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Directed compact percolation near a wall: I. Biased growth

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Abstract. The directed compact percolation cluster model of Domany and Kinzel is considered in the presence of a wall which is parallel to the growth direction and hence restricts the lateral growth of the cluster in one direction. The critical exponents are found to depend on whether the wall is wet or dry. In the former case the model is solved exactly for all the standard percolation functions and the critical behaviour is found to be the same as that for cluster growth with no wall present. With this boundary condition the cluster is completely attached to the wall and the model may also be viewed as one of symmetric compact cluster growth. In the case of a dry wall the cluster may repeatedly leave and return to the wall as it grows and in this case the percolation probability has been derived exactly by Lin and found to have a critical exponent different from that of the bulk. Lin's result is rederived and an exact formula for the percolation probability is found for a more general model in which the cluster growth is biased either towards or away from the wall. It is found that the unbiased case is special in that any bias away from the wall recovers the bulk critical exponent and a bias towards the wall produces a problem in the same class as the wet-wall model.

1. Introduction

The directed compact cluster model in the absence of a wall has been investigated by Domany and Kinzel (1984) as a limiting case of a stochastic cellular automaton. They found exact expressions for the percolation probability, the cluster length distribution and the associated critical exponents. Their work was extended to biased growth and to the mean cluster size by Essam (1989) and then to non-nodal clusters and the cluster size distribution by Essam and TanlaKishani (1990). More recently an exact form for the percolation probability of directed compact clusters with a dry wall has been conjectured by Bidaux and Privman (1991) and derived by Lin (1992).

The model is defined on a directed square lattice the sites of which are the points in the t, x plane with integer co-ordinates such that $t \geqslant 0$, $x \geqslant 0$ and $t + x$ even. The wall is represented by the sites with $x = -1$ and odd $t \geqslant 1$ which are either all wet (occupied by an atom) or all dry (unoccupied). A random cluster grows from a seed occupying m contiguous sites in the column $t = 0$ which in percolation terms is a compact source of fluid of width m. The growth rule is that the site (t, x) becomes wet (occupied) with certainty if both the sites $(t-1, x \pm 1)$ are wet, with probability p_u if $(t-1, x-1)$ is wet and $(t-1, x+1)$ is dry and with probability p_d if $(t-1, x+1)$ is wet and $(t-1, x-1)$ is dry. These conditions imply that for any column t, the wet sites will be contiguous which is why the clusters are said to be compact.

The percolation probability $P(p)$, where $p = \{p_u, p_d\}$, is the probability that the cluster never terminates. In standard percolation theory $P(p)$ is zero below a certain threshold which in the two-variable case becomes a critical curve in the p_u–p_d plane. On approaching

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the critical curve from the $P(p) > 0$ side, $P(p)$ vanishes with critical exponent β. In bulk-directed compact percolation the constraint $x \geqslant 0$ is not present and in this case it has been shown (Essam 1989) that the critical curve is the line $p_u + p_d = 1$ and $\beta = 1$ at all points on the curve.

For the dry wall, it has been shown by Lin that in the unbiased case, $p_u = p_d = p$, the percolation threshold $p_c = \frac{1}{2}$ and the critical exponent $\beta = 2$. Here we extend Lin's work to obtain an exact expression for $P(p)$ for arbitrary p_u and p_d. We show that the critical curve is now the union of two separate straight lines which meet at the unbiased critical point $(\frac{1}{2}, \frac{1}{2})$ (see figure 1). The first line is the part of the bulk curve $p_u + p_d = 1$ for which $0 \leqslant p_d \leqslant p_u$. In the region $p_d < p_u$, the growth is biased away from the wall and the model is in the same universality class as the bulk problem. At all points on this line, except the endpoint $(\frac{1}{2}, \frac{1}{2})$, β has its bulk value. The second part of the critical curve is the segment of the line $p_u = \frac{1}{2}$ for which $\frac{1}{2} \leqslant p_d \leqslant 1$. Except for the end point $(\frac{1}{2}, \frac{1}{2})$, this lies in the region $p_d > p_u$ in which there is a bias towards the surface and at all points of the critical curve in this region $\beta = 1$ again. It turns out that the other endpoint $p_d = 1$ is isomorphic to the wet wall problem and we shall call this second line the surface transition line. The wet wall problem is much simpler than the dry wall in that only one variable p_u is involved but nevertheless it determines the critical behaviour for the surface transition line. We shall see that the recurrence relations which determine the growth near a wet wall are the same as for the bulk problem which is why $\beta = 1$.

