851	2468
14	46077	24000	3970
15	98999	−72432	8356
16	174598	288912	10775
17	432642	−1017690	35227
18	564731	3903064	9280
19	2232601	−14609756	221846
20	362012	56335208	−344765
21	16189948	−217435185	2212511
22	−26593263	851023056	−6716082
23	174099539	−3351972840	29204664
24	−536319586	13322388384	−107460436
25	2400457482	−53310340434	434311674
26	−8948736489	214834999656	−1697451846
27	36950460701	−871142160820	6819637388
28	−146432266580	3553515113624	−27324761058
29	598046374572	−14574943454891	110704352472
30	−2428248089915	60089962052040	−450235182842
31	9971731818967	−248941489436604	1843371746624
32	−41046280290754	1036028744008104	−7583418314021
33	170016417511178	−4330213271481604	31359356687263
34	−706983201784077	18172134285471392	−130269792686128
35	2953533400889737	−76553535622943120	543557877573910
36	−12387332620785604	323667432366676256	−2277370904014227
37	52157739595768652	−1373173925444302666	9579078148020853
38	−220408670416384231	5844798741884835616	−40440630589263970
39	934647346871848563	−24955209713989173648	171330935250807560
40	−3976395591359356878	106865071028472444480	−728280149776067223
41	16970271504638516166	−458915807962198781844	3105524856214947297
42	−72640332507993679825	1976045229848720391376	−13282463760765322528
43	311815104049285592989	−8530511571395729344040	56972825027755567782
44	−1342115178248119879824	36916349882276312070704	−245044416740421163617
45	5791663454007644781560	−160133503456191982607681	1056711040783424874651
46	−25054654976322974240891	696181704767586459885400	−4568272373796435661454
47	108642345857722136827911	−3033191096134059989064724	19796355221796309298772
48	−472163365715479997585166	13242682030797835674099576	−85982909628630322852352
49	2056493925139861465289014	−57931613900000077033802820	374275637842510127770574
50	−8975663037610753197414789	253913241702633188283664704	−1632619098498842327897174

which, together with $S_0^-, \ldots, S_7^-$, generates all the coefficients in table 1. Accepting the correctness of this relation for all values of n, it follows that $\bar{S}_{1,0}(q)$ satisfies the following second-order differential equation:

$$
\begin{aligned}
q(1-q)^2(1-2q)^2 & \left(1+4q-4q^2\right)\bar{S}''_{1,0}(q) \\
& +2(1-q)(1-2q)\left(1-8q+11q^2+12q^3-12q^4\right)\bar{S}'_{1,0}(q) \\
& +6\left(-1+2q-7q^2+12q^3-4q^4\right)\bar{S}_{1,0}(q) \\
& = \frac{2-2q-6q^2+4q^3+30q^4-48q^5+24q^6}{(1-q)^2}.
\end{aligned} \tag{12}
$$

The regular singular points of this equation and their corresponding exponents are listed in table 2.

Table 2. Singular points and exponents for the differential equations satisfied by the mean cluster size and length. A positive exponent corresponds to an algebraically diverging solution and a zero exponent corresponds to logarithmic divergence. In the case of confluent singularities two exponents are given.

	$q=0$	$q=\frac{1}{2}$	$q=1$	$q=\frac{1\pm\sqrt{2}}{2}$	$q=\infty$
Mean size	1	1 −3	4 3	−4	−2
Mean length	0	0	3 3	−4	−1 −1

The exponents 1 and −3 at the critical point $q_c = \frac{1}{2}$ suggest that the leading asymptotic form of $\bar{S}_{1,0}(q)$ is

$$\bar{S}_{1,0}(q) \cong \frac{A^-}{1-2q} \tag{13}$$

with a possible confluent singularity $(1-2q)^3\log(1-2q)$. Substituting $\bar{S}_{1,0}(q) = A^-(q)/(1-2q)$ in (12) removes the leading singularity and gives a differential equation for $A^-(q)$ which may be used to find the amplitude. The value of $A^-(q)$ and its derivative at some small value of q may be accurately obtained from the series expansion of $\bar{S}_{1,0}(q)$ and integrating the differential equation numerically from this point to q_c determines the amplitude $A^- = A^-(q_c) = 2.895\,3042\ldots$ which is considerably higher than the low-density amplitude of 0.5. The value of A^- may also be confirmed using Padé approximants.

3. Differential approximant for the mean cluster length

The normalized mean cluster length for the bulk problem was shown in [3] to be the following simple function:

$$L_m(\text{bulk}) = \frac{m}{|1-2p|} \tag{14}$$

which is valid above and below p_c. The cluster length is defined to be the number of atoms in the shortest path from the seed to the terminal point (one more than the number of growth stages).

