Bose and Fermi walk configurations on planar graphs

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Abstract

The number, $f_n^C(H)$, of $n-$walk configurations of type C is investigated on certain two-rooted directed planar graphs H which will be always realized as plane graphs in R^2. C may be Bose or Fermi as defined by Inui and Katori. Both types of configuration are collections of non-crossing walks which follow the directed paths between the roots of the plane graph H. In the case of configurations of Fermi type each walk may be included only once. The number $f_n^{Bose}(H)$ is shown to be a polynomial in n of degree $n_{max}-1$ where n_{max} is the maximum number of walks in a Fermi configuration. The coefficient of the highest power of n in this polynomial is simply related to the number of maximal Fermi walk configurations. It is also shown that $n_{max}=c(H)+1$ where $c(H)$ is the number of finite faces on H. Extension of these results to multi-rooted graphs is also discussed.

When H is the union of paths between two sites of the directed square lattice subject to various boundary conditions Kreweras showed that the number of Bose configurations is equal to the number of $n-$element multi-chains on segments of Young's lattice. He expressed this number as a determinant the elements of which are polynomials in n. We evaluate this determinant by the method of LU decomposition in the case of "watermelon" configurations above a wall. In this case the polynomial is a product of linear factors but on introducing a second wall the polynomial does not completely factorise but has a factor which is the number of watermelon configurations on the largest rectangular subgraph.

The number of two-rooted "star" configurations is found to be the product of the numbers of watermelon configurations on the three rectangular subgraphs into which it may be partitioned.

PACS numbers: 05.50.+q, 05.70.fh, 61.41.+e **Key words: Fermi walks,**

Bose walks, vicious walkers, interacting random walks, lattice paths, partially ordered sets, Young diagrams.

1 Introduction

1.1 Background and definitions

The subject of interacting random walks on a directed square lattice was introduced by Fisher in his *Boltzmann Medal* lecture [13] where a number of physical applications are described. He introduced the term "vicious" to describe an interaction where if two walkers meet then they annihilate one another. Thus for the walkers to continue they must avoid one another. Vicious walker configurations also arise in the theory of plane polymer networks [11], [12] where the critical exponents which describe the asymptotic behaviour of long chains depend on the topology of the network. The continuum analogue of vicious walker configurations is non-intersecting Brownian paths which are relevant [21] to random matrix theory. Vicious walker configurations correspond to Bose configurations considered here (see below). Fermi configurations arise in directed percolation theory [20].

Inui and Katori [20] considered the space-time trajectories of n random walks which, at each tick of a clock, make unit steps to the left or right on a one-dimensional lattice. The walkers all start at the origin and may occupy the same site (thus known as friendly walkers) but may not pass one another. In space-time the walkers follow paths on a directed square lattice and their trajectories are non-crossing. Suppose that the vertices of the one dimensional lattice are $(i, -i)$, where i is an integer, and the time axis is in the direction $(-1, -1)$. If all walkers make ℓ positive steps and w negative steps their walks will be from the top right to the bottom left corner of the $\ell \times w$ rectangular grid $W_{\ell,w}$ an example of which is shown in figure 1(a). The set of walks in the case $\ell = w = 3$ is shown in figure 2. Two types of configuration were considered in [20]. *Bose configurations* need only satisfy the condition that the walks are non-crossing, in addition *Fermi configurations* are such that no two walks have the same trajectory. If the walks are also restricted to the sites $i \geq 0$ of the one-dimensional lattice the trajectories are confined to the sites of the square lattice with $y \geq x$ as shown in figure 1(b). We call these *walks above a wall*. Introducing a second restriction $i \leq h$ reduces the region of the square lattice available to the trajectories as shown in figures 1(c) and (f).

Here we extend the concept of Bose and Fermi walk configurations on standard grids to those which follow paths between the roots of a directed plane two-rooted coverable graph H.

Definition 1. *The plane two-rooted graph H will be said to be coverable (by paths) if*

1. *every arc belongs to at least one directed path between the roots,*
2. *it has no directed cycles.*

A plane graph H has no crossing of arcs so that the non-crossing condition for Bose and Fermi configurations can be applied to pairs of walks on H. Graphs of this type arise in directed percolation theory [2].

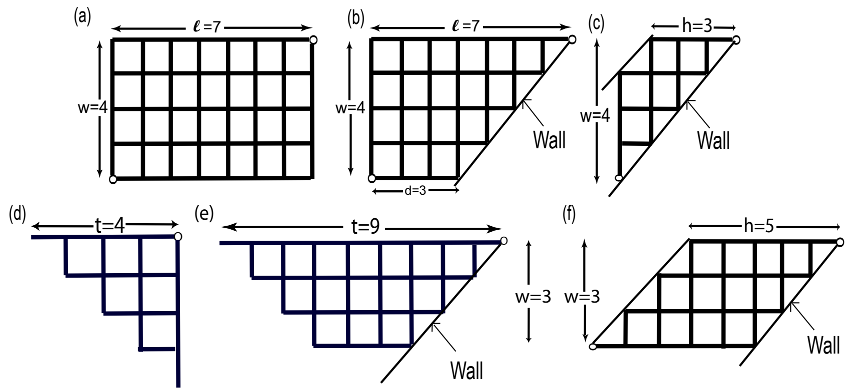


Figure 1: *The graphs(a)* $W_{7,4}$ *(b)* $\overline{W}_{7,4}$ *(c)* $\overline{\overline{W}}^{\downarrow}_{3,4}$ *(d)* S_4 *(e)* $\overline{S}_{9,3}$ *(f)* $\overline{\overline{W}}^{\uparrow}_{5,3}$. *The graphs are considered to be directed to the left and down.*

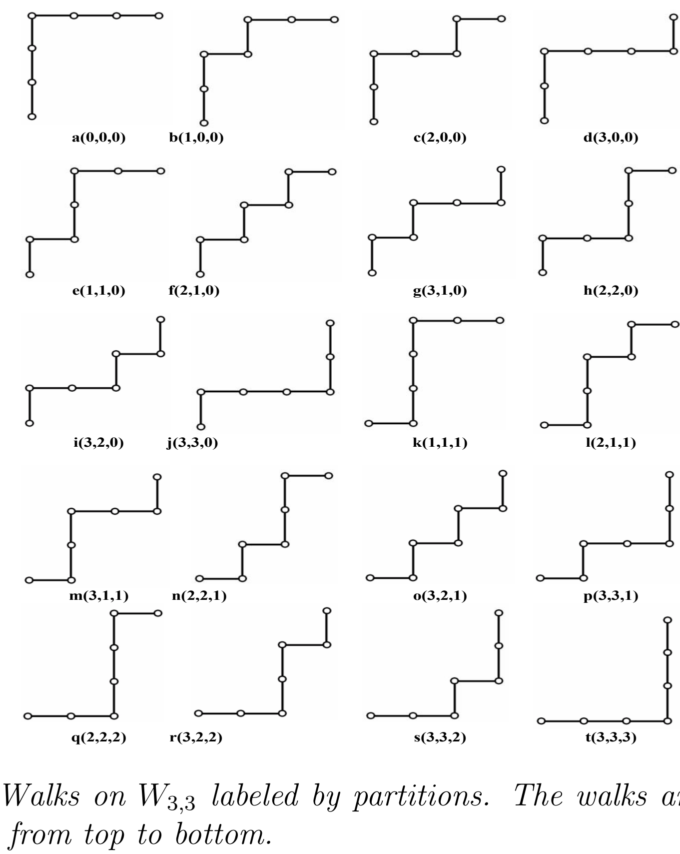


Figure 2: *Walks on* $W_{3,3}$ *labeled by partitions. The walks are considered to be directed from top to bottom.*

Assuming that the plane graph H is not trivial, i.e. consisting of a single walk, there will be two bounding paths connecting its source and sink roots and they will be allocated the complementary labels *lowest* and *highest*. This enables us to introduce a partial order for the set of walks between the roots. If walks w_1 and w_2 do not cross and are distinct, then one is greater than the other with the ordering induced by the choice of bounding path labels.

For the case of multi-rooted graphs with several sinks, the corresponding coverability condition for H is that each of its edges is part of at least one walk between the source and a sink.

In the case that H is a subgraph of the square lattice the walks all have the same length but in general the walks may have different numbers of steps depending on length of the paths between the roots. This would arise when thinking of the Bose walk configurations as integer flows [2]. Graphs (a), (b), (c) and (f) in figure 1 will be denoted $W_{\ell,w}$, $\overline{W}_{\ell,w}$, $\overline{\overline{W}}^{\downarrow}_{h,w}$ and $\overline{\overline{W}}^{\uparrow}_{h,w}$ respectively.

The numbers of $n-$walk Bose and Fermi configurations on the graph H will be denoted by $f_n^{Bose}(H)$ and $f_n^{Fermi}(H)$ respectively.