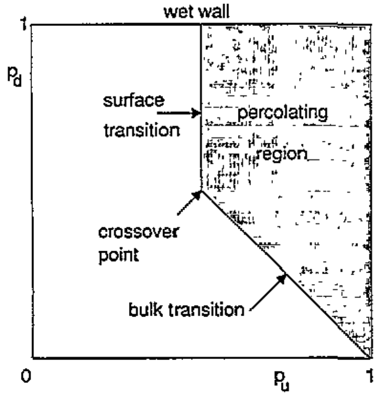

Figure 1. The critical curve for directed compact percolation near a wall.

Recurrence relations are obtained for the probability distribution function, $r_t(p)$, of the cluster length. This is defined as the probability that the cluster terminates after exactly t growth stages and can be used to determine $P(p)$. The wet surface problem is solved in detail in section 2 where in addition to the the percolation probability and the moments of the cluster length distribution we also determine the mean cluster size. An exact expression for the percolation probability of the general dry wall problem is found in section 3.

2. Wet wall and symmetric clusters

A typical compact cluster with a wet wall is shown in figure 2(a). We suppose for simplicity that the source consists of the m sites $(0, x)$ where $x = 0, 2, \ldots, 2m - 2$ and note that the

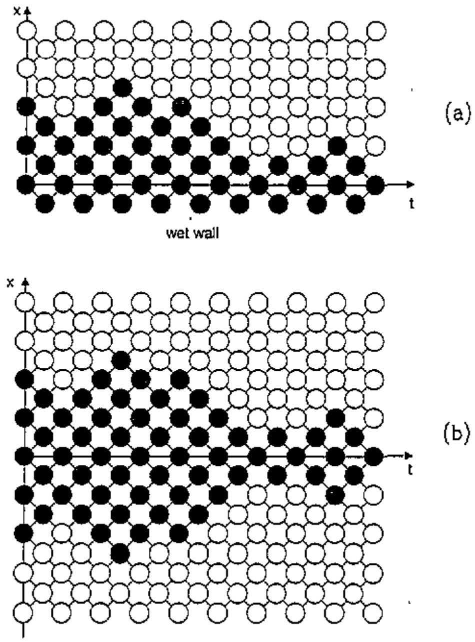

Figure 2. A typical directed compact cluster near a wet wall with seed width $m = 3$.

cluster can only terminate after an even number of growth stages so that $r_t(m) = 0$ for t odd. At each stage of the growth the upper edge moves up with probablity p_u and down with probability $q_u = 1 - p_u$ and is therefore an asymmetric random walk. The lower edge moves up and down on alternate stages. Thus after two successive growth stages the height of the lower edge is unchanged so that the cluster width has either increased by 1 (probability p_u^2), decreased by 1 (probability q_u^2) or has not changed (probability $2p_uq_u$). If in each case we multiply by the probability that the cluster grows for a further $t-2$ stages and then terminates, we obtain the recurrence relations

$$r_t(m) = p_u^2 r_{t-2}(m+1) + 2p_uq_ur_{t-2}(m) + q_u^2 r_{t-2}(m-1) \qquad m \geqslant 2 \tag{2.1}$$

$$r_t(1) = p_u^2 r_{t-2}(2) + p_uq_ur_{t-2}(1). \tag{2.2}$$

If the cluster initially has unity width, it will terminate immediately with probability q_u, so that

$$r_0(m) = \begin{cases} q_u & m = 1 \\ 0 & m \geqslant 2\,. \end{cases} \tag{2.3}$$

Defining the moment generating function by

$$R_m(z) = \sum_{t'=0}^{\infty} r_{2t'}(m)\mathrm{e}^{-t'z} \tag{2.4}$$

we get

$$R_m(z) = \mathrm{e}^{-z}(p_u^2 R_{m+1}(z) + 2p_uq_u R_m(z) + q_u^2 R_{m-1}(z)) \qquad m \geqslant 2 \tag{2.5}$$

and

$$R_1(z) - r_0(1) = e^{-z}(p_u^2 R_2(z) + p_u q_u R_1(z)). \tag{2.6}$$

Substituting $p_d = p_u$, $c = p_u^2$ and $d = q_u^2$ in the recurrence relation for compact percolation clusters without a surface (Essam 1989) gives an equation identical to (2.5). The solution of (2.5) satisfying boundary condition (2.6) is