The recurrence relations satisfied by the unnormalized mean cluster length $\bar{L}_{m,y}(p)$ in the presence of a dry wall are the same as those for the mean size, except that the factor m

in (1) and (4) is replaced by unity. This apparent simplification in fact makes the problem more complicated and we have been unable to obtain a closed form solution even below p_c.

These modified recurrence relations have therefore been used to obtain the coefficients L_n^+ and L_n^- in the series expansions of $\bar{L}_{1,0}$ in powers of p and q respectively. The first 50 terms are listed in table 1.

We find that the coefficients L_n^+ at least as far as L_{50}^+ are generated from $L_0^+ \cdots L_3^+$ by the following recurrence relation:

$$(9+6n+n^2)L_n^+ - (18+11n-n^2)L_{n-1}^+ + (6+24n-14n^2)L_{n-2}^+ \\ +(48-68n+20n^2)L_{n-3}^+ - 8(9-6n+n^2)L_{n-4}^+ = 0 \qquad (15)$$

which implies that $L_{1,0}(p)$ satisfies the following differential equation:

$$p^2(1-p)(1-2p)(1+4p-4p^2)L_{1,0}''(p) + p(7-8p-46p^2+72p^3-24p^4)L_{1,0}'(p) \\ +(9-28p-2p^2+24p^3-8p^4)L_{1,0}(p) = 9-12p+12p^2. \qquad (16)$$

The coefficients L_n^- satisfy the difference equation

$$n^2L_n^- + (24-17n-2n^2)L_{n-1}^- - (264-159n+14n^2)L_{n-2}^- \\ +(1080-563n+64n^2)L_{n-3}^- - (2184-1001n+111n^2)L_{n-4}^- \\ +(2352-960n+98n^2)L_{n-5}^- - (1296-476n+44n^2)L_{n-6}^- \\ +(288-96n+8n^2)L_{n-7}^- = 0 \qquad (17)$$

resulting in the differential equation for $L_{1,0}(q)$, namely

$$q\,(1-2q)\,(1-q)^2\left(1+4q-4q^2\right)\bar{L}_{1,0}''(q) \\ +(1-q)\left(1-20q+26q^2+24q^3-24q^4\right)\bar{L}_{1,0}'(q) \\ +\left(5+8q-22q^2-8q^3+8q^4\right)\bar{L}_{1,0}(q) \\ = \frac{1+2q-21q^2+20q^3-12q^4}{(1-q)^2}. \qquad (18)$$

Notice that replacing q by $1-p$ in the left-hand side of this equation reproduces the left-hand side of the corresponding low-density differential equation (16), but this is not the case for the right-hand side. This means that if equation (18) has a singular point at $q=q_s$ then equation (16) will have a singular point at $p=1-q_s$ with the same exponents. In table 2 we therefore list only the exponents for the singular points in the q variable. The zero exponent corresponding to the critical point suggests the following asymptotic form:

$$\bar{L}_{1,0}(p) \cong B^{\pm}\log|1-2q| + C^{\pm} \qquad (19)$$

where the superscripts + and − refer to the approach from above and below q_c, respectively. The constants have been estimated by solving the differential equation and curve fitting close to the critical point which yields the following values:

$$B^- = B^+ = -2.547 = 8/\pi \qquad C^- = 4.097 \qquad C^+ = -3.901 = C^- - 8. \qquad (20)$$

The value $8/\pi$ for the B's and the integer value 8 for the difference of the C's fit the data to within our estimated accuracy.

A similar analysis of $L_{m,y}(p)$ for other small values of m and y still yields a second-order differential equation but with coefficients of higher degree. The logarithmic singularity at p_c also occurs for these values.

The fact that the same differential operator arises for both the high- and low-density mean cluster length lead us to ask whether the same is true for the mean size. Substituting $q = 1 - p$ in the left-hand side of (12) and then substituting the closed form (10) of $S_{1,0}(p)$ into the result shows that it does indeed satisfy the differential equation

$$\begin{aligned} p^2(1-p)(1-2p)^2(1+4p-4p^2)S''_{1,0}(p) \\ -2p(1-2p)(-4+2p+25p^2-36p^3+12p^4)S'_{1,0}(p) \\ +6(2-8p+5p^2+4p^3-4p^4)S_{1,0}(p) \\ = 12-28p+22p^2-8p^4. \end{aligned} \tag{21}$$

Of course, since $S_{1,0}(p)$ is a rational function it also satisfies a simpler first-order homogeneous differential equation.

Finally we note that the denominator $(1-q)^2$ on the right-hand side of (12) does not occur in the differential equation for the normalized mean size $S_{1,0}(p)$ which is related to $\bar{S}_{1,0}(q)$ by

$$\bar{S}_{1,0}(q) = \frac{(q-q^2-q^3)S_{1,0}(q)}{(1-q)^3}. \tag{22}$$

However, the symmetry of the left-hand side of the equation about the critical point is lost by this substitution. Similar remarks apply to the normalized mean length.

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