Universal Tournaments

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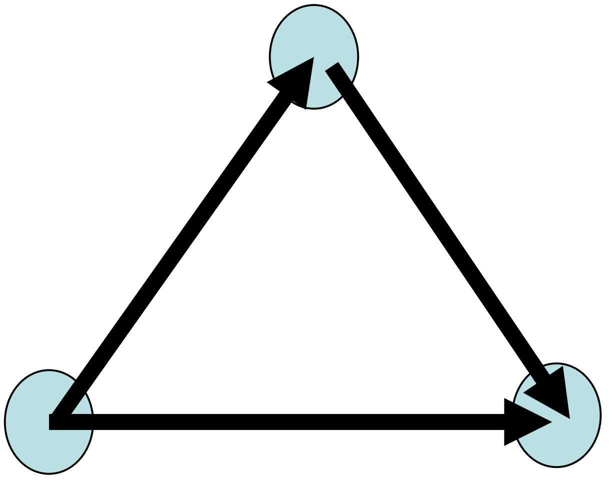
I Gutin and Tournaments

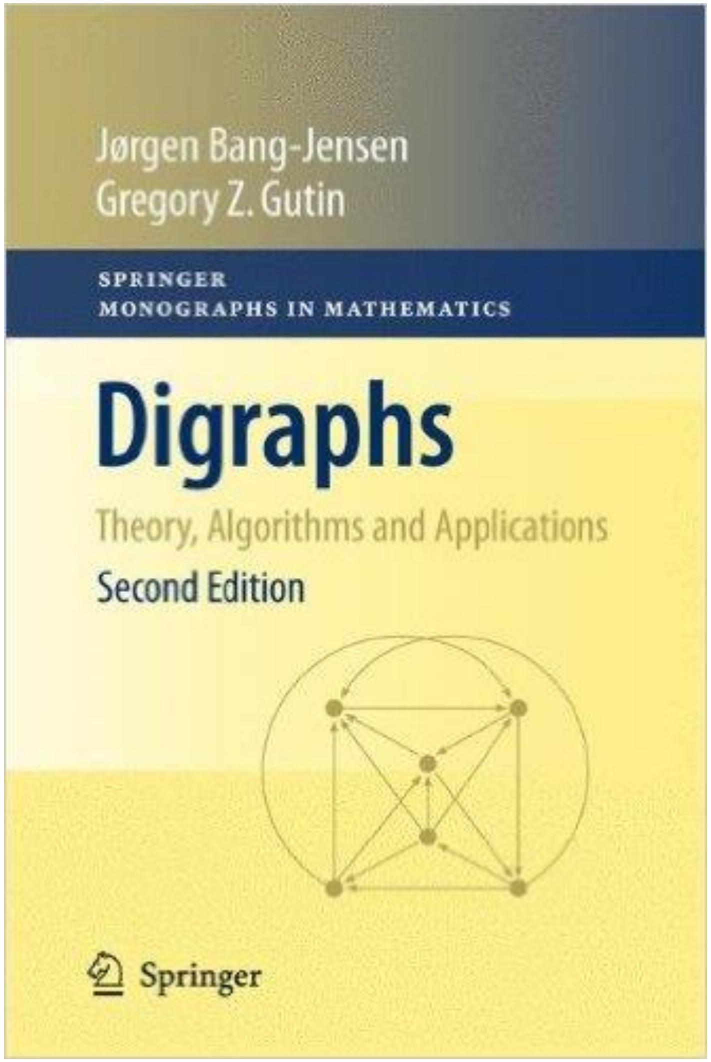
A tournament is an oriented complete graph



Zvi (=Gregory) has 13 papers in MathSciNet in which the word tournament appears in the title

and 11 more in which it appears in the abstract

In his book with Bang-Jensen (first edition) the word tournament appears 879 times !

