

k -radius Sequences

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Joint work with James F. McKee



19th May 2011

Overview

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- 4 Logarithms
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An example

A 5-ary 2-radius sequence of length 7 is:

0, 1, 2, 3, 4, 0, 1

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Definition (Jaromczyk, Lonc 2004)

Let $F = \{0, 1, \dots, n-1\}$. An n -ary k -radius sequence is a finite sequence

$$a_0, a_1, \dots, a_{m-1}$$

over the alphabet F with the following property:

For all $x, y \in F$, there exist $i, j \in \{0, 1, \dots, m-1\}$ such that $a_i = x$, $a_j = y$ and $|i - j| \leq k$.

An application

An 8-ary 3-radius sequence:

0, 1, 2, 3, 4, 5, 6, 7, 0, 1, 2, 4, 5, 6, 3, 7

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| | Time | | | | | | | | | | | | | |
|--------|------|---|---|---|---|---|---|---|---|---|---|---|---|--|
| Mem. 1 | 0 | 4 | 4 | 4 | 4 | 0 | 0 | 0 | 0 | 5 | 5 | 5 | 5 | |
| Mem. 2 | 1 | 1 | 5 | 5 | 5 | 5 | 1 | 1 | 1 | 1 | 6 | 6 | 6 | |
| Mem. 3 | 2 | 2 | 2 | 6 | 6 | 6 | 6 | 2 | 2 | 2 | 2 | 3 | 3 | |
| Mem. 4 | 3 | 3 | 3 | 3 | 7 | 7 | 7 | 7 | 4 | 4 | 4 | 4 | 7 | |

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Main Question

Let $f_k(n)$ be the length of the shortest n -ary k -radius sequence. What can we say about this function?

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- In the final position, load all the remaining images in turn.
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Essentially best possible: at most k new pairs 'covered' with each new caching.

1-radius sequences

Theorem (Ghosh 1975)

$$f_1(n) = \begin{cases} \binom{n}{2} + 1 & \text{when } n \text{ is odd;} \\ \binom{n}{2} + n/2 & \text{when } n \text{ is even.} \end{cases}$$

Simple bounds

For fixed k , the function $f_k(n)$ grows like $\binom{n}{2}$:

Lemma

We have

$$\frac{1}{k} \binom{n}{2} \leq f_k(n) \leq \binom{n}{2} + O(n)$$

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The upper bound: A 1-radius sequence is a k -radius sequence: use Ghosh 1975.

The lower bound: There are less than $kf_k(n)$ pairs $\{a_i, a_{i+\delta}\}$ where $1 \leq \delta \leq k$. They must cover all $\binom{n}{2}$ subsets of F of size 2. □

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$$\mathbf{t}_d = 0, d, 2d, 3d, \dots, (p-1)d, 0, d, \dots, (k-1)d.$$

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Distinct $x, y \in F$ occur within a distance k of each other in \mathbf{t}_d whenever

$$x - y \in \{\pm d, \pm 2d, \pm 3d, \dots, \pm kd\} = d \cdot \pm[1, k].$$

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Remark: Can easily improve the length to $|D|(p + k - 1) + 1$.

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Proof.

For all sufficiently large n , there is a prime $p \equiv 5 \pmod{8}$ with $n \leq p \leq n + n^{0.525}$. Our optimal covering shows

$$f_2(n) \leq f_2(p) \leq \frac{1}{2} \binom{p}{2} + O(p) = \frac{1}{2} \binom{n}{2} + O(n^{1.525}).$$



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- Problem: we don't know much about $\langle 2, 3, \dots, k \rangle$.
- Solution: construct tilings of \mathbb{Z}^r , where $r = \pi(k)$.

Logarithms

Definition (Galovich and Stein, 1981)

A **logarithm of length k** is a bijection $f : [1, k] \rightarrow \mathbb{Z}_k$ such that for all $a, b \in [1, k]$

$$f(ab) = f(a) + f(b) \pmod k$$

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Example

Suppose $k + 1$ is prime, and let α be a primitive root modulo $k + 1$. The 'discrete logarithm' map f is a logarithm, where f maps $i \in [1, k]$ to the unique $x \in \mathbb{Z}_k$ such that $i = \alpha^x$.

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Remark: We can get a better error term when a logarithm with an extra property exists.

Computational results on logarithms

Logarithms of length k exist when $k + 1$ is prime, and when $2k + 1$ is prime.

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The values of k with $k \leq 300$ with no logarithm of length k are:

195, 205, 208, 211, 212, 214, 217, 218, 220, 227, 229, 235, 242, 244, 246, 247, 248, 252, 253, 255, 257, 258, 259, 263, 264, 265, 266, 267, 269, 271, 274, 275, 279, 283, 286, 287, 289, 290, 291, 294, 295, 297, 298

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Do good k -radius sequences exist for these values? Can probabilistic methods help?

Hypergraphs

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- The **codegree** $\text{codeg}(u, v)$ of a pair of vertices is the number of hyperedges containing both u and v .

The Rödl Nibble

Suppose we have a large r -uniform d -regular hypergraph Γ . Suppose the codegree of a pair of vertices is always much smaller than d . Then there is a covering of the vertices by hyperedges that is about as small as we could hope for (about $|V|/r$ hyperedges).

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Theorem

Fix an integer r and a positive real number δ . Then there exists an integer n_0 and a positive real number δ' with the following property.

Let Γ be an r -uniform hypergraph on n vertices, where $n \geq n_0$. Suppose that all vertices of Γ have degree d for some integer d . Let $c = \max \text{codeg}(u, v)$, where the maximum is taken over all pairs of distinct vertices $u, v \in \Gamma$. If $c \leq \delta' d$, then there exists a hyperedge cover consisting of at most $(1 + \delta)n/r$ hyperedges.

Defining a hypergraph

Fix a large integer ℓ . We want to build a k -radius sequence from 'partial permutations' of length ℓ .

Let F be such that $|F| = n$. Define a hypergraph $\Gamma = (V, E)$ as follows:

- The vertices are the $\binom{n}{2}$ unordered pairs of elements of F .
- The hyperedges are the $n(n-1)\cdots(n-(\ell-1)) \approx n^\ell$ sequences of ℓ distinct elements over F .
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- So we can apply the Rödl Nibble to Γ .

The existence of k -radius sequences

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- Coverings of V by x hyperedges give rise to k -radius sequences of length $x\ell$. (Concatenate the sequences.)
- We get a k -radius sequence of length approximately $\binom{n}{2}/k$.

An answer

A probabilistic result:

Theorem (SRB)

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A recent explicit construction:

Theorem (Jaromczyk, Lonc, Truszczyński)

Let k, ϵ be fixed and $n \rightarrow \infty$. Then

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Some questions

Open Problem

Find optimal k -radius sequences.

Jaromczyk, Lonc and Truszczyński have optimal examples of 2-radius sequences when $n = 2p$.

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Generalise constructions from pairs of elements of F to larger subsets of F .

Open Problem

Consider packing rather than covering problems.

Some Links

This talk will appear soon on my home page:

<http://www.ma.rhul.ac.uk/sblackburn>

S.R. Blackburn and J.F. McKee, 'Constructing k -radius sequences',
Mathematics of Computation to appear:

<http://arxiv.org/abs/1006.5812>

S.R. Blackburn, 'The existence of k -radius sequences':

<http://arxiv.org/abs/1101.1172>