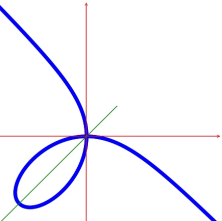
Perfect Forests in Digraphs

Gregory Gutin birthday conference

Anders Yeo

Royal Holloway

January 13, 2017



This talk

First I would like to thank Gregory and the organizers for inviting me here.

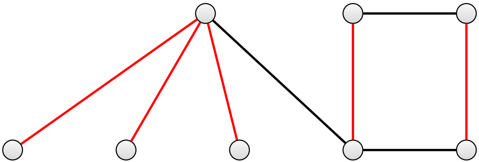
And wish Gregory a happy birthday!

I will in this talk discuss a result we obtained on one of my visits to Gregory.

Definitions

A spanning subgraph F of a graph G is called perfect if the following holds.

- F is a spanning forest.
- The degree $d_F(x)$ of each vertex x in F is odd.
- Each tree of F is an induced subgraph of G.



This generalizes matchings.

Undirected graphs

Alex Scott (Graphs & Combin., 2001) proved the following for connected graphs G.

G contains a perfect forest

$$\Updownarrow$$

G has an even number of vertices.

Any graph has an even number of vertices of odd degree, so "$\Downarrow$" is easy.

"$\Uparrow$" is also not difficult to prove....

Proof of Scott's result

We first note that if F is a forest where every degree is odd and contiains the minimum possible number of edges over all such forests, then F is a perfect forest.

Proof: If some tree, T, in F is not induced pick an edge uv not in T with $u, v \in V(T)$.

Add uv to T and delete all edges on the unique (u, v)-path in T.

This gives us a forest where all degrees are odd and with fewer edges than F, a contradiction. QED.

To find a perfect forest we just need to find a forest where all degrees are odd.

Proof of Scott's result

Let G be a connected graph of even order.

We will prove the theorem by induction on $|E(G)|$.

Clearly the theorem holds if $|E(G)| = 1$.

We may assume G is a tree (otherwise delete an edge on a cycle and use induction) rooted at some r.

If all vertices have odd degree we are done, so let x have even degree and maximum depth (clearly $x \neq r$).

Delete the edge from x to his parent and use induction on the two remaining connected components. QED.

This was the obligatory proof of the talk!

Alternative proof of Scott's result

As before let G be a connected graph of even order.

We will again prove the theorem by induction on $|E(G)|$, so can assume that G is a tree rooted at some r.

Let u be a leaf at maximum depth and let v be the parent of u.

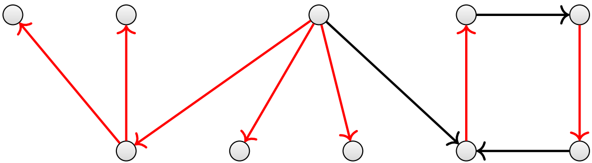
If v is adjacent to anonther leaf w, then remove u and w and after using induction add u and w to the tree containing v.

If u is the only leaf adjacent to v then remove u and v and after using induction the edge uv becomes a tree in the forest. QED.

Generalizations to directed graphs, 1

We first define a perfect out-forest, F, of a digraph D.

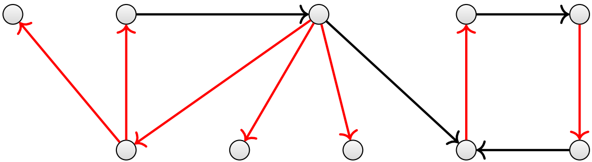
- F is a spanning out-forest of D.
- The degree of each vertex in the underlying graph of F is odd.
- Each out-tree of F is an induced subgraph of D.



Generalizations to directed graphs, 2

We define an almost perfect out-forest, F, of a digraph D.

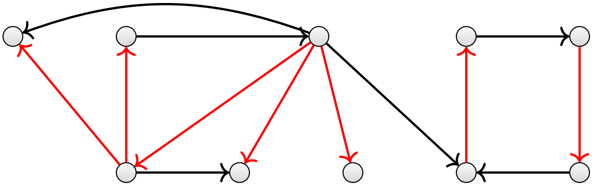
- F is a spanning out-forest of D.
- The degree of each vertex in the underlying graph of F is odd.
- Every arc in D is either in F, goes between different out-trees in F or goes from a vertex to an ancestor of that vertex in an out-tree in F.



Generalizations to directed graphs, 3

We define a weak perfect out-forest, F, of a digraph D.

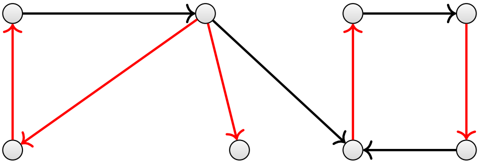
- F is a spanning out-forest of D.
- The degree of each vertex in the underlying graph of F is odd.



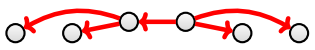
Generalizations to directed graphs, 4

We define an even out-forest, F, of a digraph D.