$$R_m(z) = l(z)^{m-\frac{1}{2}} e^z \tag{2.7}$$

where $l(z)$ is the root of the quadratic

$$cl^2 + (1 - c - d - e^z)l + d = 0 \tag{2.8}$$

which remains bounded as $z \to \infty$. The corresponding solution in the absence of a surface, which we here denote by $R_m(z, \text{bulk})$, differs from $R_m(z)$ only by the factor $l(z)^{1/2}$

$$R_m(z, \text{bulk}) = \text{l}(z)^m e^z. \tag{2.9}$$

We may therefore take over the previous results for the bulk problem Essam (1989) by replacing m by $m - \frac{1}{2}$. In particular the percolation probability $P_m(p) = 1 - R_m(0)$ is zero for $p_u < \frac{1}{2}$ and for $p_u \geqslant \frac{1}{2}$

$$P_m(p) = 1 - (q_u/p_u)^{2m-1} \simeq (4m-2)(1-2p_u) \tag{2.10}$$

hence the critical probability $p_c = \frac{1}{2}$ and the critical exponent $\beta = 1$. The moments of the cluster length distribution are defined by

$$\mu_k(m, p) = \langle t^k \rangle = \sum_{t'=0}^{\infty} (2t')^k r_{2t'}(m) = (-2)^k R_m^{(k)}(0) \tag{2.11}$$

The mean length of finite clusters is given by

$$L_m(p) = 1 + \mu_1(m, p)/\mu_0(m, p) = 1 - 2(\text{d}/\text{d}z)\log[R_m(z)]|_{z=0} = (2m-1)/|2p_u - 1|. \tag{2.12}$$

This is symmetric about the critical value $p_u = \frac{1}{2}$ and diverges with critical exponent $t = 1$, as for the bulk. The asymptotic form of the higher order moments may also be deduced from the bulk result (Essam 1989) and we find

$$\mu_k(m, p) = \frac{(2m-1)(2k-2)!}{(k-1)!2^{k-1}|2p_u - 1|^{2k-1}} \qquad m \geqslant 2 \tag{2.13}$$

which implies the existence of a scaling length with exponent $\nu_{\Uparrow} = 2$.

In considering the cluster size distribution, where the size of a cluster is the number of sites it occupies, we calculate only the first moment. Following the method of Essam(1989), the mean size $Sm(p_u)$ of clusters below the percolation threshold satisfies the equation

$$S_m(p_u) = p_u^2 S_{m+1}(p_u) + 2p_u q_u S_m(p_u) + q_u^2 S_{m-1}(p_u) + 2m - 1 + p \qquad m \geqslant 2 \tag{2.14}$$

together with the boundary condition

$$S_1(p_u) = p_u^2 S_2(p_u) + p_u q_u S_1(p_u) + 1 + p. \tag{2.15}$$

With the further requirement that S_m remain bounded as $m \to \infty$, these relations may be solved to yield

$$2S_m(p_u) = \frac{m - 2p_u(1-p_u)}{(1-2p_u)^2} + \frac{2m^2 - m}{|1-2p_u|}. \tag{2.16}$$

This expression is symmetric about the critical point (cf the mean length) and in the percolating region it determines the mean cluster size given that the cluster is finite, which includes a normalizing factor of $1/R_m(0)$. See Essam (1989) for discussion of this symmetry property.

Any compact cluster attached to the wet wall may be made symmetric by adding to it the sites obtained by reflecting it in the $x = 0$ line (see figure 2(*b*)). The critical exponents for symmetric compact clusters are therefore the same as for all compact clusters on the same base. At first sight, this fact is a little surprising since the symmetric clusters form only a vanishingly small subset of all compact clusters.

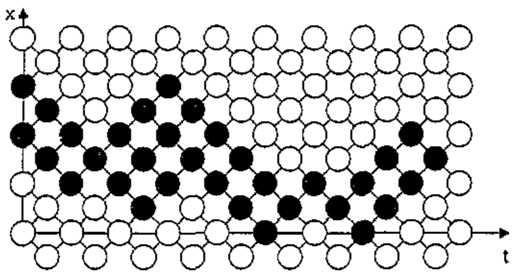

Figure 3. A typical directed compact cluster near a dry wall with seed width $m = 2$.