Kreweras [27] coded the square lattice walks by listing the positions of the vertical steps measured horizontally from the lower left corner. If the positions are listed in the order in which they occur in the walk they form a weakly decreasing sequence known as a partition. See figure 2 for the case $\ell = w = 3$. These partitions form a partially ordered set (POSET) which is Young's lattice developed in [38], see also [33], p. 288. Fermi configurations correspond to chains and Bose configurations correspond to multi-chains on this POSET. Kreweras [27] expressed the number of n element multi-chains in a chosen segment of this POSET as a determinant (3.3) the binomial coefficient elements of which are polynomials in n. The dimension of the determinant is the same for all n and is equal to the number of vertical steps w. The number of $n-$walk Bose configurations on the $\ell \times w$ rectangle is equal to the number of n element multi-chains for the segment bounded by the partitions $\{0, 0, \ldots, 0\}$ and $\{\ell, \ell, \ldots, \ell\}$ where each partition has w parts. Call this segment $P_{\ell,w}$. See figure 3 for three examples with $w = 3$. For $\ell < 3$ only partitions with parts $\leq \ell$ are included in the POSET. In the case $\ell = 2\,(\ell = 1)$ the corresponding walks are obtained from those in figure 2 by removing the first step (two steps).

For walks above a wall (figure 1(b)) the upper partition is replaced by $\{\ell-1, \ell-2, \ldots, \ell-w\}$. In general for a segment bounded by partitions μ and λ the number of multi-chains is equal to the number of Bose configurations of walks from the top right to the bottom left corner of the skew Young diagram $Y_{\lambda/\mu}$ [14] (see for example figure 4(a)).

1.2 Main results

Notice that Kreweras's determinant (3.3) is a polynomial in n which implies that the number of $n-$walk Bose configurations on a general Young diagram also has this *polynomial property*. The polynomial property for the square lattice was derived in [2] by expressing the number of multi-chains on the

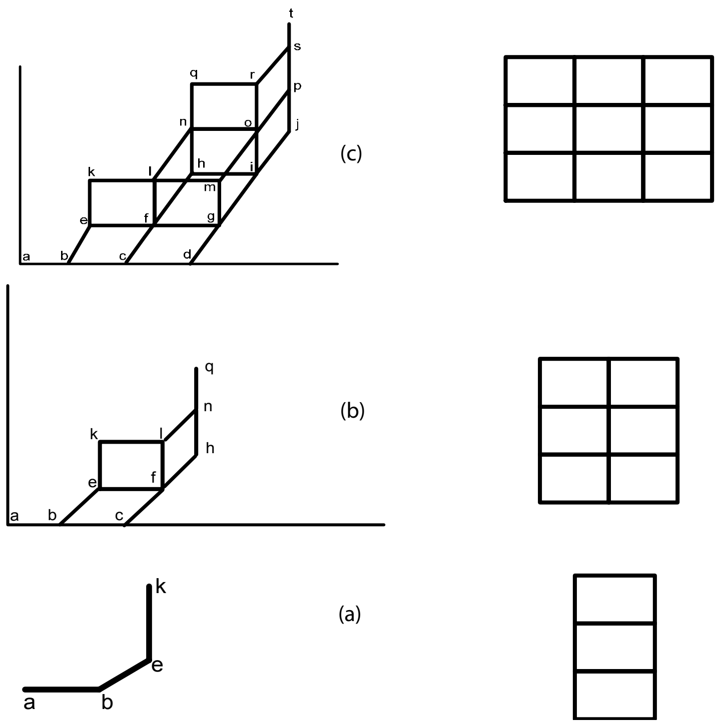


Figure 3: *The Hasse diagrams (a) $P_{1,3}$, (b) $P_{2,3}$ and (c) $P_{3,3}$.*

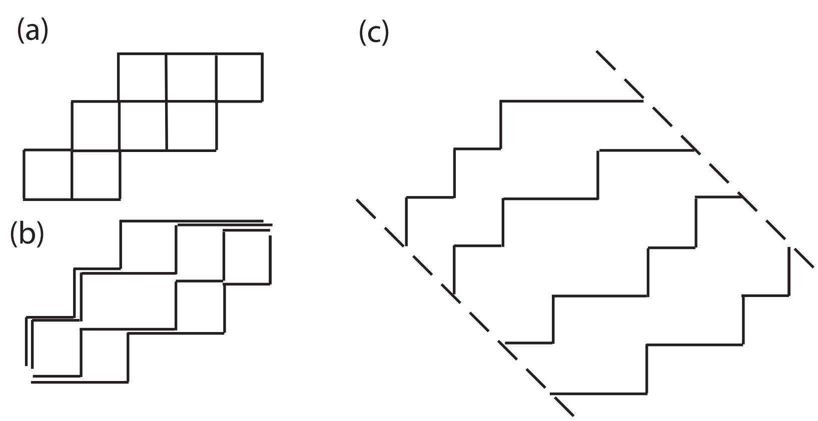
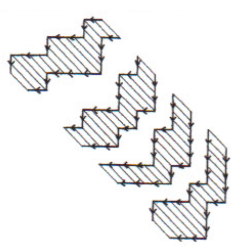


Figure 4: *(a) The skew Young diagram defined by partitions* $\{5,4,2\}$ *and* $\{2,1,0\}$*, (b) A Bose/Fermi configuration of four walks on (a), (c) The corresponding vicious walker configuration.*

POSET of walks in terms of the number of chains. The same method applies to a coverable graph H by noting that the walks on H also form a POSET $P(H)$. In section 2.1 (Proposition 1) the polynomial property for H is proven directly using a relation [20] between the number of Bose and Fermi configurations which is extended to multi-rooted graphs in section 2.2 (Proposition 2). It is also shown that the degree of the polynomial is equal to $n_{max} - 1$ where n_{max} is the maximum number of walks in a Fermi configuration. The coefficient of the highest power of n determines the number of maximal Fermi walk configurations. Further it is shown that $n_{max} = c(H) + 1$ where $c(H)$ is the number finite faces of H or the number of independent cycles. Thus $f_n^{Bose}(H)$ is a polynomial in n of degree $c(H)$. The result is illustrated by considering the graph consisting of a chain of multi-arcs (see figure 7).

On the square lattice there is a bijection between Bose and vicious walker configurations obtained by shifting the i^{th} walk from the bottom of a Bose configuration through a distance $\sqrt{2}(i-1)$ in the direction $(-1, 1)$ (see figure 4 for example). An early example of this bijection may be found in [10]. It follows that the number of $n-$walk vicious walker configurations is also polynomial in n. The vicious walker configurations which biject to Bose configurations on $W_{\ell,w}$ were known as *watermelon* configurations [13], hence the W label.

Figures 1(d) and (e) are examples of graphs formed by the union of $t-$step paths beginning at the root with no constraint on the endpoints. In case (e) the paths have at most w vertical steps and must not cross the wall $y = x$. Graphs of type (d) are denoted by S_t and those of type (e) by $\bar{S}_{t,w}$. It is assumed that $t \geq 2w$. The vicious walker configurations which biject to the Bose configurations on S_t were known [13] as *star* configurations. Another set of vicious walker configurations, known as *banana* configurations (cf. [21]), biject to Bose configurations on S_t with the constraint that the number of walks terminating at each endpoint must be even. An example of a banana configuration is shown in figure 5.



1	1	1	1	1	1	2	2
2	2	2	2	2	3	3	5
3	3	3	4	4	4	7	8
5	5	5	5	6	7		
6	8						

Figure 5: *A banana configuration with $t = 8$ steps which bijects to a Bose configuration on S_8 in which the possible endpoints have multiplicities $\{0, 0, 0, 2, 4, 2, 0, 0\}$. The table is the corresponding Young tableau.*

The number of vicious walker configurations for the square lattice is known for several boundary conditions [2], [7], [10], [18], [22], [25], [26]. In some cases these results are not manifestly polynomials in n as they are predicted to be by part (a) of proposition 1. However we have reorganised them into products of factors which are linear in n and they are listed in the first part of table 1. We have then used proposition 1(a) to determine the numbers of maximal Fermi walk configurations which are listed in the second part of table 1. The different boundary conditions described above also occur in the case of non-intersecting Brownian paths where they correspond to different types of random matrix , see for example [21] and [9]. They also naturally occur in polymer network theory [11] [12] where the chains may be confined by physical boundaries.