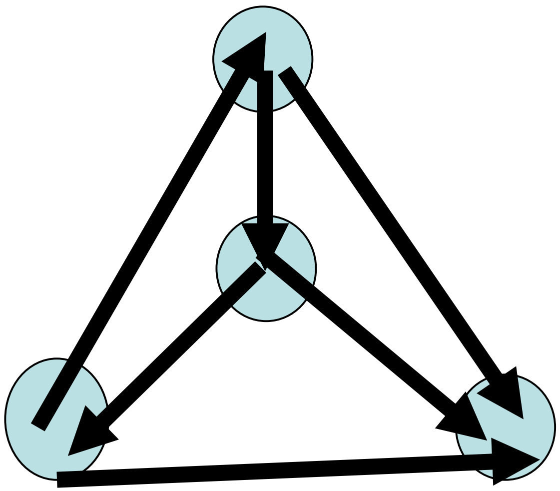


II Universal Tournaments

A tournament is k-universal if it contains every k-vertex tournament as an induced subgraph

f(k)=minimum possible number of vertices of a k-universal tournament

Example: f(3)=4



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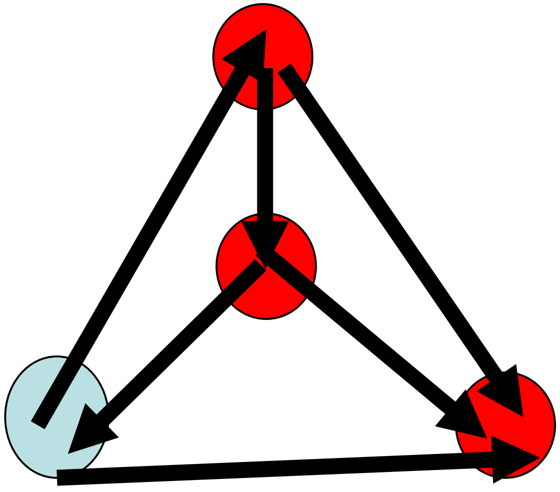

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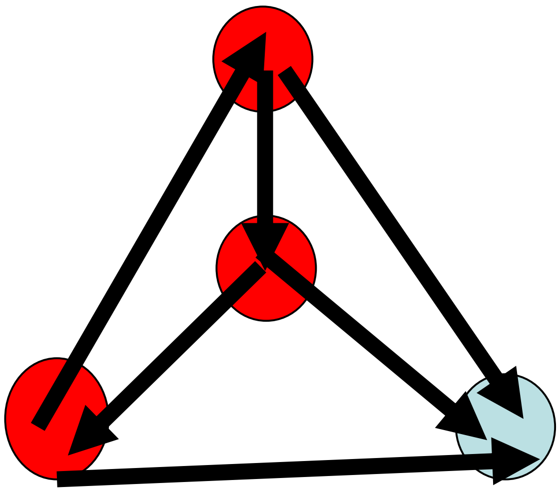
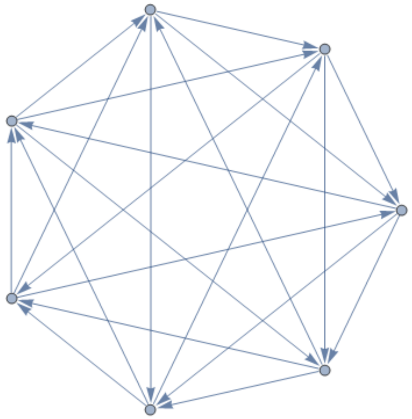


Moon (68): $f(k) = ?$

Graham and Spencer (71): if $p > k^2 \, 2^{2k-2}$ is a prime which is 3 (mod 4), then the quadratic residue tournament T_p is k-universal (and more)

The vertices of T_p are the residues modulo p, (i,j) is a directed edge iff i-j is a quadratic residue



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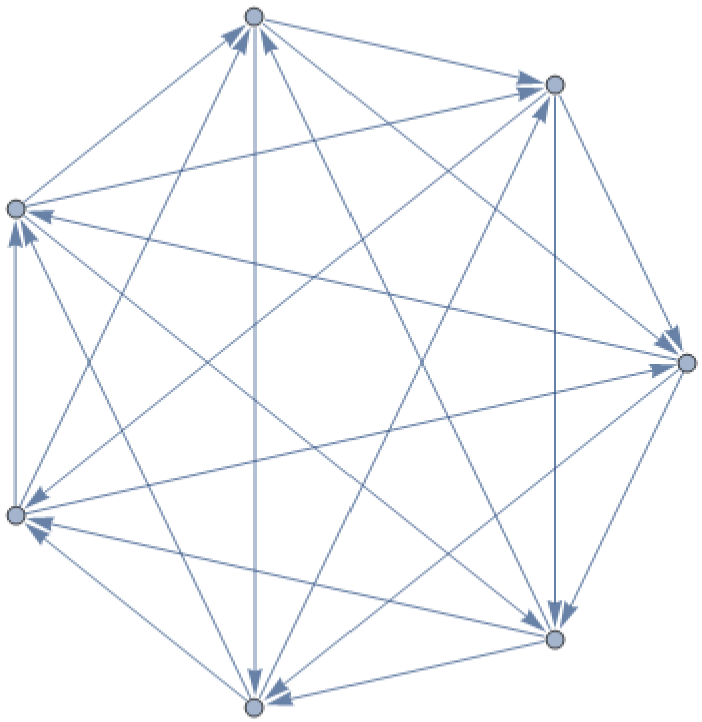


