The Probability that a Pair of Elements in a Finite Group are Conjugate

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The Conjugacy Probability

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Question

What can we say about $\kappa(G)$?

Groups with $\kappa(G)$ small

If G is abelian, all conjugacy classes have size 1 so

$$\kappa(G) = \frac{1}{|G|^2}\sum_{i=1}^{k}|g_i^G|^2 = \frac{1}{|G|^2}\sum_{i=1}^{|G|}1^2 = \frac{1}{|G|}.$$

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Proof.

Use: $|G|/|Z(G)| \geq 4$ for a non-abelian group; and $|g_i^G| \geq 2$ for a non-central conjugacy class. □

Groups with $\kappa(G)$ large

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Let G be a group with $\kappa(G) \geq 1/4$. Then one of the following holds:

- $|G| \leq 4$;
- $G \cong A_4, C_7 \rtimes C_3, S_4$ *or* A_5;
- $G \cong A \rtimes C_2$ *where A is abelian of odd order.*

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In the last case

$$\kappa(G) = \frac{1}{|G|^2}\left(1^2 + \frac{|A|-1}{2}2^2 + |G \setminus A|^2\right) = \frac{1}{4} + \frac{1}{|G|} - \frac{1}{|G|^2}.$$

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Corollary

If $\kappa(G) > 1/2$, then $\kappa(G) = 1$.

A sketch proof: I

Let $c_1 \leq c_2 \leq \cdots \leq c_k$, where $c_i = \mathrm{Cent}_G(g_i)$. We have that

$$\sum_{i=1}^{k} \frac{1}{c_i} = \sum_{i=1}^{k} \frac{|g_i^G|}{|G|} = 1$$

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Since $\kappa(G) \geq 1/4$, we know G has a centraliser of size 4 or less.
If $c_1 = 3$ or $c_1 = 4$, we show there are a finite number of possibilities for G. We use the theory of Frobenius groups, and a result of Feit and Thompson (1962) classifying groups with a centraliser of order 3.

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P has a normal 2-complement A. So $G \cong A \rtimes C_2$, and $|A|$ is odd.

C_2 acts fixed-point freely, so A is abelian and x acts by sending $a \in A$ to its inverse.

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Let $C_\kappa = 1258160557017484564/(12!)^2 \approx 5.48355$. *Then* $\kappa(S_n) \leq C_\kappa/n^2$

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Theorem (Flajolet, Fusy, Gourdon, Panario, Pouyenne, 2006)

Define $A_\kappa = \sum_{m=0}^{\infty} \kappa(S_m)$. *Then* $\kappa(S_n) \sim A_\kappa/n^2$ *as* $n \to \infty$.

Elementary proof using previous theorem. Easy computations show $4.2 < A_\kappa < 4.3$. In fact, $A_\kappa \approx 4.26340\,35141\,52669$.

Commuting conjugates

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Computations show $6.188 < A_\lambda < 6.471$

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Question

Can anything be said about the probability that another law holds in a group. For example, the probability that an element has order dividing 4, or 5 or 6?

Some Links

This talk will appear soon on my home page:

`http://www.ma.rhul.ac.uk/sblackburn`

S.R. Blackburn, J.R. Britnell, M. Wildon 'The probability that a pair of group elements are conjugate', in preparation (check my home page or the Arxiv).