watermelons	stars
$f_n^{Bose}(W_{\ell,w}) = \prod_{k=1}^{w} \frac{(n+k)_\ell}{(k)_\ell}$ $f_n^{Bose}(\overline{W}_{\ell,w}) = f_n^{Bose}(W_{d,w}) \prod_{k=1}^{w} \frac{(2n+d+k)_{k-1}}{(d+k)_{k-1}}$	$f_n^{Bose}(S_t) = \prod_{k=1}^{t} \frac{(n+k)_k}{(k)_k}$ $f_n^{Bose}(\overline{S}_{t,w}) = f_n^{Bose}(W_{t-w,w})$
$f_{n_{max}}^{Fermi}(W_{\ell,w}) = \frac{(\ell w)!}{\prod_{k=1}^{w}(k)_\ell}$ $f_{n_{max}}^{Fermi}(\overline{W}_{\ell,w}) = \frac{2^{\frac{1}{2}w(w-1)}(wd+\frac{1}{2}w(w-1))!}{\prod_{k=1}^{w}(k)_{d+k-1}}$	$f_{n_{max}}^{Fermi}(S_t) = \frac{(\frac{1}{2}t(t+1))!}{\prod_{k=1}^{t}(k)_k}$ $f_{n_{max}}^{Fermi}(\overline{S}_{t,w}) = f_{n_{max}}^{Fermi}(W_{t-w,w})$

Table 1: *The numbers of n-walk Bose and maximal Fermi configurations. Note that $d = \ell - w$.*

In section 3.1 we evaluate Kreweras's determinant for wateremelon configurations on the rectangle and above a wall by the method of LU decomposition. In the former case the determinant is easily evaluated by row and column operations but we have been unable to do this in the latter. The resulting formulae form part of table 1. These results have been obtained before [12] [7] [22] in the context of vicious walks using the Lindström-Gessel-Viennot determinant [28] [15] which does not have elements which are polynomial in n. The LU method has also been investigated for walks between two walls but fails to give a general result. Evaluation of the determinant for small examples shows that the complete factorisation of the polynomial into linear factors with rational coefficients which occurs in the first two examples is not repeated when a second wall is present. However some surprising factorisations do occur.

There is no single determinant formula for the number of star configurations which has elements which are polynomial in n. Single determinantal formulae do exist, [30] and [36], being squares of Pfaffians. In section 3.2 we discuss the polynomial property of the results in table 1 and review implicit bijections between star and watermelon configurations.

In section 3.3 we consider an extension of watermelon configurations in which n_1 of the $t-$step walks make w_1 vertical steps and $n_2 = n - n_1$ of the walks make w_2 vertical steps. The corresponding Bose configurations are such that the walks terminate at one of two different vertices. We call these configurations 2$-$*point stars*. The union of the possible paths for these walks gives rise to a graph $W_{\ell_1,w_1}^{\ell_2,w_2}$ with one initial root and two final roots. See for example figure 6. This is a special case of the multi-rooted graph considered in section 2.2 where it is shown that the number of configurations is polynomial in the n_i variables. The number of 2$-$point stars as a polynomial in n_1

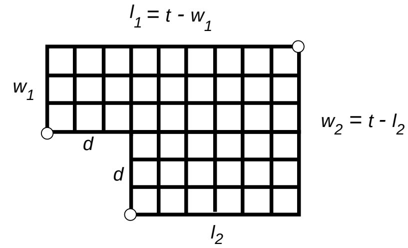


Figure 6: *The two-point star graph* $W_{9,3}^{6,6}$

and n_2 factorises in a surprisingly simple way.

$$f_{n_1,n_2}^{Bose}(W_{\ell_1,w_1}^{\ell_2,w_2}) = \frac{f_{n_1}^{Bose}(W_{d,w_1})f_{n_2}^{Bose}(W_{\ell_2,d})f_{n+d}^{Bose}(W_{\ell_2,w_1})}{f_d^{Bose}(W_{\ell_2,w_1})}, \tag{1.1}$$

where $d = \ell_1 - \ell_2 = w_2 - w_1$.

The result is derived from the general formula [12] for stars in which the i^{th} walk makes q_i vertical steps. The degree of $f_n^{Bose}(W_{\ell,w})$ is $c(W_{\ell,w}) = \ell w$ which means that $f_{n_1,n_2}^{Bose}(W_{\ell_1,w_1}^{\ell_2,w_2})$ has degree $dw_1 + \ell_2 w_1 = \ell_1 w_1 = c_1$ in n_1 and similarly degree $\ell_2 w_2 = c_2$ in n_2. c_i is the number of finite faces in the graph formed from the union of paths which terminate at (ℓ_i, w_i). This is as predicted by proposition 2.

2 The number of Bose and Fermi configurations for a coverable plane graph

2.1 The polynomial property of $f_n^{Bose}(H)$

The observation that the number of walks in a Fermi configuration is finite leads to part (a) of the following proposition.

Proposition 1. *Let n_{max} be the maximum number of walks in a Fermi configuration on the coverable plane graph H then:*

(a) $f_n^{Bose}(H)$ is a polynomial in n of degree $n_{max} - 1$ and the coefficient of $n^{n_{max}-1}$ is $f_{n_{max}}^{Fermi}(H)/(n_{max}-1)!$.

(b) $n_{max} = c(H) + 1$ where $c(H)$ is the number of finite faces of H.

Proof.

(a) Any $n-$walk Bose configuration corresponds to a unique Fermi configuration which uses the same paths. The corresponding configuration is obtained by replacing walks which use the same path by a single walk. The number of Bose configurations which correspond to a given $k-$walk Fermi configuration is obtained by distributing the n walks over the k paths used by the Fermi walks such that each path is used at least once. Hence [20]

$$f_n^{Bose}(H) = \sum_{k=1}^{n_{max}} \binom{n-1}{k-1} f_k^{Fermi}(H) \tag{2.1}$$

$$= \sum_{k=1}^{n_{max}} \frac{(n-k+1)_{k-1}}{(k-1)!} f_k^{Fermi}(H) \tag{2.2}$$

where the Pochhammer symbol is defined by $(a)_k \equiv a(a+1)\ldots(a+k-1)$. The k^{th} term of the sum is a polynomial in n of degree $k-1$ so the highest power of n comes from the term $k = n_{max}$ with coefficient $f_{n_{max}}^{Fermi}(H)/(n_{max}-1)!$.

Note the formula 2.2 also holds for the enumeration of Bose walks for $n = 0$, i.e. $F_0^{Bose}(H) = 1$, even though the summation is for positive configurations of Fermi walks, see lemma 1 following Proposition 3.

(b) Imagine that all arcs of H are directed from right to left. All maximal Fermi walk configurations include the walk which follows the uppermost path. Label this walk zero. We will label the remaining walks $1, 2, \ldots$ from the top down. A maximal configuration may be constructed as follows. For $i \geq 1$, an i^{th} walk may be obtained by choosing a face adjacent to walk $i-1$ on two consecutive sides and diverting this walk from above to below the chosen face. Label the walk and the chosen face i. The process terminates when all the faces are labelled so the number of walks is $c(H)+1$. Every walk uses a different path and no further walks may be inserted so the configuration is maximal.

□

Note: In the case of the square lattice this sequential labelling of the faces produces a standard Young tableau. In [3] the numbers of maximal Fermi walk configurations for the examples in figures 1 (a), (b), (d) and (e) were obtained by enumerating the corresponding numbers of standard Young tableaux. The results agree with those in table 1 obtained using proposition 1(a).

Example **Series parallel**

Consider a graph $V_{\ell,p}$ which is the series combination of ℓ multiedges of multiplicity p. The n walks must pass through every multi-edge and the number of configurations for each multi-edge is the number of weak p-compositions of n [33].

$$f_n^{Bose}(V_{\ell,p}) = \binom{n+p-1}{p-1}^{\ell}. \tag{2.3}$$

The number of finite faces $c(V_{\ell,p}) = (p-1)\ell$ which is also the degree of $f_n^{Bose}(V_{\ell,p})$ in agreement with proposition $1(b)$. Using proposition $1(a)$ the

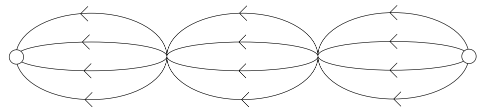


Figure 7: *The graph* $V_{3,4}$

number of maximal Fermi configurations is

$$f^{Fermi}_{n_{max}}(V_{\ell,p}) = \frac{[(p-1)\ell]!}{[(p-1)!]^{\ell}} \tag{2.4}$$

2.2 Extension to multi-rooted graphs

Suppose that H_s is a directed graph with a single source u and s sinks $\{v_1, v_2 \dots v_s\}$. We consider walk configurations in which the walks start at the source and end at one of the sinks. There are two cases.

(a) **The n walks end at any one of the sinks.**

H_s may be converted to a two-rooted graph H by adding a sink vertex v and a path from each sink of H_s to v. The numbers of Bose and Fermi walk configurations are respectively the same for both H and H_s so proposition 1 may be used. It follows that $f^{Bose}_n(H_s)$ is a polynomial in n of degree $c(H) = c(H_s) + s - 1$ since $s - 1$ additional faces are created in the conversion.