The proof is based on Weil's character sum estimate

Moon (1968): $2^{(k-1)/2} \le f(k) \le O(k2^{k/2})$

Alstrup, Kaplan, Thorup and Zwick (2015):

$$f(k)=\Theta(2^{k/2})$$

(The ratio between the upper and lower bounds is roughly 22)

New: $f(k)=(1+o(1))\, 2^{(k-1)/2}$

III Groups and Random Tournaments

The random tournament T(n) on n vertices is obtained from the complete graph on n vertices by orienting each edge, randomly, uniformly and independently

The expected number of copies of a k-vertex tournament H with A automorphisms in T(n) is

$$\binom{n}{k}\left(\frac{k!}{A}\right)\ 2^{-\frac{k(k-1)}{2}}$$

For A=1 and $n=(1+o(1))2^{(k-1)/2}$ this is $(1+o(1))^k >> 1$

The group of automorphisms of a tournament is of odd order, hence by the Feit-Thompson Theorem (62) it is solvable.

Using this, Dixon (67) proved that the number of automorphisms of any k-vertex tournament is at most $3^{(k-1)/2}$ (with equality iff it is the iterated cyclic triangle).

Combining Dixon's result with some probabilistic arguments it can be shown that the random tournament on

$$(\sqrt{3}+o(1))2^{(k-1)/2}$$

vertices is, with high probability, k-universal.

This is tight up to the o(1) term.

IV The tight result: proof ideas

T(k)=family of all k-vertex tournaments

f(k)=minimum number of vertices of a k-universal tournament

Theorem A:

$$2^{\frac{k-1}{2}} \leq f(k) \leq 2^{\frac{k-1}{2}}\left(1 + O\left(\frac{(\log k)^{\frac{3}{2}}}{\sqrt{k}}\right)\right)$$

The lower bound (Moon (68)):
T = k-universal tournament with n=f(k) vertices

The number of induced subgraphs of T on k vertices is $\binom{n}{k}$. Thus

$$\binom{\boldsymbol{n}}{\boldsymbol{k}} \geq |\boldsymbol{T}(\boldsymbol{k})| \geq \frac{\boldsymbol{2}^{\boldsymbol{k}(\boldsymbol{k}-\boldsymbol{1})/\boldsymbol{2}}}{\boldsymbol{k}!}$$

Note that equality can hold only if T contains every member of T(k) exactly once.

Equality can hold only if T contains every member of F(k) exactly once.

However:

Proposition: every tournament with $\mathbf{2}^{\boldsymbol{\Omega}(\boldsymbol{k})}$ vertices contains a collection of $\mathbf{2}^{\boldsymbol{\Omega}(\boldsymbol{k}^{\mathbf{2}})}$ pairwise disjoint families of induced k–vertex subgraphs, each family of cardinality $\mathbf{2}^{\boldsymbol{\Omega}(\boldsymbol{k})}$, where the induced subgraphs in each family are isomorphic.

Note: If the number of vertices of T is

$$\big(1 + o(1)\big)2^{(k-1)/2}$$

most members of T(k) cannot appear more than $\mathbf{2}^{\boldsymbol{o}(\boldsymbol{k})}$ times

Can random tournaments help ?

