Percolation theory on multirooted directed graphs

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The multiroot connectedness $P_{u\mathbf{v}}$ of a directed graph G between the vertex u and the collection of vertices $\mathbf{v} = \{v_1,...,v_n\}$ is the probability that there are (directed) paths from u to each of the vertices v_i, $i = 1,...,n$, when each edge and vertex has a given probability of being independently deleted. The properties of the coefficients in the expansion

$P_{u\mathbf{v}}(G) = \Sigma_{A'\subseteq A}\vec{d}_{u\mathbf{v}}(G')\Pi_{a\in A'}p_a\,\Pi_{w\in V'}p_w$,

where A and V are the arc and vertex sets of G respectively, $p_a(p_w)$ is the probability that an arc a (vertex w) is not deleted, and G' is the arc set A' together with its incident vertices V', are considered. The values of $\vec{d}_{u\mathbf{v}}(G)$ are characterized as follows: $\vec{d}_{u\mathbf{v}}(G)$ is shown to be nonzero if and only if G is coverable by paths and has no directed circuit. For this case $\vec{d}_{u\mathbf{v}}(G') = (-1)^{t_{u\mathbf{v}}+n} = (-1)^{\nu(G)}$, where $t_{u\mathbf{v}}$ is the maximal number of independent directed paths between u and the set $\mathbf{v}$, and $t_{u\mathbf{v}}$ is shown to be equal to $\nu(G)+n$, where $\nu(G)=|E|-|V|+1$ is the cyclomatic number of G.

INTRODUCTION

This paper generalizes the results in Ref. 1 on two rooted directed graphs to directed graphs with many roots. The weights associated with such graphs and their properties proved here are required in a forthcoming paper[2] on percolation theory in a gas. The result $\vec{d}_{u\mathbf{v}}(G) = (-1)^{\nu(G)}$ is also used in developing a relation between graphical expansions and renormalization for the percolation problem.[3]

Many of the proofs of the results given here follow those given in Ref. 1 and so they are omitted. Only the differences will be emphasized between the many-root and two-rooted situations.

A completely different proof of Theorem 5 on the directed–undirected d weight relation is given[4] in contrast to the inductive proof in Ref. 1.

Consider a graph G with input vertex u and output vertices $\boldsymbol{v}=\{v_1,...,v_n\}$ and suppose that a subset of the edges and vertices is deleted. G is sometimes denoted by $G_{u\,v}$ if we wish to emphasize the root points of G. Let S_{u,v_i} be the set of all self-avoiding paths (following the arrows if G is directed) from u to v_i on G. For $s\in S_{u,v_i}$ define the indicator random variable

$$\gamma(s)=\begin{cases}1, & \text{if } s \text{ is open,}\\ 0, & \text{if } s \text{ is closed,}\end{cases}$$

in a given state of multilation of G. Now

$$\gamma_i\equiv\sum_{\phi\subset S_i\subseteq S_{u,v_i}}(-1)^{|S_i|+1}\prod_{s\in S_i}\gamma(s) \tag{2.1}$$

is one or zero according as there is at least one open path or no open path from u to v_i. The pair-connectedness is therefore given between u and v_i by

$$P_{u,v_i}=\langle\gamma_i\rangle=\sum_{\phi\subset S_i\subseteq S_{u,v_i}}(-1)^{|S_i|+1}\langle\prod_{s\in S_i}\gamma(s)\rangle.$$

Now let $g(S_i)$, $V(S_i)$, $E(S_i)$, $A(S_i)$ be the graph, the vertex set, the edge set, and the arc set obtained by taking the union of the paths in S_i. Thus if p_α is the probability that element α is not deleted, then

$$\langle\prod_{s\in S_i}\gamma(s)\rangle=\prod_{v\in V(S_i)}p_v\cdot\prod_{e\in E(S_i)}p_e,$$

since this is the probability that all the paths in S_i are open. The result may be expressed as a sum over subgraphs of G by grouping together all S_i for which $E(S_i)=E'$, and $g(S_i)=G'$; thus

$$P_{u,v_i}=\sum_{E'\subseteq E}d_{u,v_i}(G')\prod_{v\in V'}p_v\prod_{e\in E'}p_e,$$

where the "d weight" is given by

$$d_{u,v_i}(G')=\sum_{\substack{\phi\subset S_i\subseteq S_{u,v_i}\\ g(S_i)=G'}}(-1)^{|S_i|+1}.$$

For the directed case read A for E and $\vec{d}$ for d.

The extension to many outputs is now straightforward.