(b) **n_i of the walks end at the i^{th} sink**

The following is an extension of proposition 1 to the case when the number of walks terminating at each root is specified.

Proposition 2. *Let $f^{(C)}_{n_1,n_2,\dots,n_s}(H_s)$ be the number of walk configurations, subject to constraint C, such that n_i walks terminate at v_i then $f^{(Bose)}_{n_1,n_2,\dots,n_s}(H_s)$ is a polynomial in n_i of degree c_i where c_i is the number of finite faces in the graph formed by the union of all paths from u to v_i.*

Proof. Any Bose configuration corresponds to unique a Fermi configuration which uses the same paths. The required number of Bose configurations which correspond to a Fermi configuration in which k_i walks terminate at v_i is obtained by distributing n_i walks over the k_i paths used by the Fermi walks such that each path is used at least once. Hence

$$f^{Bose}_{n_1,\dots,n_s}(H_s) = \sum_{k_1=1}^{c_1+1} \cdots \sum_{k_s=1}^{c_s+1} f^{Fermi}_{k_1,\dots,k_s}(H_s) \prod_{i=1}^{s} \binom{n_i - 1}{k_i - 1} \tag{2.5}$$

$$= \sum_{k_1=1}^{c_1+1} \cdots \sum_{k_s=1}^{c_s+1} f^{Fermi}_{k_1,\dots,k_s}(H_s) \prod_{i=1}^{s} \frac{(n_i - k_i + 1)_{k_i - 1}}{(k_i - 1)!} \tag{2.6}$$

$\square$

Example **A graph H_2 having two sinks.** For the graph in figure 8a we find the numbers of Fermi configurations in table 2 by direct enumeration. Using equation (2.5) gives the result

$$f^{Bose}_{n_1,n_2}(H_2) = \frac{1}{2}(1+n_1)(1+n_2)(2+n_1+n_2). \tag{2.7}$$

Evaluation of this formula for small integers n_1 and n_2 gives the results in table 2. This is a special case of the formula derived in the next section.

Summing over n_1 and n_2 with the constraint $n_1 + n_2 = n$ gives

$$\frac{1}{12}(1+n)(2+n)^2(3+n)$$

which is the number of $n-$walk Bose configurations with no constraint on which of the two sinks the walks terminate. With no constraint, $n_{max} = 5$ and there are two maximal Fermi configurations in agreement with proposition 1.

		Fermi						Bose			
k_2	k_1	0	1	2	3	n_2	n_1	0	1	2	3
0		1	3	3	1	0		1	3	6	10
1		3	8	7	2	1		3	8	15	24
2		3	7	5	1	2		6	15	27	42
3		1	2	1	0	3		10	24	42	64

Table 2: *$f^{Fermi}_{k_1,k_2}$ and $f^{Bose}_{n_1,n_2}$ for the graph in figure 8(a).*

It should be noted that the numbers in the row $n_2 = 0$ of table 2 which were obtained by evaluating formula (2.6) are given by $f^{Bose}_{n_1}(H_1)$, where H_1 is the union of all paths from u to v_1. This is despite the fact that the summation in 2.6 refers only to configurations of positive numbers of Fermi walks. The reduced configurations are obtained by deleting the edges which only have walks to the sink i where $n_i = 0$. We now show that these reductions hold generally for arbitrary $n = n_1 + n_2$ and the observation extends to a general plane coverable multi-rooted graph.

Proposition 3. *With $f^{Bose}_{n_1,n_2,\ldots,n_s}(H_s)$ defined by (2.5)*

$$f^{Bose}_{n_1,n_2,\ldots,n_{s-1},0}(H_s) = f^{Bose}_{n_1,n_2,\ldots,n_{s-1}}(H_{s-1})$$

where H_{s-1} is the union of all paths which terminate at one of the first $s-1$ sinks.

Proof. Consider the case of just two sinks. The extension is clear. From (2.5)

$$f^{Bose}_{n_1,0}(H_2) = \sum_{k_1=1}^{c_1+1} \binom{n_1-1}{k_1-1} \sum_{k_2=1}^{c_2+1} (-1)^{k_2-1} f^{Fermi}_{k_1,k_2}(H_2) \tag{2.8}$$

Let P_i be the set of paths from u to v_i. For $p \in P_1$ let $K_1(p)$ be the subgraph obtained by taking the union of all paths in P_1 above p (excluding p) and let $K_2(p)$ be the subgraph formed by the union of all paths in P_2 excluding paths which are partly in $K_1(p)$. By partitioning the Fermi k_1-subsets of paths from u to v_1 according to the lowest path used

$$f_{k_1,k_2}^{Fermi}(H_2) = \sum_{p \in P_1} f_{k_1-1}^{Fermi}(K_1(p)) f_{k_2}^{Fermi}(K_2(p)) \tag{2.9}$$

Substituting in (2.8) and using

$$\sum_{k_2=1}^{c_2+1} (-1)^{k_2-1} f_{k_2}^{Fermi}(K_2(p)) = 1, \tag{2.10}$$

see lemma 1 below, we obtain

$$f_{n_1,0}^{Bose}(H_2) = \sum_{k_1=1}^{c_1+1} \binom{n_1-1}{k_1-1} \sum_{p \in P_1} f_{k_1-1}^{Fermi}(K_1(p)) \tag{2.11}$$

$$= \sum_{k_1=1}^{c_1+1} \binom{n_1-1}{k_1-1} f_{k_1}^{Fermi}(H_1) = f_{n_1}^{Bose}(H_1) \tag{2.12}$$

□

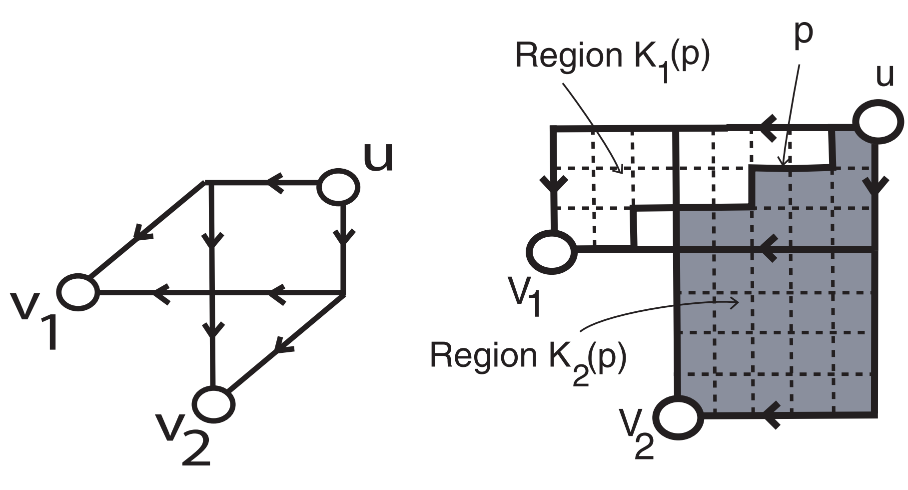


Figure 8: (a) A multi-rooted graph with paths from vertex u to either v_1 or v_2; (b) The split of the two intersecting walk rectangles into regions with initial vertices u_1 the final vertices v_1 and v_2 respectively.

Note that setting $n_2 = 0$ in (1.1) and using the proposition 3 gives the identity

$$f_{n_1}^{Bose}(W_{d,w_1}) f_{n_1+d}^{Bose}(W_{\ell_2,w_1}) = f_{n_1}^{Bose}(W_{d+\ell_2,w_1}) f_d^{Bose}(W_{\ell_2,w_1}) \tag{2.13}$$

Lemma 1. *The number $f_k^{Fermi}(H)$ of k-tuples of Fermi walk configurations on a two-rooted directed plane graph H satisfy*

$$\sum_{k\geq 1}(-1)^{k-1} f_k^{Fermi}(H) = 1.$$

Proof. Consider the POSET $\mathcal{F}$ of walks between two vertices of a graph H where the ordering on $\mathcal{F}$ is such that if walk w is *above* walk w' and distinct in at least one edge, then $w > w'$. A Fermi $k-$walk configuration is a k-subset $\{w_1, w_2, \dots w_k\}$ of $\mathcal{F}$ which is a proper chain in $\mathcal{F}$, i.e a collection of k distinct walks with, say $w_1 > w_2 > \cdots > w_k$.

Stanley 35 proves that: if P is a POSET with Möbius function μ, with minimum element $\bar{0}$, and maximum element $\bar{1}$ then

$$\mu(\bar{0},\bar{1}) = \sum_{k\geq 0}(-1)^k c_k \tag{2.14}$$

where c_k is the number of chains of length k of the type $\bar{0} = x_0 < x_1 < \cdots < x_k = \bar{1}$. We observe that c_0=0 unless $|P| = 1$ and when $|P| > 1$, then $c_1 = 1$ with the only chain $\bar{0} = x_0 < x_1 = \bar{1}$.

