Prolific IPP Codes

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Joint work with Tuvi Etzion and Siaw-Lynn Ng

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Overview

Some definitions

Notation: F is a finite set of size q, and ℓ is a positive integer.

Definition

Let $x, y \in F^\ell$. The set of descendants of $\{x, y\}$ is the set:

$$\mathrm{Desc}(\{x, y\}) = \{d \in F^n \mid \forall i \in \{1, 2, \ldots, \ell\},\ d_i \in \{x_i, y_i\}\}.$$

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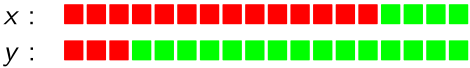

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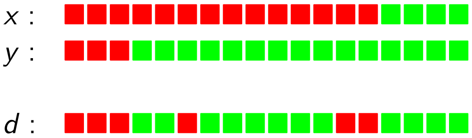

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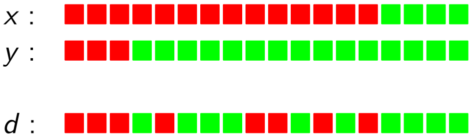

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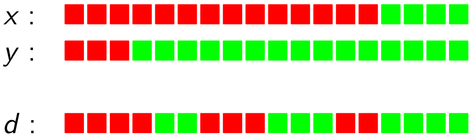

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Let $C \subseteq F^\ell$ be the *repetition code* over F of length ℓ.

$\mathrm{Desc}(C)$: the set of all words involving no more than 2 distinct symbols.

Definitions

Definition

Let $C \subseteq F^{\ell}$ and let $d \in \mathrm{Desc}(C)$.

- We say that $\{x, y\} \subseteq C$ is a (possible) parent set for d if $d \in \mathrm{Desc}(\{x, y\})$.

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Let $C \subseteq F^{\ell}$ and let $d \in \mathrm{Desc}(C)$.

- We say that $\{x, y\} \subseteq C$ is a (possible) parent set for d if $d \in \mathrm{Desc}(\{x, y\})$.
- We say that d has an identifiable parent if the intersection of all parent sets of d is non-trivial.

Definitions

Example

$$C = \{{\color{red}\blacksquare\blacksquare\blacksquare}, {\color{blue}\blacksquare\blacksquare\blacksquare}, {\color{green}\blacksquare\blacksquare\blacksquare}, {\color{red}\blacksquare}{\color{blue}\blacksquare}{\color{green}\blacksquare}\} \text{ and } d = {\color{red}\blacksquare\blacksquare}{\color{green}\blacksquare}.$$

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$C = \{$, , , $\}$ and $d =$ .

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Definitions

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$C = \{$🟥🟥🟥, 🟦🟦🟦, 🟩🟩🟩, 🟥🟦🟩$\}$ and $d =$ 🟥🟥🟩.

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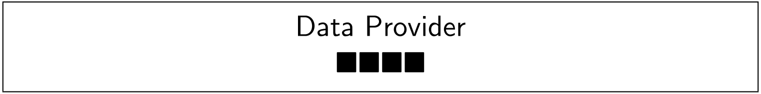
Definition

A code C has the Identifiable Parent Property (C is an IPP code) if every $d \in \mathrm{Desc}(C)$ has an identifiable parent.

An application: copyright protection



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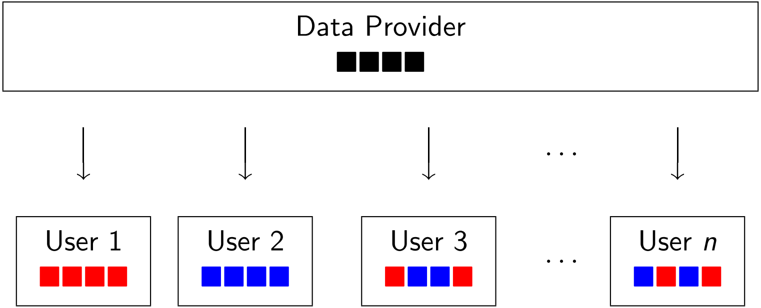

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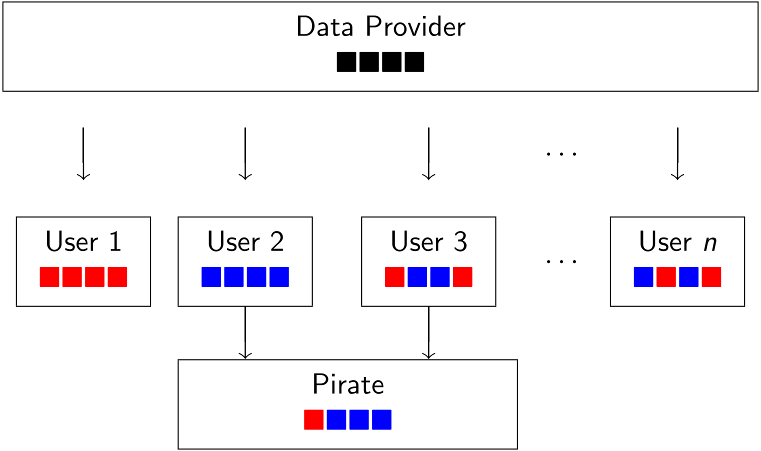

Connections with error correcting codes

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Question

Is there an analogue of a perfect error correcting code, for IPP codes?

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Definition

An IPP code is prolific if $\mathrm{Desc}(C) = F^{\ell}$.

Examples of prolific IPP codes

Some trivial examples

- The code F of length 1.

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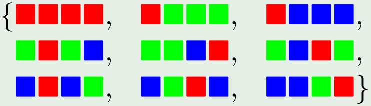
Example

The ternary Hamming code of length 4:



Bounds on the size of a prolific IPP code

Theorem

Let C be a q-ary prolific IPP code of length ℓ with m codewords. Then

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(Due to Hollmann et al.) Let C be a q-ary IPP code of length ℓ with m codewords. Then

$$m \leq 3q^{\lceil \ell/3 \rceil}.$$

Non-existence results

Corollary

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There are no more examples of prolific IPP codes when $\ell \leq 5$.

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Theorem

There are no more examples of prolific IPP codes that are equivalent to linear codes.

Questions and problems

Conjecture

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There are no more examples of prolific IPP codes.

- Can you prove there are no 3-ary examples?
- Can you prove there are no 8-ary examples?
- No examples for sufficiently large ℓ?