$P_{u,v}$ = probability that there is a path from u to all of the output vertices

$$=\langle\prod_{i=1}^{n}\gamma_i\rangle=\sum_{E'\subseteq E}d_{u,v}(g')\prod_{v\in V'}p_v\prod_{e\in E'}p_e \tag{2.2}$$

where

$$d_{u,v}(G')=\prod_{i=1}^{n}\sum_{\substack{\phi\subset S_i\subseteq S_{u,v_i}\\ g(\cup_{i=1}^{n}S_i)=G'}}(-1)^{|S_i|+1}. \tag{2.3}$$

If there does not exist $S=\bigcup_{i=1}^{n}S_i$ such that $g(S)=G'$, then $\vec{d}_{u\,v}(G')=0$ from (2.3).

To establish a contraction–deletion rule for $\vec{d}_{u\,v}$ weights let $p_a=1, p_w=1$ in (2.2). Then

$$\vec{\gamma}_{u\,v}(G)=\sum_{A'\subseteq A}\vec{d}_{u\,v}(G'), \tag{2.4}$$

where the connectedness indicator $\vec{\gamma}_{u\,v}(G)$ is 1 if there are paths from u to each v_i of $G_{u\,v}$ and zero otherwise.

Since the set of all subsets of the arc set A form a lattice, (2.4) may be inverted[2] to give

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$$\vec{d}_{uv}(G)=\sum_{A'\subseteq A}(-1)^{|A\setminus A'|}\vec{\gamma}_{uv}(G') . \tag{2.5}$$

Now consider a particular arc $a\in A$ and divide the sum according as $a\in A'$ and $a\notin A'$. We get

$$\vec{d}_{uv}\sum_{\substack{A'\subseteq A\\ a\in A'}}(-1)^{|A\setminus A'|}\vec{\gamma}_{uv}(G')+\sum_{\substack{A'\subseteq A\\ a\notin A'}}(-1)^{|A\setminus A'|}\vec{\gamma}_{uv}(G'). \tag{2.6}$$

Let G^γ be the graph G contracted along the arc a; that is, G^γ is the graph $G\setminus\{a\}$ with the adjacent vertices of a identified. Also we define G^δ to be the graph G with arc a deleted. If G' is a subgraph of G, we define $G^{\gamma'}$ to be the graph with the arc and vertex set of G' with arc a contracted, and $G^{\delta'}$ to be G' less the arc a.

Thus if we ensure the arc a is oriented out of u it is clear that $\vec{\gamma}_{uv}(G')=\vec{\gamma}_{uv}(G^{\gamma'})$ when $a\in A'$. This true even if the arc a has a root v_i as an adjacent vertex if we assume that when $u=v_i$, then u and v_i are connected. Furthermore in the second summation G' does not contain the arc a and so $\vec{\gamma}_{uv}(G')=\vec{\gamma}_{uv}(G^{\delta'})$ since $G'=G^{\delta'}$.

It follows that

$$\vec{d}_{uv}(G)=\vec{d}_{uv}(G^\gamma)-\vec{d}_{uv}(G^\delta) \tag{2.7}$$

subject to a being oriented out of u.

Remark: When $u=v_i$ on contraction, $\vec{d}_{uv}(G^\gamma)=\vec{d}_{uv'}(G^\gamma)$, where $v'=v\setminus\{v_i\}$ if we define $\vec{\gamma}_{uu}(G^\gamma)=1$.

2. STATEMENT OF RESULTS

As stated above the d weight of a graph which is not coverable is zero. The following theorems apply to the d weight of a coverable multirooted graph G.

Theorem 1: The directed d weight is ± 1 or 0.

Theorem 2: The directed d weight is zero if an only if G has a circuit.

Definition: A collection $\mathscr{C}=\{\pi_i|i=1,...,n\}$ of (directed) paths on a coverable directed graph $G_{u\boldsymbol{v}}$ is said to be independent if the matrix $M=[m_{ij}]$ has maximal row rank where $\pi_i=\sum_{j=1}^{l}m_{ij}a_j$, $\pi_i\in\mathscr{C}$, and $a_j,j=1,...,l$ is the collection of arcs in G_{uv}, and $m_{ij}\in\mathbb{Q}$. Such a collection $\mathscr{C}$ is said to be *maximal* if every path not in $\mathscr{C}$ is dependent on the elements of $\mathscr{C}$. Obviously the number of paths in such a class is an invariant of G_{uv}. We will call such a set a maximal independent set (MIS) for G_{uv}.

Theorem 3: If the directed graph has no circuit, then

$$\vec{d}_{uv}(G)=(-1)^{t_{uv}+n}$$

where t_{uv} is the order of a MIS for G_{uv}

Theorem 4: For a directed graph G with no circuit $t_{uv}=\nu(G)+n$, where $\nu(G)=|E|-|V|+1$ is the cyclomatic number for G.

