Gregory: The "Tree" of Knowledge

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Gregory Gutin's 60th Birthday Conference, 2017,
London, January 8, 2017

Birthday Quote

You leave everyone in awe. At 60, you have more going on in your life than most people have in a whole lifetime. You're an inspiration to me and everyone who knows you! You leave us all wondering...what's next for this fascinating person!?! You are awesome!!

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Happy 60th Birthday Gregory!

My Articles with Gregory

- Parameterized Algorithms for Directed Maximum Leaf Problems.
- Spanning Directed Trees with Many Leaves.
- Algorithm for finding k-vertex out-trees and its application to k-internal out-branching problem.
- Parameterized Study of the Test Cover Problem.
- Parameterized complexity of MaxSat Above Average.
- Fixed-Parameter Tractability of Satisfying Beyond the Number of Variables

- Noga Alon, Fedor V. Fomin, Gregory Gutin, Michael Krivelevich, Saket Saurabh: Parameterized Algorithms for Directed Maximum Leaf Problems. ICALP 2007: 352-362
- Noga Alon, Fedor V. Fomin, Gregory Gutin, Michael Krivelevich, Saket Saurabh: Spanning Directed Trees with Many Leaves. SIAM J. Discrete Math. 23(1): 466-476 (2009)
- Nathann Cohen, Fedor V. Fomin, Gregory Gutin, Eun Jung Kim, Saket Saurabh, Anders Yeo: Algorithm for finding k-vertex out-trees and its application to k-internal out-branching problem. J. Comput. Syst. Sci. 76(7): 650-662 (2010)
- Robert Crowston, Gregory Gutin, Mark Jones, Saket Saurabh, Anders Yeo: Parameterized Study of the Test Cover Problem. MFCS 2012: 283-295
- Robert Crowston, Gregory Gutin, Mark Jones, Venkatesh Raman, Saket Saurabh: Parameterized complexity of MaxSat Above Average. Theor. Comput. Sci. 511: 77-84 (2013)
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- Fixed-Parameter Tractability of Satisfying Beyond the Number of Variables Used the algorithm for Tree Subgraph Isomorphism

So I hope my connection with Gregory and Trees is more clearer now!

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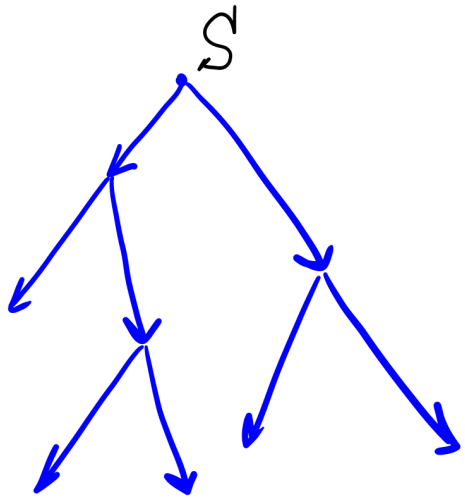
Out-Trees and Out-Branchings

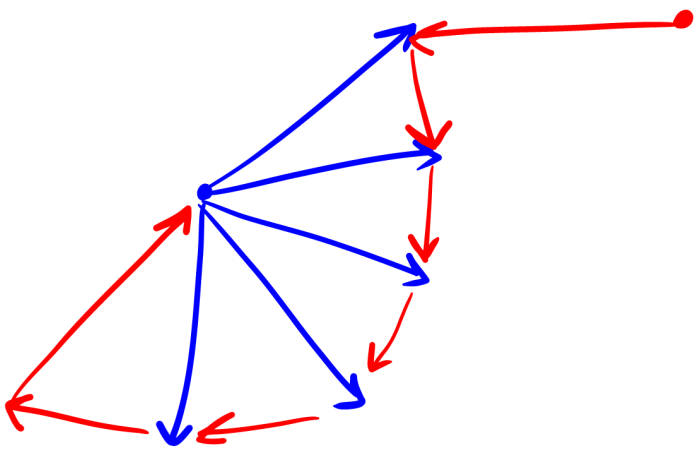


We say that a subdigraph T of a digraph D is an out-tree if T is an oriented tree with only one vertex s of in-degree zero (its root).
The vertices of T of out-degree zero are leaves.

Out-Trees and Out-Branchings

If T is a spanning out-tree, i.e. $V(T) = V(D)$, then T is an out-branching of D.