Note the chains in Stanley's theorem are special in that they always include $\bar{0}$ and $\bar{1}$. To relate the counts of Fermi walk configurations $f_k^{Fermi}(H)$ and the c_k we attach (i.e. add) a $\bar{0}$ and and a $\bar{1}$ to the Fermi partially ordered set $\mathcal{F}$ to make an enlarged POSET $\bar{\mathcal{F}}$. The element $\bar{0}$ is less than the lowest walk and the $\bar{1}$ is greater than the highest walk. All other partial orderings in $\bar{\mathcal{F}}$ are induced from $\mathcal{F}$. We then observe that there is a $1-1$ correspondence between the fixed-end chains of length $k+1$ in $\bar{\mathcal{F}}$ and the free chains containing k elements in $\mathcal{F}$. Using Stanley's result with $P = \bar{\mathcal{F}}$, we have

$$\mu(\bar{0},\bar{1}) = \sum_{k\geq 0}(-1)^k c_k = -1 + \sum_{k\geq 1}(-1)^{k-1} f_k^{Fermi}(H), \tag{2.15}$$

which gives the result

$$\sum_{k\geq 1}(-1)^{k-1} f_k^{Fermi}(H) = \mu(\bar{0},\bar{1}) + 1. \tag{2.16}$$

Checking the evaluation of the Möbius function for μ on the POSET $\bar{\mathcal{F}}$ we find that $\mu(\bar{0},\bar{1}) = 0$.

Note the Möbius function of all such $\bar{\mathcal{F}}$ has the property of $\mu(\bar{0},\bar{1}) = 0$ follows from the fact that the POSET $\bar{\mathcal{F}}$ has a single element of rank 1, namely the lowest walk $\bar{w}$ in H. Thus $\mu(\bar{0},\bar{0}) = 1$ by definition, and $\mu(\bar{0},\bar{w}) = -1$. All other walks satisfy $w > \bar{w} > \bar{0}$ and therefore satisfy $\mu(\bar{0},w) = 0$. In particular, $\mu(\bar{0},\bar{1}) = 0$. □

Note that lemma 1 also validates equation 2.2 for $n = 0$.

3 Application to the directed square lattice

3.1 Evaluation of Kreweras's determinant

Kreweras [27] considered the segment of Young's lattice $P_{\lambda/\mu}$ bounded by two partitions which we denote by μ and λ with $\mu_i \leq \lambda_i$. Kreweras proved the following results, (3.1) and (3.3), for the number of saturated chains (puissance) and the number of $n-$element multichains in $P_{\lambda/\mu}$ (in the case $n = 1$ the latter number was called the richesse). These are respectively the number of maximal Fermi walk configurations and the number of $n-$walk Bose configurations on the graph $H = \lambda/\mu$ which is the union of all paths corresponding to partitions in $P_{\lambda/\mu}$. λ/μ is a skew Young diagram [14].

$$f_{n_{max}}^{Fermi}(\lambda/\mu) = c(\lambda/\mu)! \det\left[\frac{1}{(\lambda_i - \mu_j - i + j)!}\right]_{1\leq i,j\leq w} \tag{3.1}$$

where the number of finite faces in the diagram of shape λ/μ is given by

$$c(\lambda/\mu) = \sum_{i=1}^{w}(\lambda_i - \mu_i). \tag{3.2}$$

$$\begin{aligned} f_{n}^{Bose}(\lambda/\mu) &= \det\left[\binom{\lambda_i - \mu_j + n}{i - j + n}\right]_{1\leq i,j\leq w} &(3.3)\\ &= \det\left[\frac{(i-j+n+1)_{\lambda_i-\mu_j-i+j}}{(\lambda_i - \mu_j - i + j)!}\right]_{1\leq i,j\leq w} &(3.4)\end{aligned}$$

The determinant (3.4) is clearly a polynomial in n of degree at most $c(\lambda/\mu)$ which is consistent with proposition 1(b). Equation (3.1) was derived independently by Kreweras but follows from (3.4) using proposition 1(a).

In the case $\mu_i = 0$ the determinant (3.1) may be reduced to a single product

$$\begin{aligned} f_{n_{max}}^{Fermi}(\lambda/0) &= \frac{c(\lambda/0)!}{\prod_{i=1}^{w}(\lambda_i + w - i)!}\det\left[(\lambda_i - i + 2)_{w-j}\right]_{1\leq i,j\leq w} &(3.5)\\ &= \frac{c(\lambda/0)!\prod_{1\leq i<j\leq w}(\lambda_i - \lambda_j + j - i)}{\prod_{i=1}^{w}(\lambda_i + w - i)!} &(3.6)\end{aligned}$$

This is applied below to graphs (a) and (b) in figure 1. Macdonald, [29] 2^{nd} ed. page 10 Example 1, shows that this formula is equivalent to the *hook length* formula

$$f_{n_{max}}^{Fermi}(\lambda/0) = \frac{c(\lambda/0)!}{\prod_{x\in\lambda} h(x)} \tag{3.7}$$

where the hook length $h(x)$ of cell x of the Young diagram λ is one more than the number of cells to the right or below x.

Example 1. In the case $\lambda_i = \ell, \mu_i = 0$ all possible directed paths on the rectangular grid $W_{\ell,w}$ may be used for walks and the corresponding vicious walker configurations are known as watermelons [13]. Substituting $\lambda_i = \ell$ in (3.6) immediately gives the number of maximal Fermi configurations in agreement with the result in table 1 obtained using proposition 1.

From (3.3)

$$f_n^{Bose}(W_{\ell,w}) = \det\left[\binom{\ell+n}{\ell+i-j}\right]_{1\le i,j\le w} \tag{3.8}$$

which may be simply evaluated using row and column operations to give

$$f_n^{Bose}(W_{\ell,w}) = \prod_{i=1}^{w} \frac{(n+i)_\ell}{(i)_\ell}. \tag{3.9}$$

Kreweras derives the following equivalent expression ([27] equation (54)).

$$f_n^{Bose}(W_{\ell,w}) = \frac{\ell!!w!!}{(\ell+w)!!}\prod_{j=1}^{w}\frac{(n+\ell+j-1)!}{(n+j-1)!} \tag{3.10}$$

where $k!! = \prod_{j=1}^{k}(j-1)!$.

In example 2 below, evaluation of the relevant determinant by row and column operations is not a simple matter. Instead the method of *LU* factorisation [8] will be used. The method is widely used by Andrews in the enumeration of plane partitions, for example see [1]. The application to combinatorics is described in detail by Krattenthaler [24]. We illustrate the method by first deriving (3.9). Notice that the elements of the required determinant are independent of w and using *Mathematica* to perform the decomposition for $w = 9$ we conjectured the general result

$$f_n^{Bose}(W_{\ell,w}) = \det(LU) \tag{3.11}$$

where L and U have dimension w with elements independent of w and given by

$$L_{ij} = \binom{n}{i-j}\frac{(j)_\ell}{(i)_\ell} \qquad \text{and} \qquad U_{ij} = \binom{\ell}{j-i}\frac{(n+j)_{i-j+\ell}}{(j)_{i-j+\ell}} \tag{3.12}$$

where the binomial coefficients are taken to be zero outside their usual limits. Thus L is lower triangular with unit diagonal elements and U is upper triangular. Hence $\det(LU) = \prod_{i=1}^{w} U_{ii}$ from which (3.9) follows.

To prove this conjecture let $LU = m$ so that

$$m_{ij} = \sum_{1\le k\le min(i,j)} L_{i,k}U_{kj}. \tag{3.13}$$

The summand is hypergeometric in both indices j and k and we have used Paule and Schorn's *Mathematica* implementation [31] of Zeilberger's algorithm [39], [40] to prove that the sum satisfies the recurrence relation

$$m_{ij} = \frac{\ell+i-j+1}{n+j-i}m_{i,j-1} \tag{3.14}$$

which is the recurrence relation satisfied by $\binom{\ell+n}{\ell+i-j}$. Also $m_{i,1} = L_{i1}U_{11} = \binom{\ell+n}{\ell+i-1}$ so by induction on j

$$m_{ij} = \binom{\ell+n}{\ell+i-j} \tag{3.15}$$

which proves the conjecture. An excellent account of Zeilberger's algorithm is given in the lecture notes of Wilf [37].

Equation (3.9) may also be obtained [12] using an alternative determinant [28], [15] but the polynomial dependence on n is not so obvious since it appears as the dimension of the determinant.

