

# The Probability that a Pair of Elements in a Finite Group are Conjugate

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## Question

What can we say about  $\kappa(G)$ ?

## Groups with $\kappa(G)$ small

If  $G$  is abelian, all conjugacy classes have size 1 so

$$\kappa(G) = \frac{1}{|G|^2} \sum_{i=1}^k |g_i^G|^2 = \frac{1}{|G|^2} \sum_{i=1}^{|G|} 1^2 = \frac{1}{|G|}.$$

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### Proof.

Use:  $|G|/|Z(G)| \geq 4$  for a non-abelian group; and  $|g_i^G| \geq 2$  for a non-central conjugacy class. □

## Groups with $\kappa(G)$ large

### Theorem

Let  $G$  be a group with  $\kappa(G) \geq 1/4$ . Then one of the following holds:

- $|G| \leq 4$ ;
- $G \cong A_4, C_7 \rtimes C_3, S_4$  or  $A_5$ ;
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In the last case

$$\kappa(G) = \frac{1}{|G|^2} \left( 1^2 + \frac{|A|-1}{2} 2^2 + |G \setminus A|^2 \right) = \frac{1}{4} + \frac{1}{|G|} - \frac{1}{|G|^2}.$$

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### Corollary

If  $\kappa(G) > 1/2$ , then  $\kappa(G) = 1$ .

## A sketch proof: I

Suppose  $|G| > 4$ .

Let  $c_1 \leq c_2 \leq \dots \leq c_k$ , where  $c_i = \text{Cent}_G(g_i)$ . We have that

$$\sum_{i=1}^k \frac{1}{c_i} = \sum_{i=1}^k \frac{|g_i^G|}{|G|} = 1$$

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So  $c_1 = 2$  or  $c_1 = 3$ .

## A sketch proof: II

Suppose the smallest centraliser has size  $c_1 = 3$ . The equation

$$\frac{1}{9} + \sum_{i=2}^k \frac{1}{c_i^2} \geq \frac{1}{4}$$

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In the other two cases,  $G \cong S_4$  and  $G \cong A_5$  respectively. To prove this, we use the theory of Frobenius groups, and a result of Feit and Thompson (1962) classifying groups with a centraliser of order 3.

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$C_2$  acts fixed-point freely, so  $A$  is abelian and  $x$  acts by sending  $a \in A$  to its inverse.

## Related results

Let  $\text{cp}(G)$  be the probability that two elements from  $G$  commute.

Theorem (Gustafson 1973)

*If  $\text{cp}(G) \neq 1$ , then  $\text{cp}(G) \leq 5/8$ .*

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It follows from an observation of Erdős and Turan 1968 that

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Bertram 2006: a history of bounding  $k(G)$  from below. Keller 2011, strengthening a result of Pyber 1992, shows that

$$k(G) \geq \frac{c \log |G|}{(\log \log |G|)^7}.$$

## More results

### Theorem (C.T.C. Wall 1970)

*If  $G$  is not of exponent 2, the probability that  $x \in G$  satisfies  $x^2 = 1$  is at most  $3/4$ .*

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### Theorem (Laffey 1976)

*If  $G$  is not of exponent 3, the probability that  $x \in G$  satisfies  $x^3 = 1$  is at most  $7/9$ .*

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Or:  $\kappa(S_n)$  is the probability that two integer partitions of  $n$  are equal, if they are chosen with the cycle-type distribution.

The probability that  $g \in S_n$  is an  $n$ -cycle is  $1/n$ . So  $\kappa(S_n) \geq 1/n^2$ .

# A uniform bound

## Theorem

Let  $C_\kappa = 1258160557017484564/(12!)^2 \approx 5.48355$ . Then  $\kappa(S_n) \leq C_\kappa/n^2$  (with equality when  $n = 12$ ).

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Let  $P_k(n)$  be the probability that a permutation has all cycles of length less than  $k$ .

## Lemma

$$\kappa(S_n) \leq P_k(n)^2 + \sum_{\ell=k}^n \frac{\kappa(S_{n-\ell})}{\ell^2}.$$

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$$\begin{aligned} \kappa(S_n) &\leq P_{15}(n)^2 + \sum_{\ell=15}^{n-16} \frac{\kappa(S_{n-\ell})}{\ell^2} + \sum_{\ell=n-15}^n \frac{\kappa(S_{n-\ell})}{\ell^2} \\ &< \frac{0.03639}{n^2} + \frac{0.95018}{n^2} + \frac{4.36294}{n^2} < \frac{C_\kappa}{n^2}. \end{aligned}$$

## Why is the lemma true?

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For  $\ell$ -subsets  $X, Y$ , let  $E(X, Y)$  be the event that  $\tau$  acts as an  $\ell$ -cycle on  $X$ ,  $\sigma$  acts as an  $\ell$ -cycle on  $Y$ , and  $\tau, \sigma$  have the same cycle structure.

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$$\kappa(S_n) \leq P_k(n)^2 + \sum_{\ell=k}^n \sum_{|X|=\ell} \sum_{|Y|=\ell} \Pr(E(X, Y)).$$

## An asymptotic bound

In fact, since  $E(X, Y)$  are disjoint when  $k > n/2$ :

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### Theorem (Flajolet, Fusy, Gourdon, Panario, Pouyanne 2006)

Define  $A_\kappa = \sum_{m=0}^{\infty} \kappa(S_m)$ . Then  $\kappa(S_n) \sim A_\kappa/n^2$  as  $n \rightarrow \infty$ .

A corollary of the previous theorem and the above lemma. Proof gives  $4.2 < A_\kappa < 4.3$ . In fact,  $A_\kappa \approx 4.263403514152669$ .

## Commuting conjugates

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Specialise to  $G = S_n$ . Elements  $\tau, \sigma \in S_n$  lie in commuting conjugacy classes iff there exist partitions  $X_1, X_2, \dots, X_s$  and  $Y_1, Y_2, \dots, Y_s$  of  $\{1, 2, \dots, n\}$  such that:

- $|X_i| = |Y_i|$  for all  $i$ ;
- $\tau$  acts regularly on each  $X_i$ ;
- $\sigma$  acts regularly on each  $Y_i$ .

## A uniform result for $\rho(S_n)$

### Theorem

Let  $C_\rho = 5805523/508032 \approx 11.4275$ . Then  $\rho(S_n) \leq C_\rho/n^2$

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Computations show  $6.188 < A_\rho < 6.471$ .

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Define  $A_\rho = \sum_{m=0}^{\infty} \rho(S_m)$ . Then  $\rho(S_n) \sim A_\rho/n^2$  as  $n \rightarrow \infty$ .

Computations show  $6.188 < A_\rho < 6.471$ .

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$$\rho(S_n) \geq \sum_{\ell=k}^n \frac{\rho(S_{n-\ell})}{\ell^2}.$$

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The techniques of Flajolet et al do not work in this new situation.

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## Question

Can anything be said about the probability that another law (or formula involving quantifiers) holds in a group. For example, the probability that an element has order dividing 4, or 5 or 6?

## Some Links

This talk will appear soon on my home page:

<http://www.ma.rhul.ac.uk/sblackburn>

S.R. Blackburn, J.R. Britnell, M. Wildon 'The probability that a pair of group elements are conjugate':

<http://arxiv.org/abs/1108.1784>