Out-tree with 5 leaves



Out-branching with one leaf

Fixed Parameter Tractable (FPT) Algorithms

For decision problems with input size n, and a parameter k, (which typically is the solution size), the goal here is to design an algorithm with running time $f(k) \cdot n^{\mathcal{O}(1)}$, where f is a function of k alone.

Problems that have such an algorithm are said to be fixed parameter tractable (FPT).

Directed Maximum Leaf Out-Branching

Directed Maximum Leaf Out-Branching (DMLOB)
Input: A digraph D and a positive integer k.
Parameter: k
Question: Does there exist a an out-branching (i.e., a rooted oriented spanning tree) in a given digraph with at least k leaves?

Our Results (2006–2007)

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Structural Theorem/Win Win Theorem

Let D be a strongly connected digraph. Then in polynomial time we can either (a) obtain an out-branching of D with at least k leaves; or (b) obtain a path decomposition of width $\mathbf{pw}(G) = \mathcal{O}(k \log k)$ for the underlying undirected graph of D.

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- In the later case one can apply the algorithm running in time $2^{\mathcal{O}(\mathbf{pw}(G) \log \mathbf{pw}(G))} n^{\mathcal{O}(1)} = 2^{\mathcal{O}(k(\log k)^2)} n^{\mathcal{O}(1)}$.

Follow up work and the current best

- Bonsma and Dorn used our ideas and used tree-width and showed that DMLOB can be solved in time $2^{\mathcal{O}(k \log k)} n^{\mathcal{O}(1)}$.

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- Daligault, Gutin, Kim, and Yeo improved this to $3.72^k n^{\mathcal{O}(1)}$ time algorithm for DMLOB. This is the current best algorithm on the problem.

Kernelization

INFORMALLY: A kernelization algorithm is a polynomial-time transformation that transforms any given parameterized instance to an equivalent instance of the same problem, with size and parameter bounded by a function of the parameter.

Kernel: Formally

FORMALLY: A kernelization algorithm, or in short, a kernel for a parameterized problem $L \subseteq \Sigma^* \times \mathbb{N}$ is an algorithm that given $(x, k) \in \Sigma^* \times \mathbb{N}$, outputs in $p(|x| + k)$ time a pair $(x', k') \in \Sigma^* \times \mathbb{N}$ such that

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- $(x, k) \in L \iff (x', k') \in L$,
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where f is an arbitrary computable function, and p a polynomial. Any function f as above is referred to as the size of the kernel.

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Polynomial kernel $\implies$ f is polynomial.

Our Results and what has happened after that

Kernel(s) for problems with no kernel: On out-trees with many leaves

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- Daligault, Gutin, Kim, and Yeo showed that Rooted DMLOB admits $\mathcal{O}(k)$ kernel on acyclic digraphs. Apart from planar digraphs this is the only class which is known to admit linear kernels.

Tree/Out-Tree Subgraph Isomorphism

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Parameter: $k = |V(T)|$
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Out-Tree Subgraph Isomorphism
Input: An out-tree T and a digraph D.
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Question: Does there exist an out-tree T^* in D that is isomorphic to T?

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Input: A digraph D and a positive integer k.
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- A graph admits an out-branching with at most one leaf if and only if it has a Hamiltonian Path.

k-MLOB

- Gutin, Razgon, and Kim designed an algorithm for k-MLOB running in time $2^{\mathcal{O}(k \log k)} n^{\mathcal{O}(1)}$ and an $\mathcal{O}(k^2)$ sized kernel.

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Algorithm for finding k-vertex out-trees and its application to k-internal out-branching problem

- Designed an algorithm for Out-Tree Subgraph Isomorphism running in time 5.704^k and an algorithm for k-MLOB running in time $2^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$.

Our Method – An outline of the algorithms

We call a k-internal out-tree *minimal* if none of its proper *subtrees* is a k-internal out-tree, or minimal k-tree in short.

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- So a *witness* that we have an out-tree with k internal nodes is an out-tree with at most $2k$ nodes.
- So we enumerate all non-isomorphic trees on $2k$ nodes and use Out-Tree Subgraph Isomorphism algorithm to test whether such a tree is present or not.

Follow up work and the current best

- The best deterministic algorithm uses our structural lemma and runs in time $5.1^k n^{\mathcal{O}(1)}$ [Zehavi, ESA 2016]
- Randomized runs in time $4^k n^{\mathcal{O}(1)}$ and uses reduction to multilinear monomial testsing. [Zehavi, IPEC 2013]

Kernel – An open question?