3. Percolation probability for the dry wall boundary condition

A typical compact cluster near a dry wall is shown in figure 3. Since gaps can occur between the cluster and the wall, to obtain a recurrence relation it is necessary to consider a source which is not necessarily in contact with the wall. Thus two parameters are required to define the sites which belong to the source. We take as the first parameter the number of sites in the source m, as before, and the second parameter y is defined as the position of the centre of mass. If $y \geqslant m$, at the first growth stage either m is unchanged and y increases or decreases by unity, or y stays the same and m increases or decreases by unity. The cluster is then completed by adding any cluster of length $t - 1$. Let $r_t(m, y)$ be the probability that a cluster with source of width m and centre of mass y has exactly t growth stages before it terminates then, with $a = p_u q_d$, $b = q_u p_d$, $c = p_u p_d$ and $d = q_u q_d$

$$r_t(m, y) = ar_{t-1}(m, y+1) + br_{t-1}(m, y-1) + cr_{t-1}(m+1, y) + dr_{t-1}(m-1, y)$$

$$y \geqslant m \geqslant 1, \; t > 0 \tag{3.1}$$

$$r_t(m, m-1) = p_u r_{t-1}(m, m) + q_u r_{t-1}(m-1, m-1) \qquad m \geqslant 1, \; t > 0 \tag{3.2}$$

$$r_0(1, y) = d \qquad y \geqslant 1 \tag{3.3}$$

$$r_0(1, 0) = q_u \tag{3.4}$$

$$r_0(m, y) = 0 \qquad y \geqslant m \geqslant 2 \tag{3.5}$$

$$r_t(0, y) = 0 \qquad y \geqslant m - 1. \tag{3.6}$$

Since $a + c = p_u$ and $b + d = q_u$ equation (3.2) may be replaced by the simpler boundary condition

$$r_t(m+1, m-1) = r_t(m, m) \qquad m \geqslant 1, \; t \geqslant 0 \tag{3.2a}$$

These equations correspond to a random walk in the $m - y$ plane with an absorbing boundary at $m = 0$ and a reflecting boundary at $y = m - 1$. Introducing the moment generating function as before

$$R_{m,y}(z) = \mathrm{e}^{-z}(aR_{m,y+1}(z) + bR_{m,y-1}(z) + cR_{m+1,y}(z) + dR_{m-1,y}(z)) \qquad y \geqslant m \geqslant 1 \tag{3.7}$$

where in order to satisfy (3.3) and (3.4) we must have

$$R_{0,y}(z) = \mathrm{e}^z \tag{3.8}$$

and from (3.2a)

$$R_{m+1,m-1}(z) = R_{m,m}(z). \tag{3.9}$$

We now search for separable solutions of (3.7) in the form

$$R_{m,y}(z) = M_m(z) Y_y(z) \tag{3.10}$$

which leads to

$$cM_{m+1}(z) + (1 - c - d + s - \mathrm{e}^z)M_m(z) + dM_{m-1}(z) = 0 \tag{3.11}$$

and

$$aY_{y+1}(z) - (a + b + s)Y_y(z) + bY_{y-1}(z) = 0 \tag{3.12}$$

where s is a separation variable which may depend on z.

To find the percolation probability, it is only necessary to solve these equations with $z = 0$. The choice $s = 0$ leads to the exponential solutions

$$Mm(0) = 1 \quad \text{or} \quad (d/c)m \qquad \text{and} \qquad Y_y(0) = 1 \quad \text{or} \quad (b/a)^y \tag{3.13}$$

which are sufficient to satisfy (3.8) but not (3.9). Guided by Lin's solution (Lin 1992) we found that $s = c + d - a - b$ yields further solutions in terms of which it is possible to satisfy both boundary conditions. These are

$$M_m(0) = (q_u/p_u)^m \quad \text{or} \quad (q_d/p_d)^m \qquad \text{and} \quad Y_y(0) = (q_u/p_u)^y \quad \text{or} \quad (p_d/q_d)^y. \tag{3.14}$$

