

Distinct Difference Configurations

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Distinct Difference Configurations

Definition

A **distinct difference configuration with m dots** is a set $\{v_1, v_2, \dots, v_m\} \subseteq \mathbb{Z}^2$ such that the differences $v_i - v_j$ for $i \neq j$ are all distinct.

We abbreviate the above to $DD(m)$.

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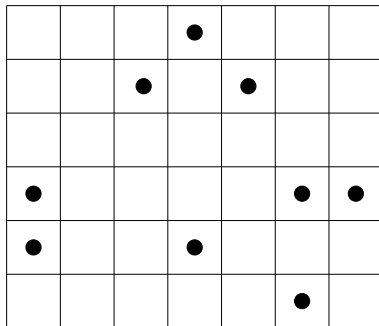
We abbreviate the above to $DD(m)$.

So $\{(0, 0), (1, 0), (0, 1)\}$ is a $DD(3)$ with differences given by:

	(0, 0)	(1, 0)	(0, 1)
(0, 0)		(1, 0)	(0, 1)
(1, 0)	(-1, 0)		(-1, 1)
(0, 1)	(0, -1)	(1, -1)	

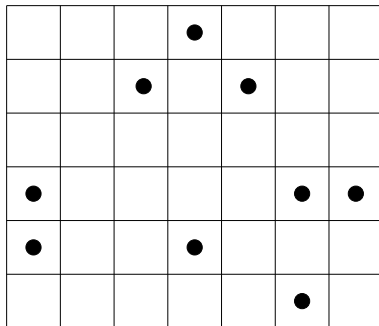
Distinct Difference Configurations

The following is a DD(9):



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All pairs of dots are at distance at most 7. This is the largest such example.

Distinct Difference Configurations

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A **distinct difference configuration with m dots of diameter d** is a $DD(m)$ such that $|v_i - v_j| \leq d$ for all i, j .

So any pair of dots is at distance d or less.

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- The example above is a $DD(9, 7)$.

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Main combinatorial question

When d is fixed, how big can m be?

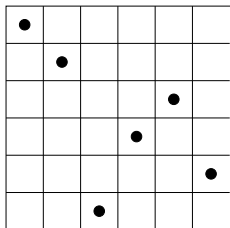
Motivation

- Key predistribution schemes in **Wireless Sensor Networks**.

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- A 2 dimensional generalisation of **B_2 -sequences**.

Some constructions



Definition

An $n \times n$ **Costas array** is a $DD(n)$ contained in $\{0, 1, \dots, n-1\}^2$ with no pair of dots parallel with the x -axis, and no pair of points parallel with the y -axis.

Costas arrays

There exist several constructions of Costas arrays.

For example let p be a prime. Let α be a primitive root mod p .

Construction

Let $n = p - 2$. Put a dot in position (i, j) when $\alpha^{i+1} + \alpha^{j+1} = 1$. Then the dots form an $n \times n$ Costas array.

A lower bound on m

Costas arrays give:

Theorem

There exists a $\text{DD}(m, d)$ with $m = \frac{1}{\sqrt{2}}d - o(d) \approx 0.70711d$.

Proof: Place an $n \times n$ square in a circle of diameter d . □

A corresponding upper bound

In fact we have an explicit construction such that:

Theorem

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Theorem

A $DD(m, d)$ must satisfy $m \leq (\sqrt{\pi}/2)d + o(d) \approx 0.88623d$.

A new distance measure

A $\overline{DD}(m, d)$ is a $DD(m)$ in which every pair of dots are at **Manhattan distance** at most d .

A new distance measure

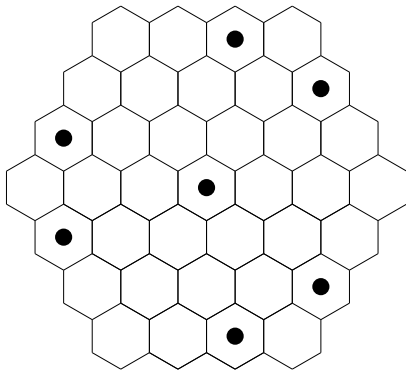
A $\overline{DD}(m, d)$ is a $DD(m)$ in which every pair of dots are at **Manhattan distance** at most d .

Theorem

There exists a $\overline{DD}(m, d)$ with $m = (1/\sqrt{2})d - o(d)$. Moreover, in any $\overline{DD}(m, d)$ we have that $\overline{DD}(m, d) \leq (1/\sqrt{2})d + o(d)$.

A new grid

A $DD^*(m)$ is a set of vectors **in the hexagonal grid** whose differences are all distinct.



The hexagonal grid

If all pairs of dots are within Euclidean distance d , we have a $DD^*(m, d)$.

Theorem

There exists a $DD^(m, d)$ with $m = 0.86819d - o(d)$. In any $DD^*(m, d)$ we have that $\overline{DD}(m, d) \leq 0.95231d + o(d)$.*

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Open Problem

Close this gap.

Hexagonal distance

If we replace Euclidean distance by **hexagonal distance** we call the resulting object a $\overline{DD}^*(m, d)$.

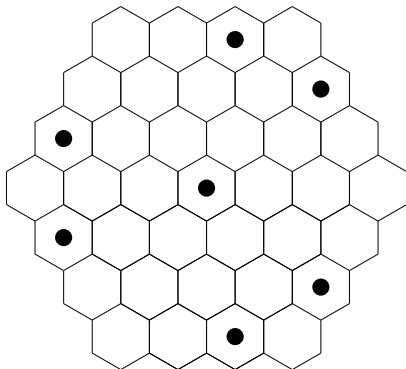
Theorem

There exists a $\overline{DD}^(m, d)$ with $m = 0.79444d - o(d)$. In any $\overline{DD}^*(m, d)$ we have that $\overline{DD}(m, d) \leq 0.86603d + o(d)$.*

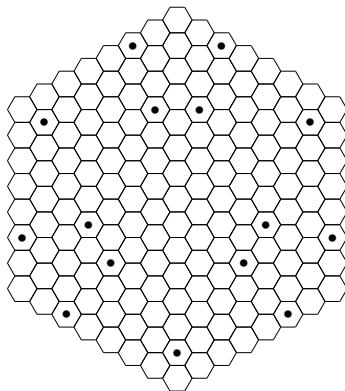
Honeycomb arrays

Definition (Golomb and Taylor 1984)

A **honeycomb array of radius r** is a $DD^*(2r + 1)$ contained in a hexagonal sphere of radius r , with the property that there is exactly one dot in every 'row', where 'rows' go in three directions.



A radius 7 honeycomb array



Honeycomb arrays

Honeycomb arrays are known to exist for $r = 0, 1, 3, 4, 7, 10, 13$ and 22 .

Theorem (Anastasia Panoui, 2009)

This list is complete for $r \leq 13$.

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A Honeycomb array is a $\overline{DD}^*(m, d)$ where $m = 2r + 1$ and $d = 2r$.
But we know that $m \leq 0.86603d + o(d)$.

So there are only finitely many honeycomb arrays!

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So there are only finitely many honeycomb arrays!

Theorem

Honeycomb arrays of radius r do not exist when $r \geq 644$.

Some links

This talk will appear soon on my homepage:

<http://www.ma.rhul.ac.uk/sblackburn>

Our preprint 'Two-dimensional patterns with distinct differences – Constructions, Bounds and Maximal Anticodes' is available at:

<http://arxiv.org/abs/0811.3832>

The following preprint explores other combinatorial properties motivated by our WSN key predistribution scheme:

<http://arxiv.org/abs/0811.3896>