- Is there linear sized vertex kernel for k-MLOB?

Spend Rest of my time on Tree Isomorphism

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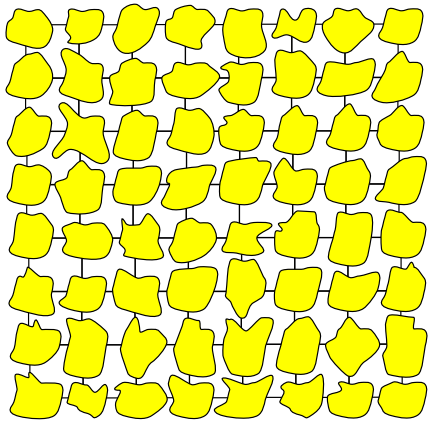
- Current best is $2^k n^{\mathcal{O}(1)}$ randomized [Koutis and Williams, ICALP 2009] and $2.618^k n^{\mathcal{O}(1)}$ deterministic (Fomin et al, JACM, 2016).

A Special Graph Class – Planar Graphs

- Tree Subgraph Isomorphism, even finding a path of length k is NP-complete on this graph class.

Grid minors and treewidth



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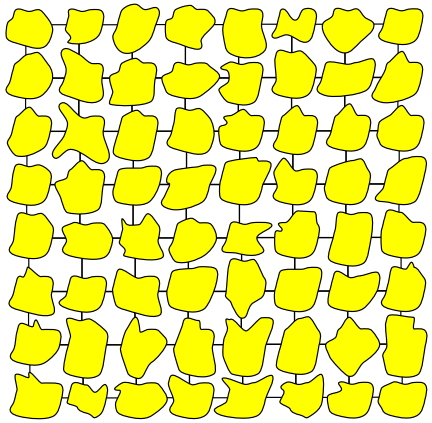

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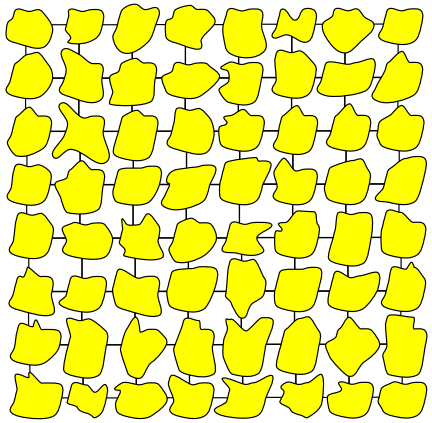

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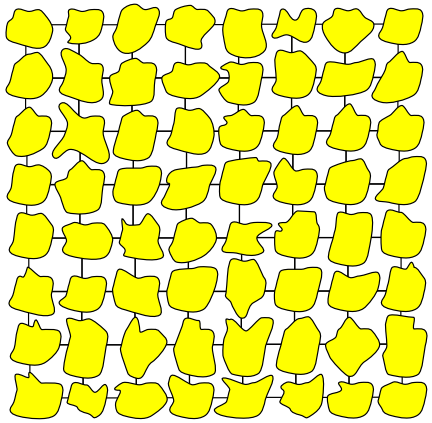

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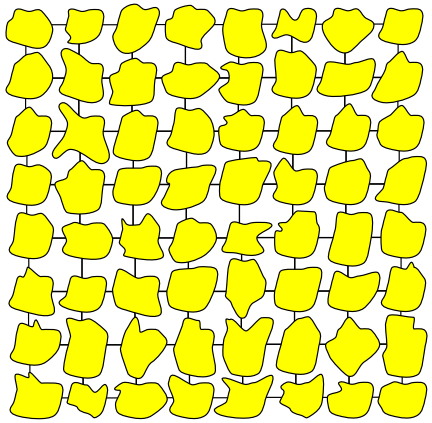

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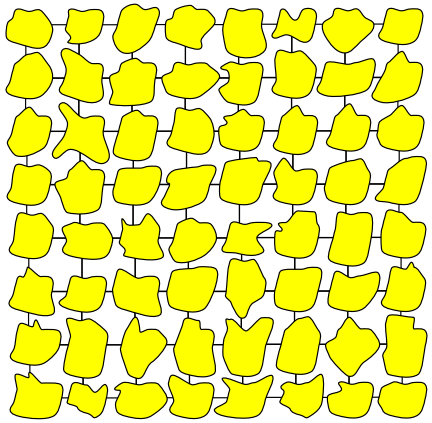

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- **Goal**: $2^{\mathcal{O}(\sqrt{k})} \cdot n$ algorithm for planar graphs.
- **Fact 1**: k-Path can be solved in time $2^{\mathcal{O}(w)} \cdot n$ on a tree decomposition of width w.
- **Fact 2**: If there is $\sqrt{k} \times \sqrt{k}$ grid minor, then there is a k-path.