Note $\vec{d}_{uv}(G)=(-1)^{\nu(G)}$ follows from Theorems 3 and 4.

Theorem 5: The undirected d weight of G is equal to the sum of the directed $\vec{d}$ weights of G over all possible orientations of G.

3. Remarks and proofs of results

The basic graph which is the terminal stage of the reduction process using the contraction–deletion rule which was described in Sec. 1 is the multiroot parallel graph: Every arc is adjacent to the input root point u and one of the output root points of $\boldsymbol{v}$.

Proof of Theorem 1: First of all is it assumed that $\boldsymbol{v}$, the set of output root points, contains sinks and at each stage in the reduction process the arc $[u,w]$ is chosen so that w is not a sink. The theorem then follows since the reduction process ensures that $\vec{d}_{uv}(G)=0$ *or* $|\vec{d}_{uv}(G)|=|\vec{d}_{uv'}(G')|$, where G' is a parallel graph with input u and output $\boldsymbol{v}_k=\{v_{i_1},...,v_{i_k}\}$, the set of sinks in $\boldsymbol{v}$. Therefore, $|d_{uv}(G)|=\pm 1$ or 0.

For the purposes of calculation the assumption that $\boldsymbol{v}_k\neq\emptyset$ is no real restriction because $\vec{d}_{uv}(G)=\vec{d}_{uv}(G_e)$ where G_e is the union of G with a single arc b attached to a root point v_i where v_i is replaced by v_e as the root point, and b is oriented from v_i to v_e.

It can be further seen that if G does not have any root points which are sinks then there is a directed circuit in G and so the first part of Theorem 2 applies which will result in $\vec{d}_{uv}(G)=0$.

Proof of Theorem 2: This proof follows the corresponding Theorem 3 in Ref. 1. The only essential difference is that if $[u,w]$, the arc upon which the contraction–deletion rule is applied, is such that w is a root, then

$$\{\pi_i'\}_1^r\cup\{\pi_i\}_1^s\cup\{\pi_i\circ\pi_j''\}_1^{s,t}$$

is a covering for G^δ.

Proof of Theorem 3: As in the proof of the first part of Theorem 2 ($\rightarrow$), we can apply the contraction–deletion to an arc a of G with vertices u and w and contain two cases:

(1) $\vec{d}_{uv}(G^\gamma)=\vec{d}_{uv}(G)$, G^γ has no circuit,

(2) $\vec{d}_{uv}(G^\delta)=-\vec{d}_{uv}(G)$, G^δ has no circuit,

where G^γ and G^δ are both coverable. When w is not a root point it is easily shown, as in Theorem 4,[1] that number of independent paths in G and G^γ are equal and differs from the number independent paths in G^δ by one.

If w is a root point say v_1, then there is a natural one–one correspondence between paths on G^γ and the set of paths on G not including the root path π consisting of the (contracted) arc a between u and v_1.

A maximal independent set of contracted paths $\mathscr{C}^\gamma$ on G^γ gives rise to an independent set $\mathscr{C}'$ on G using the 1–1 correspondence.

We claim that $\mathscr{C}=\mathscr{C}'\cup\{\pi\}$ is a maximal independent set for paths on G. The set $\mathscr{C}$ is independent because if $\pi=\Sigma_{i=1}^n\alpha_i\pi_i$, $\pi_i\in\mathscr{C}'$, $\alpha_i\in\mathbb{Q}$, then on the contracted graph G^γ, $\pi^\gamma=\Sigma_{i=1}^n\alpha_i\pi_i^\gamma$; but $\pi^\gamma=\mathbf{0}$, the π_i^γ are independent, and not all the α_i's are zero since $\pi\neq\mathbf{0}$, so we have a contradiction.

To show that $\mathscr{C}$ is maximal on G suppose there exists a path $\pi_0(\neq\pi)$ such that $\mathscr{C}\cup\{\pi_0\}$ is independent. However the contracted path π_0^γ must be a linear sum of the MIS for G^γ, that is

$$\pi_0^\gamma=\sum_{i=1}^{n}\alpha_i\pi_i^\gamma,\ \pi_i^\gamma\in C^\gamma,\ \alpha_i\in\mathbb{Q},\ \alpha_i\neq 0, \text{ for some } i.$$

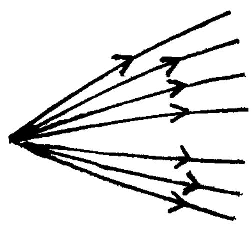
For the cases $a\in\pi_0$, $a\notin\pi_0$ the coefficients α_i at the vertex w sum to 1 and 0, respectively, therefore

$$\pi_0=\sum_{i=1}^{n}\alpha_i\pi_i, \text{ and so we have a contradiction.}$$

Thus when w is a root point the number of independent paths in G and G^γ differ by one.