Example 2. Now suppose that the walks on $W_{\ell,w}$ are not allowed to go below the "wall" $y = x$ and that $\ell - w = d \geq 0$. We refer to these as watermelon configurations above a wall (see figure 1(b) for example). Notice that first step is common to all walks and is deleted in forming the Young diagram. This is the case $\mu_i = 0$, $\lambda_i = \ell - i$ and the graph λ/μ will be denoted by $\overline{W}_{\ell,w}$ where the bar suggests the wall.

A product formula for $f_n^{Bose}(\overline{W}_{\ell,w})$ was obtained in [25] (equation 7.2) and [7] (equation 2.9). In the latter case the number of configurations having a given number of contacts with the wall was also found. These results were not obviously polynomials in n but they may be rearranged into the form given in table 1. We now rederive this formula directly from Kreweras's determinant.

We note in passing that substituting $\lambda_i = \ell - i$ into (3.6) gives the number of maximal Fermi walk configurations

$$f_{n_{max}}^{Fermi}(\overline{W}_{\ell,w}) = \frac{c! \prod_{1 \leq i < j \leq w}(2i - 2j)}{\prod_{i=1}^{w}(\ell + w - 2i)!} \tag{3.16}$$

$$= \frac{c! 2^{\frac{1}{2}w(w-1)} \prod_{j=2}^{w}(j-1)!}{\prod_{j=1}^{w}(d + 2j - 2)!} = \frac{c! 2^{\frac{1}{2}w(w-1)}}{\prod_{j=1}^{w}(j)_{d+j-1}}. \tag{3.17}$$

This also follows from proposition 1(a) using $f_n^{Bose}(\overline{W}_{\ell,w})$ from table 1 as expected.

Evaluation of Kreweras's the determinant is not as straightforward as in example 1. Substituting $\lambda_i = \ell - i$ and $\mu_i = 0$ in (3.3)

$$f_n^{Bose}(\overline{W}_{\ell,w}) = \det \left[\binom{\ell - i + n}{\ell - 2i + j} \right]_{1 \leq i,j \leq w} \tag{3.18}$$

$$= \det \left[\binom{n + i + d - 1}{2i - j + d - 1} \right]_{1 \leq i,j \leq w} \tag{3.19}$$

where we have reversed the order of rows and columns so that, for fixed d, the elements are independent of w. It is important in what follows that w appears only as the size of the determinant. Removing common factors from each row of the determinant

$$f_n^{Bose}(\overline{W}_{w+d,w}) = f_n^{Bose}(W_{d,w}) \det M^{(d)}(n) \tag{3.20}$$

where

$$M^{(d)}(n) = \left[\binom{n+i-1}{2i-j-1}\frac{(i)_d}{(2i-j)_d}\right]_{1\le i,j\le w} \tag{3.21}$$

$$= \left[\frac{(n-i+j+1)_{2i-j-1}(i)_d}{(2i-j+d-1)!}\right]_{1\le i,j\le w} \tag{3.22}$$

Following the method of example 1 leads to the following proposition.

Proposition 4. *The matrix*

$$M^{(d)}(n) = L^{(d)}(n)U^{(d)}(n)$$

where $L^{(d)}$ and $U^{(d)}$ are lower and upper triangular matrices respectively of size $w \times w$. Moreover, the lower triangular matrix $L^{(d)}$, for $i \ge j$, has elements given by

$$L^{(d)}_{i,j}(n) = \binom{n+i-1}{i-j}\frac{(n-i+j+1)_{i-j}(j+d)_{i-j}}{(2j+d-1)_{2i-2j}}. \tag{3.23}$$

Proof.
The case $d = 0$

This is the case of non-crossing Dyck paths. Computation of the *LU*-decomposition of $M^{(0)}(n)$ for small values of w leads to the further conjecture

$$U^{(0)}_{ij}(n) = \binom{i-1}{j-i}\frac{(2n+j)_{2i-j-1}}{(j)_{2i-j-1}}. \tag{3.24}$$

Both conjectures are proven, as in example 1, by using Zeilberger's algorithm to show that $L^{(0)}(n)U^{(0)}(n) = M^{(0)}(n)$.

The case $d > 0$

In contrast with the case $d = 0$, $U^{(d)}_{i,j}$ for $d > 0$ and $j > 1$ no longer has a simple factorisation. However to prove the conjecture it is only necessary show that $(L^{(d)})^{-1}(n)M^{(d)}(n)$ is upper triangular. That is to show that

$$U^{(d)}_{ij}(n) = \sum_{k=1}^{i}(L^{(d)})^{-1}_{i,k}(n)M^{(d)}_{k,j}(n), \tag{3.25}$$

is zero for $i > j$.

The inverse of $L^{(d)}(n)$ can be checked to be lower triangular with entries

$$(L^{(d)})^{-1}_{i,j}(n) = (-1)^{i-j}\binom{n+i-1}{i-j}\frac{(n)_{i-j}(j+d)_{i-j}}{(2j+d-1)_{2i-2j}}. \tag{3.26}$$

and $M^{(d)}_{k,j}(n)$ is given by (3.21) which leads to

$$U^{(d)}_{i,j}(n) = \frac{(-1)^i(i+d-1)!(i+n-1)!}{(n-1)!(2i+d-2)!}U^r_{i,j} \tag{3.27}$$

where

$$U^r_{i,j} = \sum_{k=1}^{i} \frac{(-1)^k(-2+d+2k)!(-1-k+n+i)!}{(-1+k)!(-1+d-j+2k)!(j-k+n)!(-k+i)!} \tag{3.28}$$

For the case $j = i-1$ Zeilberger's algorithm , see ([39]), gives the recurrence

$$(d+i)U^r_{i,i-1} = (2-i)U^r_{i-1,i-2} \tag{3.29}$$

and noting that $U^r_{2,1} = 0$ and $(d+i) > 0$ we deduce that $U^r_{i,i-1} = 0$ for all $i \geq 2$.

Further use of Zeilberger's algorithm gives the following recurrence relation for $U^r_{i,j}$

$$\begin{aligned}
&(-1+i)(-2+d-j+2i)(-1+d-j+2i)U^r_{i,j} = \\
&\qquad -[30-16n+d^2(3+j-2i)-56i+22ni+36i^2-8ni^2 \\
&\qquad -8i^3-j^2(-2+n+i)+j(20-11n-25i+8ni+8i^2) \\
&\qquad +d(-19+6n+24i-4ni-8i^2+j(-9+2n+6i))]U^r_{i-1,j} \\
&\qquad +(2+j-i)(-5+d+2n+2i)(d+2(-2+n+i))U^r_{i-2,j}.
\end{aligned} \tag{3.30}$$

Setting $i = j+1$ the relation expresses $U^r_{j+2,j}$ in terms of $U^r_{j+1,j}$ and $U^r_{j,j}$. $U^r_{j+1,j} = 0$, resulting from the previous recurrence, and although $U^r_{j,j}$ is not zero it has a zero coefficient. Hence $U^r_{j+2,j} = 0$.

Having shown that there are two successive zeros in moving down column j from the diagonal (3.30) shows that the next element is zero and so on which gives the desired result that $U^r_{i,j} = 0$ for any $i > j$. From (3.27) it follows that $U^{(d)}_{i,j}(n) = 0$ for $i > j$ and we conclude that $U^{(d)}$ is upper triangular.

□

It follows from the proposition that

$$\det M^{(d)}(n) = \prod_{j=1}^{w} U^{(d)}_{jj}(n) \tag{3.31}$$

The required diagonal elements are determined by setting $i = j$ in (3.27) and (3.28) which gives

$$U^{(d)}_{jj}(n) = \frac{(n)_j(j+d-1)!}{(2j+d-2)!} \sum_{k=1}^{j} \frac{(-1)^{j-k}(2k+d-2)!}{(n-k+j)(k-1)!(j-k)!(2k-j+d-1)!}. \tag{3.32}$$

Zeilberger's algorithm [39], [40] yields the following second order recurrence relation satisfied by the sum (3.32).

$$(-5+d+2i)_4(-1+d+j+2n)(-6+d+3j+2n)U^{(d)}_{jj} =$$
$$+(-5+d+2i)_2(15i^4+i^3(32d+43n-87)+j^2(24d^2+4d(-35+17n)$$
$$+40n^2-187n+186)+j(8d^3+d^2(-71+35n)+d(199-199n+44n^2)$$
$$+2(-87+133n-57n^2+6n^3))+(d-2)(d^3+2d^2(-5+3n)$$
$$+d(31-40n+12n^2)-30+62n-40n^2+8n^3))U^{(d)}_{j-1,j-1}$$
$$-(j-2)(d+j-2)(j+n-2)(d+3j+2n-3)(-5+d+2j+2n)_2U^{(d)}_{j-2,j-2}. \tag{3.33}$$

This leads to the result

$$U^{(d)}_{jj}(n) = \frac{(2n+d+j)_{j-1}}{(d+j)_{j-1}}. \tag{3.34}$$

which satisfies 3.33 and agrees with (3.32) for $j = 1$ and 2 and therefore for all j. Setting $d = 0$ gives a formula which agrees with (3.24). Combining (3.34) with (3.31) and (3.20) gives the result in table 1.