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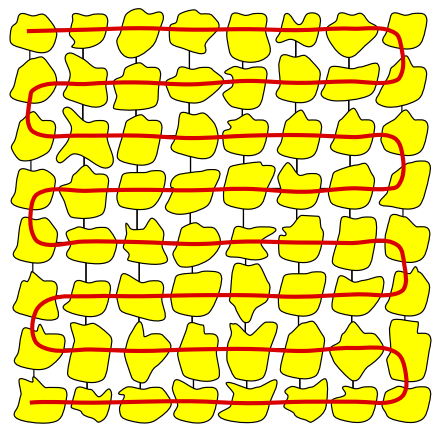

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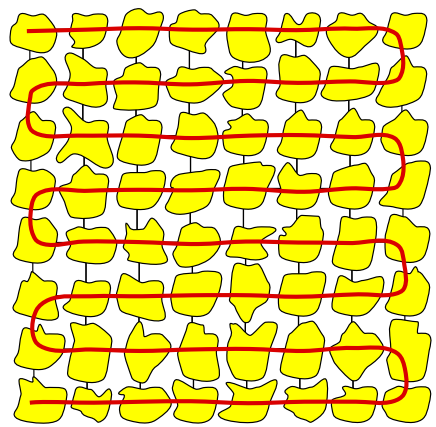
- If $\mathbf{tw} \geqslant 100\sqrt{k}$ then there is a $\sqrt{k} \times \sqrt{k}$ grid minor, so also a k-path.
- Otherwise we have a tree decomposition of width $\mathcal{O}(\sqrt{k})$. Apply dynamic programming.



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Pattern coverage

Main result

Suppose G is a planar graph on n vertices, and suppose k is a positive integer. Then there exists a family $\mathcal{F}$ of subsets of $V(G)$ such that:

(a) $|\mathcal{F}| \leqslant 2^{\tilde{\mathcal{O}}(\sqrt{k})} \cdot n^c$ for some constant c;

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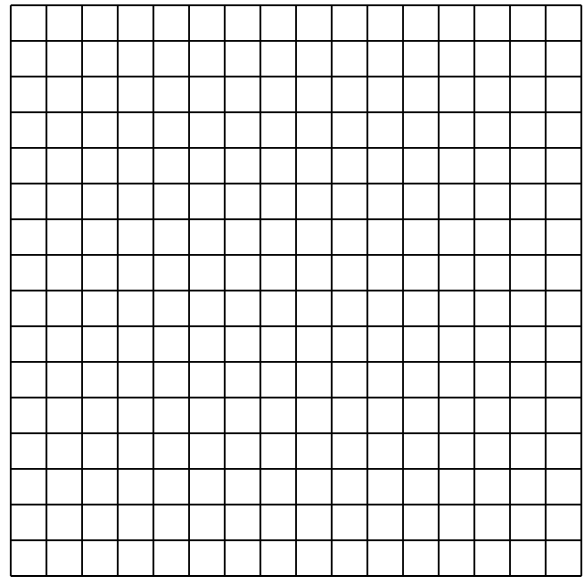
  - This running time is tight under ETH! (Bodlaender et al.)

Example: grid



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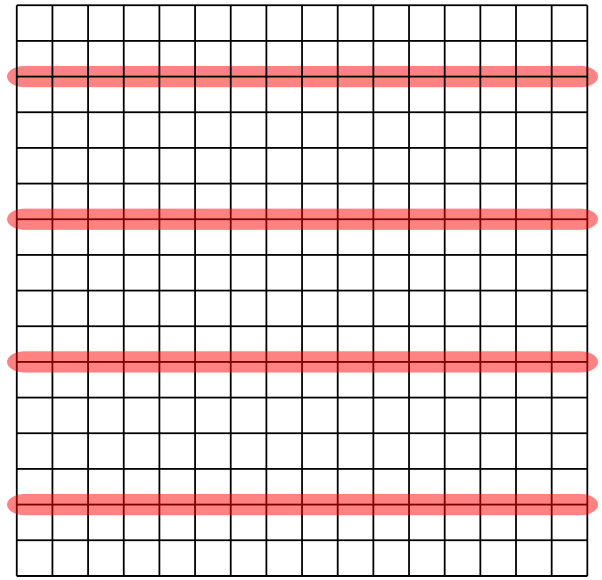