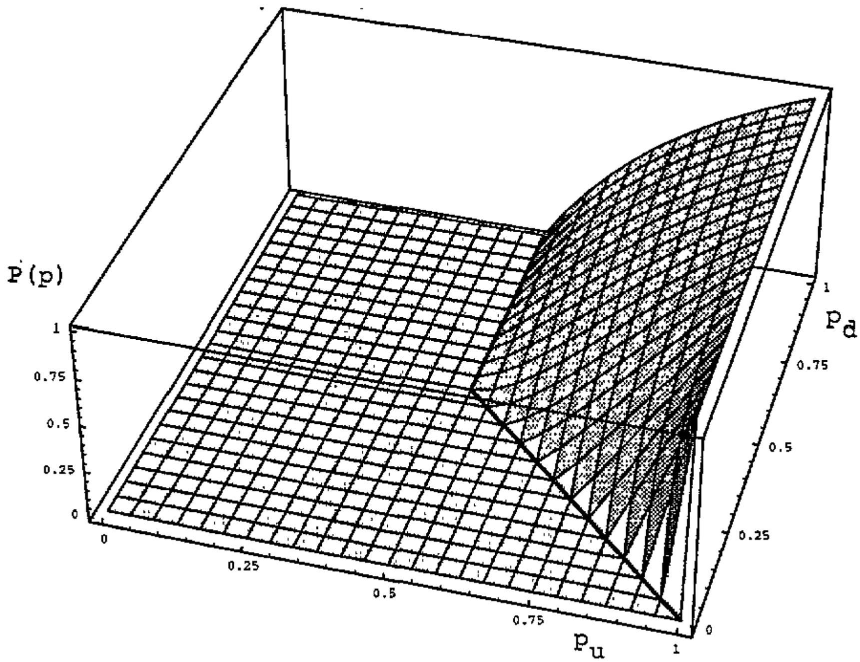

Figure 4. The percolation probability for a directed compact cluster with unity seed width, adjacent to the wall.

For the symmetric growth problem, $p_u = p_d = p$, the solutions for $M_m(0)$ in (3.14) become equal and, following standard analysis for second-order difference equations, a second independent solution is the one found by Lin (1992):

$$M_m(0) = m(q/p)^m \qquad \text{with} \quad Y_y(0) = (q/p)^y \quad \text{or} \quad (p/q)^y. \tag{3.15}$$

For the symmetric case we try to satisfy the boundary conditions with the form

$$R_{m,y}(0) = A(q/p)^{2m} + Bm(q/p)^{m+y}. \tag{3.16}$$

Imposing conditions (3.8) and (3.9) determines A and B and results in an expression for the percolation probability equivalent to that of Lin (1992)

$$P_{m,y}(p) = 1 - R_{m,y}(0) = 1 - (q/p)^{2m} - (2p-1)m(q/p)^{m+y}/p^2. \tag{3.17}$$

In the asymmetric case we replace Lin's solution by a combination of those in (3.14) which vanishes at $m = 0$, thus

$$R_{m,y}(0) = A(d/c)^m + B[(q_u/p_u)^m - (q_d/p_d)^m](q_u/p_u)^y \tag{3.18}$$

and imposing the boundary conditions, we finally obtain

$$P_{m,y}(p) = 1 - (d/c)^m - (p_u - q_d)[(q_u/p_u)^m - (q_d/p_d)^m](q_u/p_u)^{y+1}/(q_u - q_d). \tag{3.19}$$

Note that when $p_d = 1$ and $y = m - 1$, which corresponds to the lower edge of the source being at the origin, equation (3.19) reduces to the wet-wall formula (2.10).

In general, $P_{my}(p) = 0$ when either $p_u = \frac{1}{2}$ or $p_u + p_d = 1$ and the positive $P_{my}(p)$ region is to the right of both these lines (see figure 1). This may be seen explicitly in the case $m = 1$, $y = 0$ for which

$$P_{1,0}(p) = (2p_u - 1)(p_u + p_d - 1)/p_u^2 p_d. \tag{3.20}$$

The asymptotic form near the crossover point $(\frac{1}{2}, \frac{1}{2})$ for any m and y is

$$P_{m,y}(p) \simeq 8m(y+1)(2p_u - 1)(p_u + p_d - 1). \tag{3.21}$$

At a distance r from the crossover point along a line at an angle θ to the p_u axis we have

$$P_{m,y}(p) \simeq 16m(y+1)r^2 \cos\theta(\cos\theta + \sin\theta) \tag{3.22}$$

which shows that the exponent $\beta = 2$ along any such line. At any other point on the critical curve only one of the factors vanishes and $\beta = 1$. A three-dimensional plot of the percolation probability is shown in figure 4.

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