From (3.34) it may be seen that $U^{(d)}_{jj}$ satisfies the first order recurrence

$$U^{(d)}_{jj}(n) = \frac{(j+d-1)(2n+2j+d-3)_2}{(2n+j+d-1)(2j+d-3)_2}U^{(d)}_{j-1,j-1}(n). \tag{3.35}$$

Zeilberger's algorithm as implemented by Paule and Schorn [31] failed to yield the first order relation (3.35) for $U^{(d)}_{jj}$ even though it satisfies the second order relation (3.33) induced by it. A simple example which shows that the algorithm does not always produce the recurrence of lowest order is given in [31].

Example 3. Non crossing walks between two walls.

Now consider two sets of n walks which start at the origin and are not allowed to go below the wall $y = x$ or above the wall $y = x + h$. In the first case (a) the walks end on the upper wall after $2w+h$ steps and the resulting graph will be denoted $\overline{\overline{W}}^{\uparrow}_{h,w}$. See figure 1(f) for the case $h = 5, w = 3$. In the second case (b) the walks follow Dyck paths (rotated through $\frac{\pi}{4}$) ending on the lower wall after $2w$ steps. Call the graph which is the union of all such walks $\overline{\overline{W}}^{\downarrow}_{h,w}$. See figure 1(c) for the case $h = 3, w = 4$.

When $h = 2$ both graphs are special cases of the series parallel graph with $p = 2$ discussed in section 2 so $f^{Bose}_n = (n+1)^\ell$ and $f^{Fermi}_{n_{max}} = \ell!$. In case (a), $\ell = w$ and in case (b), $\ell = w - 1$.

When $h = 3$ the corresponding Young diagrams are known [34] as *staircase border strips*. There is a bijection, [34], the solution to example 7.64, between standard Young tableaux on these strips and alternating permutations of $\{1, 2, \ldots c\}$ where c is the number of faces of the strip. These are enumerated by the Euler numbers E_c which are shown, [33] section 3.16, to have generating function

$$\sum_{c\geq 0} E_c x^c/c! = \tan x + \sec x \tag{3.36}$$

and satisfy the recurrence relation

$$2E_{c+1} = \sum_{i=1}^{c} \binom{c}{i} E_i E_{c-i} \qquad c \geq 1 \tag{3.37}$$

The proof of part (b) of proposition 1 describes a correspondence between standard Young tableaux and maximal Fermi walk configurations which leads to the results

$$f_{n_{max}}^{Fermi}(\overline{\overline{W}}^{\uparrow}_{3,w}) = E_{2w} \qquad \text{and} \qquad f_{n_{max}}^{Fermi}(\overline{\overline{W}}^{\downarrow}_{3,w}) = E_{2w-3}. \tag{3.38}$$

Extension of these results to f_n^{Bose} is unlikely to be found since the polynomials do not completely factorise over the integers (see below).

In case (a) the relevant Young diagram has $\lambda_i = w + h - i - 1$ and $\mu_i = w - i$ so

$$f_n^{Bose}(\overline{\overline{W}}^{\uparrow}_{h,w}) = \det\left[\binom{h-i+j+n-1}{i-j+n}\right]_{1\leq i,j\leq w} \tag{3.39}$$

$$= \det\left[\frac{(i-j+n+1)_{h-2(i-j)-1}}{(h-2(i-j)-1)!}\right]_{1\leq i,j\leq w} \tag{3.40}$$

Notice that the elements of the matrix are independent of w and that, for $2i \leq h$, the i^{th} row has factor $(n+i)_{h-2i+1}$. It follows using (3.9) that for $2w \leq h$ the determinant has a factor $f_n^{Bose}(W_{h-w,w})$. In the case $2w > h$ the factor is $f_n^{Bose}(W_{\lceil \frac{h}{2}\rceil,\lfloor \frac{h}{2}\rfloor})$. It is interesting that the factor is the Bose polynomial for the largest rectangular graph which is part of the Young diagram. For example, in the case $h = 5, w = 3$ (see figure 1(f))

$$\frac{f_n^{Bose}(\overline{\overline{W}}^{\uparrow}_{5,3})}{f_n^{Bose}(W_{3,2})} = \frac{(100800 + 220800n + 205210n^2 + 103545n^3 + 29917n^4 + 4695n^5 + 313n^6)}{100800}. \tag{3.41}$$

$\overline{\overline{W}}^{\uparrow}_{5,3}$ has 12 faces and $f_n^{Bose}(\overline{\overline{W}}^{\uparrow}_{5,3})$ is a polynomial of degree 12 in agreement with proposition 1 but complete factorisation over the integers fails to occur.

In case (b), if $h \geq w$ the second wall is never crossed so $f_n^{Bose}(\overline{\overline{W}}^{\downarrow}_{h,w}) = f_n^{Bose}(\overline{W}_{w,w})$. Assuming $h < w$ the corresponding skew Young diagram has $\lambda_i = w - i, \mu_i = w - h - i + 1$ for $1 \leq i \leq w - h$ and $\mu_i = 0$ for $w - h + 1 \leq i \leq w$. *Mathematica* evaluation of the resulting determinant (3.3) in the case $h = 3, w = 4$ gives

$$f_n^{Bose}(\overline{\overline{W}}^{\downarrow}_{3,4}) = \frac{1}{30}(1+n)(2+n)(3+2n)(5+6n+2n^2) \tag{3.42}$$

$$= \frac{1}{5}(5+6n+2n^2) f_n^{Bose}(\overline{W}_{3,3}). \tag{3.43}$$

Inspection of results for greater values of w and h leads to the conjecture that $f_n^{Bose}(\overline{W}_{h,h})$ is a factor of $f_n^{Bose}(\overline{\overline{W}}^{\downarrow}_{h,w})$. In this case the factor is the Bose polynomial for the largest single surface subgraph.

3.2 The polynomial property for star configurations

As was noted before the number of star configurations has not been expressed as a single determinant with elements that are polynomial in n. However proposition 1 may still be applied by converting the graphs S_t and $\bar{S}_{t,w}$ into two-rooted graphs by connecting the possible terminal vertices to an additional root vertex. In the case of S_t the resulting graph has an additional t faces making the total $\frac{1}{2}t(t+1)$. For $\bar{S}_{t,w}$ the number of additional faces is w so that the total number is $w(t-w)$. In both cases the number is in agreement with the degree of the formula in table 1. We now recall the surprising bijective relations between the numbers of star and watermelon configurations which also lead to the polynomial property.

3.2.1 The number of star configurations

In the case of S_t a conjectured formula [2] (see table 1) was proved [25] by showing a correspondence with plane partitions the number of which was conjectured by Bender and Knuth [5] and proved by Gordon [17]. An alternative approach via Young tableaux shows the following equalities.

$$f^{Bose}_{2m+1}(S_t) = 2^t f^{Bose,banana}_{2m,t}(S_t) = 2^t f^{Bose}_m(\overline{W}_{t+1,t+1}) \tag{3.44}$$

where $f^{banana}_{2m,t}$ is the number of banana configurations consisting of $2m$ walks having t steps. Because $f^{Bose}_m(\overline{W}_{t+1,t+1})$ is polynomial in m the equality determines $f^{Bose}_n(S_t)$ for all n with the result in table 1. The number of banana configurations is

$$f^{Bose,banana}_{2m,t}(S_t) = \prod_{1\le i\le j\le t} \frac{2m+i+j}{i+j} = \prod_{k=1}^{t+1} \frac{(2m+k)_{k-1}}{(k)_{k-1}}. \tag{3.45}$$

The bijection between square lattice walk configurations and Young tableaux, which leads to (3.44), is illustrated in [18]. Briefly, if the steps of each walk are labelled 1 to t, the i^{th} column of the corresponding tableau is a list of the labels of the vertical steps of the i^{th} walk from the bottom. The tableau entries are strictly increasing down the columns and weakly increasing along the rows. The tableau for the banana configuration in figure 5 is also shown in the figure. In general the number of banana configurations of $2m-$walks, each with t steps, is equal to the number of tableaux with entries $\le t$ having rows of even length $\le 2m$ (the inequality is due to the fact that some of the walks may have no left steps). In [10] the number of such tableaux (actually the transposed tableaux) is denoted by $b_{t,m}$ and the tableaux are shown to biject to nested Dyck paths of length $2t+2$, which are the Bose configurations on $\overline{W}_{t+1,t+1}$. The bijection is illustrated diagrammatically on page 95 of Krattenthaler's book [23]. The first part of (3.44) follows from a Pieri formula [14].