- F is a spanning out-forest of D.
- Every out-tree in F has even order.



Relation between definitions.

F is a perfect out-forest. ()

$\Downarrow$

F is an almost perfect out-forest. ()

$\Downarrow$

F is a weak perfect out-forest. ()

$\Downarrow$

F is an even out-forest. ()

Classes of digraphs

We furthermore consider 3 families of connected digraphs.

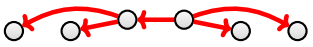
$\mathscr{D}^{st}$: The class of all strongly connected digraphs of even order.

$\mathscr{D}^{u}$: The class of all connected digraphs of even order which contain only one initial strong component.

$\mathscr{D}$: The class of all connected digraphs of even order.

Note that any digraph in $\mathscr{D}^{u}$ contains some vertex, say u, such that for any vertex in the digraph it can be reached from u by a directed path.

Summary of results

	$\mathscr{D}^{st}$	$\mathscr{D}^{u}$	$\mathscr{D}$
perfect out-forest 	*NP*-hard to decide	*NP*-hard to decide	*NP*-hard to decide
almost perfect out-forest 	always exists	always exists	may not exist, but can be decided in polynomial time
weak perfect out-forest 	always exists	always exists	may not exist, but can be decided in polynomial time
even out-forest 	always exists	always exists	may not exist, but can be decided in polynomial time

Quiz: Which part generalizes the undirected result?

Generalizing the undirected result

Let G be a connected undirected graph of even order.

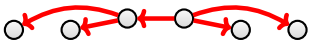
D is obtained by replacing every edge by a directed 2-cycle.

D is strong.

A perfect forest in G is equivalent to an almost perfect out-forest in D.

So the answer is....

Generalizing the undirected result

	$\mathscr{D}^{st}$	$\mathscr{D}^{u}$	$\mathscr{D}$
perfect out-forest 	*NP*-hard to decide	*NP*-hard to decide	*NP*-hard to decide
almost perfect out-forest 	always exists	always exists	may not exist, but can be decided in polynomial time
weak perfect out-forest 	always exists	always exists	may not exist, but can be decided in polynomial time
even out-forest 	always exists	always exists	may not exist, but can be decided in polynomial time

Finding an almost perfect-forest in $\mathscr{D}^{st}$ generalizes Scott's result.

Proof techniques

Theorem: Every $D \in \mathscr{D}$ either contains all three of the below or none of the below.

- Almost perfect out-forest.
- Weak perfect out-forest.
- Even out-forest.

F is an almost perfect out-forest. ()

$\Downarrow$ $\Uparrow$ (not too difficult to prove)

F is a weak perfect out-forest. ()

$\Downarrow$ $\Uparrow$ (similar to the undirected proof)

F is an even out-forest. ()

Proof techniques

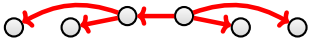
- Almost perfect out-forest. ()
- Weak perfect out-forest. ()
- Even out-forest. ()

Clearly these all exist in $\mathscr{D}^{st}$ and $\mathscr{D}^{u}$. Why?

But maybe not in $\mathscr{D}$. Why?

It turns out that it is easiest to then show that there is a polynomial time algorithm to determine if a digraph has a weak perfect out-forest (not trivial).

Summary

	$\mathscr{D}^{st}$	$\mathscr{D}^{u}$	$\mathscr{D}$
perfect out-forest 	*NP*-hard to decide	*NP*-hard to decide	*NP*-hard to decide
almost perfect out-forest 	always exists	always exists	may not exist, but can be decided in polynomial time
weak perfect out-forest 	always exists	always exists	may not exist, but can be decided in polynomial time
even out-forest 	always exists	always exists	may not exist, but can be decided in polynomial time

Considered so far.

Perfect out-forest

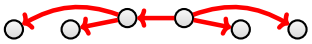
We can reduce 3-Dimension Matching to finding a perfect out-forest in a strong digraph (of even order).

This implies that finding a perfect out-forest in $\mathscr{D}^{st}$ is *NP*-hard.

This implies that finding a perfect out-forest in $\mathscr{D}^{u}$ is *NP*-hard.

This implies that finding a perfect out-forest in $\mathscr{D}$ is *NP*-hard.

Summary

	$\mathscr{D}^{st}$	$\mathscr{D}^{u}$	$\mathscr{D}$
perfect out-forest 	*NP*-hard to decide	*NP*-hard to decide	*NP*-hard to decide
almost perfect out-forest 	always exists	always exists	may not exist, but can be decided in polynomial time
weak perfect out-forest 	always exists	always exists	may not exist, but can be decided in polynomial time
even out-forest 	always exists	always exists	may not exist, but can be decided in polynomial time

Considered so far.

The end

This was just one of the many interesting problems Gregory thought of looking at.

And which gave a nice extension of an "undirected" result to digraphs.

Thanks Gregory! And happy birthday!!

The end

Thanks again to the organizers.

Happy birthday Gregory!

Any Questions?