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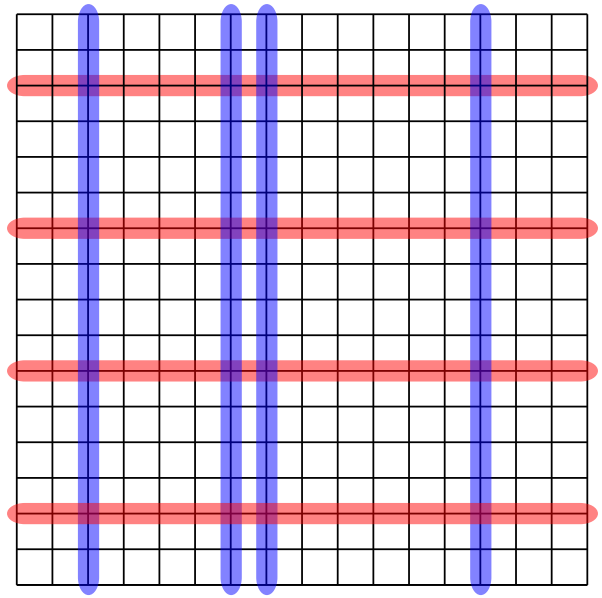

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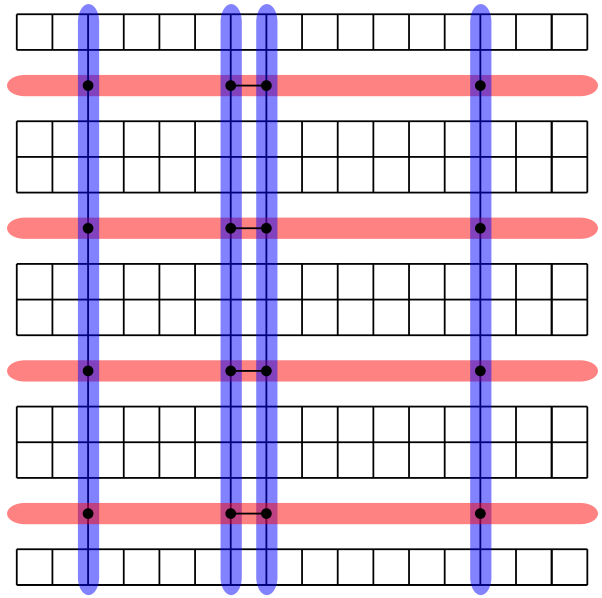

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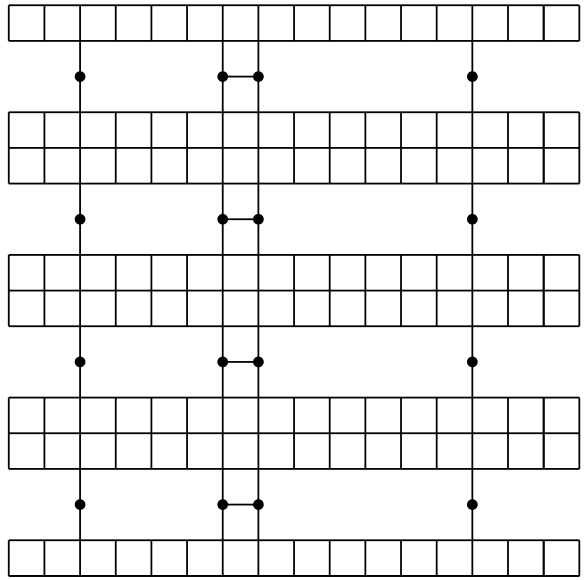

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- In total, $\sqrt{k} \cdot \binom{k}{\sqrt{k}} = 2^{\mathcal{O}(\sqrt{k} \log k)}$ possible guesses.

Once More Trees

- k-Distinct In- and Out-Branchings (finding an out-branching and in-branching in a digraph D that do not share at least k arcs in common). Bang-Jensen, SS and Simonsen showed that the problem is FPT on strongly connected digraphs.

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Thank You!

Any Questions?