The other cases follow as in Theorem 4.[1]

Repeated application of the contraction–deletion process results in a parallel graph H of the following type:



Note there will in general be only k of the original n roots, say $v'=\{v_1,...,v_k\}$, the set of sinks which is nonempty by the no-circuit property. Moreover $d_{uv}(H)=+1=(-1)^{t_{uv_k}+k}$, where $t_{uv_k}(=k)$ is the number of independent paths in H.

Furthermore in the reduction process from G to H the number of independent paths has changed by one: (a) for every reduction by deletion, (b) for every reduction by contraction of a root point.

Therefore, the total change in the number of independent paths $t_{uv}-t_{uv_k}=(n-k)+\#$ of deletions

and hence the number of sign changes in the $\vec{d}$ weight for the sequence of graphs from G to H is

$$t_{uv}-t_{uv_k}-(n-k),$$

since $n-k$ is the number of roots contracted.

Therefore,

$$\vec{d}_{uv}(G)=(-1)^{t_{uv_k}+k+t_{uv}-t_{uv_k}-(n-k)}$$

$$=(-1)^{t_{uv}+n},$$

where t_{uv} is the number of independent paths in G.

Proof of Theorem 4: Let $\Pi=\{\pi_i|i=1,...,l\}$ be a MIS for G. The generalization required here is that G_0 be a subgraph of G consisting of a subset $\{\pi_i|i=1,...,n\}$ of Π covering all the root points. We can always take G_0 to be a tree.

The order of a MIS for G_0 is $n=|v|$. Moreover it is easy to check that $\vec{d}_{uv}(G)=+1$. Let $\rho_1,...,\rho_k$ be paths in Π which are not in G_0.

Define a sequence of subgraphs $G_i=G_0\cup\{\cup_{j=1}^{i}\rho_j\}$. For the sequence G_i define $\mu(G_i)=|E_i|-|V_i|+1+n$. We note $\mu(G_0)=n$ which is the number of paths in a MIS for G_0.

The argument now closely follows Theorem 5[1] and we obtain $\mu(G)=\mu(G_k)=t_{uv}$. Now $\mu(G)=\nu(G)+n$, where $\nu(G)$ is the cyclomatic number.[5] Therefore,

$$\vec{d}_{uv}(G)=(-1)^{t_{uv}+n}=(-1)^{\nu(G)+n+n}=(-1)^{\nu(G)}.$$

Proof of Theorem 5: Let G be an undirected graph and $\mathscr{D}(G)$ be the set of graphs obtained by directing G in all possible ways. Now consider the paths on G as directed from u to v_i. Any subset of S_i which covers G either covers some $\vec{g}\in\mathscr{D}(G)$ or $g(S_i)$ has at least one loop of length 2. Let $\mathscr{L}(G)$ be the set of graphs obtained from g by directing it in all possible ways and replacing at least one edge by a loop of length 2. Thus

$$d_{uv_i}(G)=\sum_{\vec{g}\in\mathscr{D}(G)}\ \sum_{\substack{\phi\subset S_i\subseteq S_{uv_i}\\ g(S_i)=\vec{g}}}(-1)^{|S_i|+1}$$

$$\times\sum_{\vec{g}\in\mathscr{L}(G)}\ \sum_{\substack{\phi\subset S_i\subseteq S_{uv_i}\\ g(S_i)=\vec{g}}}(-1)^{|S_i|+1}. \tag{4.1}$$

But for $\vec{g}\in\mathscr{L}(G)$,

$$\sum_{\substack{\phi\subset S_i\subseteq S_{uv_i}\\ g(S_i)=\vec{g}}}(-1)^{|S_i|+1}=0 \tag{4.2}$$

since it is the $\vec{d}$ weight of a directed graph with a loop and the result follows by substituting (4.1) with condition (4.2) in (2.3).

[1]D. K. Arrowsmith and J. W. Essam, "Percolation theory on directed graphs," J. Math. Phys. **18**, 235–8 (1977).
[2]J. W. Essam and A. Coniglio, "Percolation Theory in a Gas," J. Phys. A: Math. Gen. **10**, 1917–26 (1977).
[3]J. W. Essam and C. M. Place, "Low Density Expansion of the Pair Connectedness for Percolation Models," Ann. Israel. Phys. Soc. **2**, 882 (1978).
[4]Private communication from J. W. Essam.
[5]C. Berge, *The Theory of Graphs* (Methuen, London, 1962).