3.2.2 The number of star configurations above a wall

The graph $\overline{S}_{t,w}, t \geq 2w$ may be converted to a shifted Young diagram $\overline{Y}_{t,w}$ by removing the common first step and adding staircase to the left hand side. The diagram has trapezoidal shape with top edge of length $t-1$ and width w. The number of faces in this diagram is $c(\overline{Y}_{t,w}) = w(t-w)$ which is the same as for $W_{t-w,w}$. Theorem 1 of Proctor [32] states that the number of shifted plane partitions on $\overline{Y}_{t,w}$ is equal to the number of ordinary plane partitions on $W_{t-w,w}$ with part size bounded by n in both cases. Using a correspondence between plane partitions, potentials, flows and Bose configurations [4] it follows that

$$f_n^{Bose}(\overline{S}_{t,w}) = f_n^{Bose}(W_{t-w,w}). \tag{3.46}$$

It is possible that this relation can be proved bijectively but Proctor found the question of such a bijection to be a complete mystery. However Haiman [19] obtained a bijection between standard Young tableaux on $\overline{Y}_{t,w}$ and $W_{t-w,w}$. By the construction described in the proof of proposition 1 this yields a bijection between maximal Fermi walk configurations on $\overline{S}_{t,w}$ and $W_{t-w,w}$ so that

$$f_{n_{max}}^{Fermi}(\overline{S}_{t,w}) = f_{n_{max}}^{Fermi}(W_{t-w,w}). \tag{3.47}$$

3.3 The number of two-point stars

In [12] it is shown that the number of vicious walk configurations on the square lattice in which the i^{th} walk makes q_i vertical steps (in the current description) is given by

$$w_t(q_1, q_2, \ldots, q_n) = \prod_{1 \leq i < j \leq n} (q_j - q_i + j - i) \prod_{k=1}^{n} \frac{(t+k-1)!}{(q_k+k-1)!(t-q_k+n-k)!} \tag{3.48}$$

The number of watermelon configurations is obtained by setting $q_i = w$ and $t - q_i = \ell$

$$f_n^{Bose}(W_{\ell,w}) = \prod_{j=1}^{n} \frac{(j-1)!(\ell+w+j-1)!}{(w+j-1)!(\ell+j-1)!} \tag{3.49}$$

which may be rearranged to give the formula in table 1.

For two-point stars (see figure 6) we set $q_i = w_1$ for $i = 1$ to n_1 and $q_i = w_2$ for $i = n_1 + 1$ to n with $w_1 \leq w_2$

$$\prod_{1 \leq i < j \leq n} (q_j - q_i + j - i) = \prod_{i=1}^{n_1} (i-1)!(d+i)_{n_2} \prod_{j=1}^{n_2} (j-1)! \tag{3.50}$$

where $d = w_2 - w_1$. The number of two-point stars may be written as

$f^{Bose}_{n_1,n_2}(W^{\ell_2,w_2}_{\ell_1,w_1}) = f_1 f_2$ where, with $\ell_i = t - w_i$

$$f_1 = \prod_{j=1}^{n_1} \frac{(j-1)!(\ell_1+w_1+j-1)!(d+j)_{n_2}}{(w_1+j-1)!(\ell_1+n_2+j-1)!} \tag{3.51}$$

$$= f^{Bose}_{n_1}(W_{d,w_1}) \prod_{j=1}^{n_1} \frac{(d+j+w_1)_{\ell_2}(d+j)_{n_2}}{(d+j)_{\ell_2+n_2}} \tag{3.52}$$

$$= f^{Bose}_{n_1}(W_{d,w_1}) \prod_{j=1}^{n_1} \frac{(d+j+w_1)_{\ell_2}}{(d+j+n_2)_{\ell_2}} \tag{3.53}$$

and

$$f_2 = \prod_{j=1}^{n_2} \frac{(\ell_2+d+w_1+n_1+j-1)!(j-1)!}{(\ell_2+j-1)!(d+w_1+n_1+j-1)!} \tag{3.54}$$

$$= f^{Bose}_{n_2}(W_{d,\ell_2}) \prod_{j=1}^{n_2} \frac{(\ell_2+d+j)_{n_1+w_1}}{(d+j)_{n_1+w_1}} \tag{3.55}$$

$$= f^{Bose}_{n_2}(W_{d,\ell_2}) \prod_{j=1}^{n_2} \frac{(d+j+w_1+n_1)_{\ell_2}}{(d+j)_{\ell_2}}. \tag{3.56}$$

which leads to

$$f^{Bose}_{n_1,n_2}(W^{\ell_2,w_2}_{\ell_1,w_1}) = f^{Bose}_{n_1}(W_{d,w_1}) f^{Bose}_{n_2}(W_{\ell_2,d}) \prod_{j=1}^{\ell_2} \frac{(d+n+j)_{w_1}}{(d+j)_{w_1}} \tag{3.57}$$

$$= \frac{f^{Bose}_{n_1}(W_{d,w_1}) f^{Bose}_{n_2}(W_{\ell_2,d}) f^{Bose}_{n+d}(W_{\ell_2,w_1})}{f^{Bose}_{d}(W_{\ell_2,w_1})} \tag{3.58}$$

where we have used the formula for $f^{Bose}_n(W_{\ell,w})$ in table 1. We note that $f^{Bose}_n(W_{\ell,w})$ is invariant under any permutation of the variables ℓ, w and n, a result of duality [6]. It is interesting that the three numerator factors correspond to the three rectangular areas into which $W^{\ell_2,w_2}_{\ell_1,w_1}$ naturally divides (see figure 6). The third factor is polynomial in n relating to the fact that all n walks pass through the corresponding area. The polynomial dependence on n_1 and n_2 is in agreement with proposition 2. For example the highest power of n_1 is $c(W_{d,w_1}) + c(W_{\ell_2,w_1})$. The denominator normalises the formula to unity when $n_1 = n_2 = 0$.

4 Discussion

Previous results [2] for Bose and Fermi configurations on the directed square lattice have been extended to two-rooted coverable plane graphs. In particular the number of n-walk Bose configurations is shown to be a polynomial in n whose degree is the number of finite faces of the underlying graph. Kreweras [27] showed that the number of $n-$element multichains on the segment of Young's lattice $P_{\lambda/\mu}$ bounded by the partitions μ and λ could be expressed as a determinant the elements of which are polynomials in n. He also showed that the paths on the rectangular graph $W_{\ell,w}$ can be mapped to partitions so that multichains on Young's lattice correspond to Bose walk configurations on $W_{\ell,w}$. Multichains on the segment $P_{\lambda/\mu}$ correspond to Bose configurations on the skew Young diagram λ/μ the number of which is a polynomial in n in agreement with our general result.

Previous results for vicious walks, which correspond to Bose configurations, were based on the Lindstöm-Gessel-Viennot determinant [28],[16] which has dimension equal to the number of walks and is therefore not manifestly polynomial in n. We have arranged these into polynomial form and summarised the results in table 1. In the case of Dyck walks between two walls the following formula obtained by Krattenthaler *et al* [26] is not easily so arranged.

$$f_n^{Bose}(\overline{\overline{W}}^{\downarrow}_{h,w}) = \sum_{k_1,k_2,\ldots,k_n=-\infty}^{\infty} \prod_{i=1}^{n} A_i \prod_{1\le i<j\le n} B_{i,j} \tag{4.1}$$

where

$$A_i = \frac{(2i-1+2k_i(h+2n))(2w+2i-2)!}{(w+i+n-1+k_i(h+2n))!(w-i+n-k_i(h+2n))!} \tag{4.2}$$

and

$$B_{i,j} = [j-i+(h+2n)(k_j-k_i)][i+j-1+(h+2n)(k_i+k_j)] \tag{4.3}$$

We have applied Kreweras's determinant formula to the special subgraphs $\overline{W}_{\ell,w}$ and $\overline{\overline{W}}^{\downarrow}_{h,w}$ corresponding respectively to walks above the wall $y=x$ and between two walls $y=x$ and $y=x+h$. It is shown that $f_n^{Bose}(\overline{W}_{\ell,w})$ completely factorises with rational coefficients and for $\ell > w$ has a factor $f_n^{Bose}(W_{\ell-w,w})$. $f_n^{Bose}(\overline{\overline{W}}^{\downarrow}_{h,w})$ has no such complete factorisation but when $h \le w$ it is found to have a factor $f_n^{Bose}(\overline{W}_{h,h})$.

The polynomial property of the number of Bose configurations for two-rooted graphs has been extended to graphs with several terminal roots where the number n_i of walks terminating at the i^{th} root is given. This is illustrated by the exact formula for two-point stars on the square lattice which shows a remarkable factorisation into the Bose polynomials for rectangles into which it naturally partitions.

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