Cryptanalysis of the Critical Group

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23rd October 2008

Overview

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- The discrete log problem should be difficult.
- G is known as a platform group.

A dollar-firing game

Let $\Gamma = (V, E)$ be a graph. Let $q \in V$ be a vertex.
A configuration: a function $s : V \to \mathbb{Z}$ such that $s(v) \geq 0$ for $v \neq q$, and $\sum_v s(v) = 0$.
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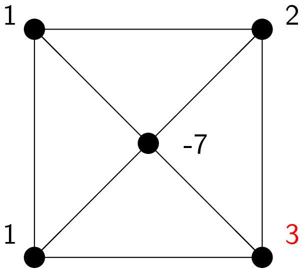

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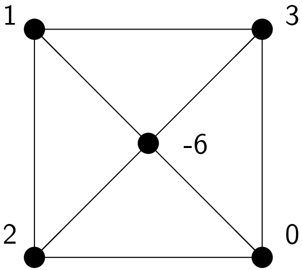

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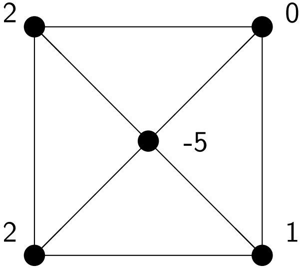

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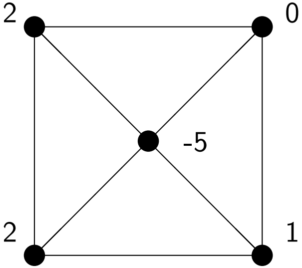

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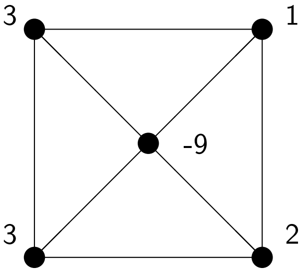

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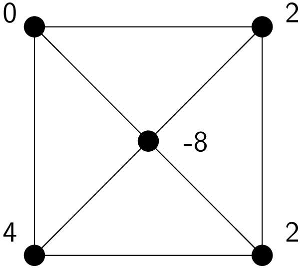

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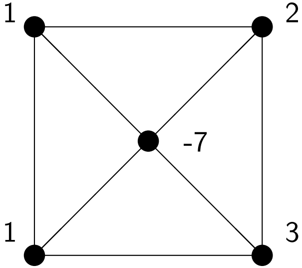

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- The critical group is finite and abelian.
- Addition may be carried out using at most $O(|V|^3)$ firings. [van den Heuvel, *Combin. Probab. Comput.* 2001]

The proposed platform group

Take the wheel graph with vertices $v_1, v_2, \ldots, v_{2n+2}$ on the 'rim', and a 'hub' vertex q.
Remove the spoke at v_{2n+2}, to obtain a graph $W^\dagger$.

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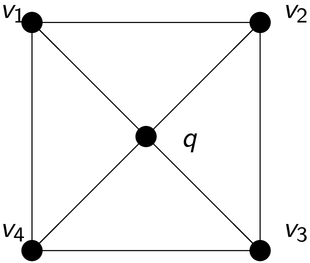

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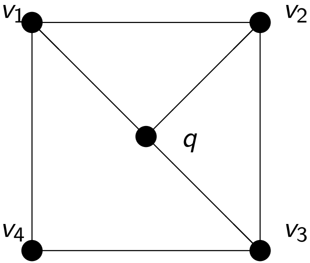

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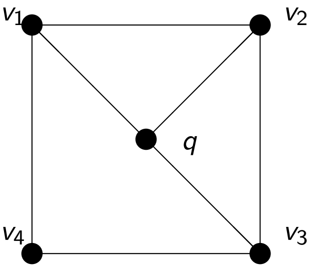

Proposed as a platform group by [Biggs, *Bull. LMS.* 2007]

Advantages of $\mathcal{K}(W^{\dagger})$

- Cyclic.
- Concrete representation of elements: $O(n)$ bits.
- Efficient addition: $O(n^3)$ operations.
- Exponential order: $2\ell_{2n+1}f_{2n+2}$ elements.

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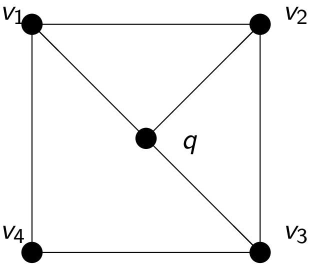
Answer: No.

Another perspective on $\mathcal{K}(W^\dagger)$:



'Configurations': $s_1 v_1 + s_2 v_2 + s_3 v_3 + s_4 v_4$ where $s_i \in \mathbb{Z}$ (free abelian group of rank $|V| - 1 = 4$).

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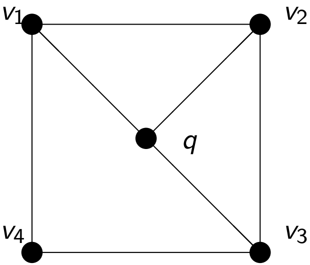

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Firings: $|V|$ relations.

$$\begin{array}{ll} 3v_1 = v_2 + v_4 & 3v_2 = v_1 + v_3 \\ 3v_3 = v_2 + v_4 & 2v_4 = v_1 + v_3 \end{array}$$

plus the dependent relation $0 = v_1 + v_2 + v_3$.

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Theorem

For any connected graph Γ, $\mathrm{Pic}(\Gamma) \cong \mathcal{K}(\Gamma)$.

(See Biggs, *Bull. LMS*, 1997)

The relations of $\mathrm{Pic}(W^\dagger)$

The relations matrix of the Picard group of $W^\dagger$:

$$Q'' = \begin{pmatrix} 1 & & & & & 1 & -2 \\ -3 & 1 & & & & & 1 \\ 1 & -3 & 1 & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & 1 & -3 & 1 & & \\ & & & 1 & -3 & 1 & \\ & & & & 1 & -3 & 1 \end{pmatrix}$$

A cryptanalysis

- Compute the Smith Normal Form A of the relations matrix Q'':
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- The discrete log problem in G is trivial to solve.

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- The special structure of Q'' can be used to reduce the complexity of the attack to $O(n)$ integer operations.
- The $O(n^3)$ attack applies to any graph, not just $W^\dagger$.

Some Links

This talk will appear soon on my home page:

`http://www.cs.rhbnc.ac.uk/~simonb/`

The paper 'Cryptanalysing the critical group' is available at:

`http://eprint.iacr.org/2008/170`