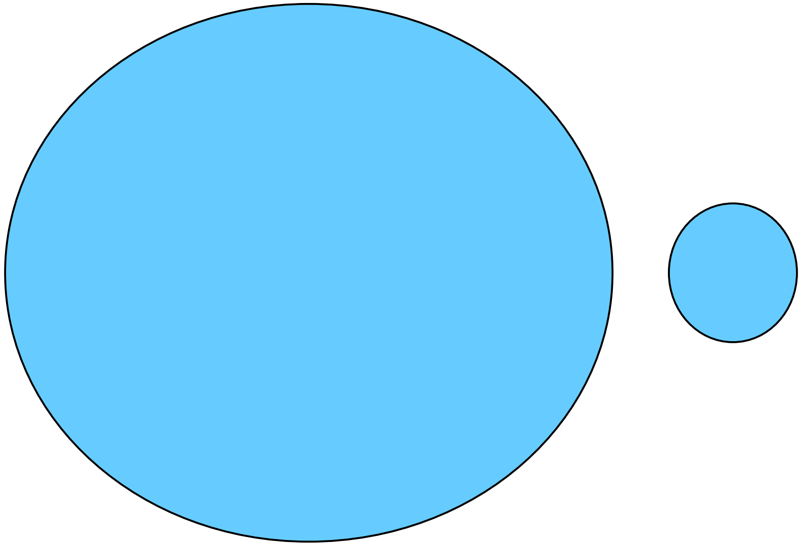
The bad news: With high probability, the random tournament T of size n=$(1+o(1))2^{(k-1)/2}$ does not contain any k vertex tournament with $2^{\Omega(k)}$ automorphisms. Indeed, the expected number of copies of any such k-vertex tournaments is o(1).

The good news: The expected number of copies of any k vertex tournament with no nontrivial automorphisms in the random T with $\mathrm{n} = (1 + \mathrm{o}(1))\ 2^{(k-1)/2}$ vertices is

$$\boldsymbol{\binom{n}{k} k!\ 2^{-\frac{k(k-1)}{2}} = (1+o(1))^{k} \gg 1}$$

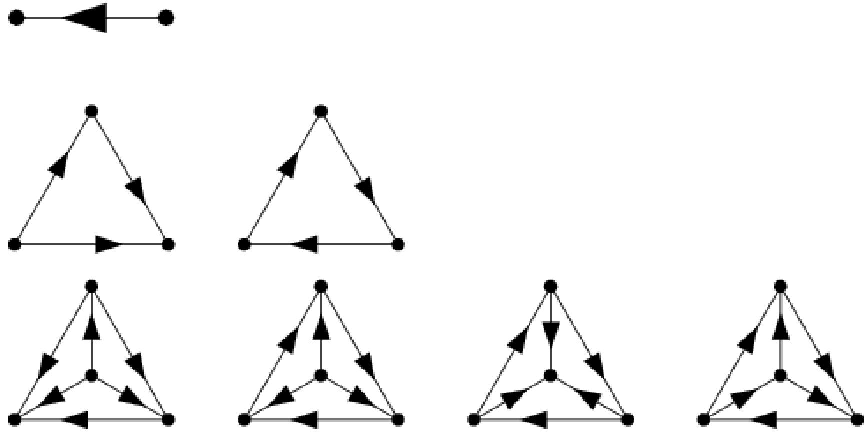
The plan: prove the existence of a k-universal tournament T for with **$n=(1+o(1))2^{(k-1)/2}$** vertices, where T consists of two vertex disjoint parts:

A random part containing all “asymmetric” k-vertex tournaments, and a much smaller structured part containing all k-vertex “symmetric” ones.



Definition: A k-vertex tournament is called asymmetric if it contains no induced subtournament with at least k^{4m} automorphisms, where $m = 2\sqrt{k \log k}$

If it is not asymmetric, it is called symmetric



Theorem 1: With high probability, the random tournament T(n), where

$$n=2^{\frac{k-1}{2}}\left(1+c\left(\frac{(\log k)^{\frac{3}{2}}}{\sqrt{k}}\right)\right)$$

contains every k-vertex asymmetric tournament as an induced subgraph.

The proof bounds the probability that a fixed asymmetric H is not an induced subgraph by applying Talagrand's Inequality to the random variable counting the maximum cardinality of a family of nearly disjoint induced copies of H in T(n).

What about the symmetric tournaments ?

Can we apply group theoretic results about the structure of large subgroups of the symmetric group to deduce some useful information about these ?

Many examples of symmetric tournaments: Take a random tournament and replace each one of

$$4m \log k = 8\sqrt{k \log k} \log k$$

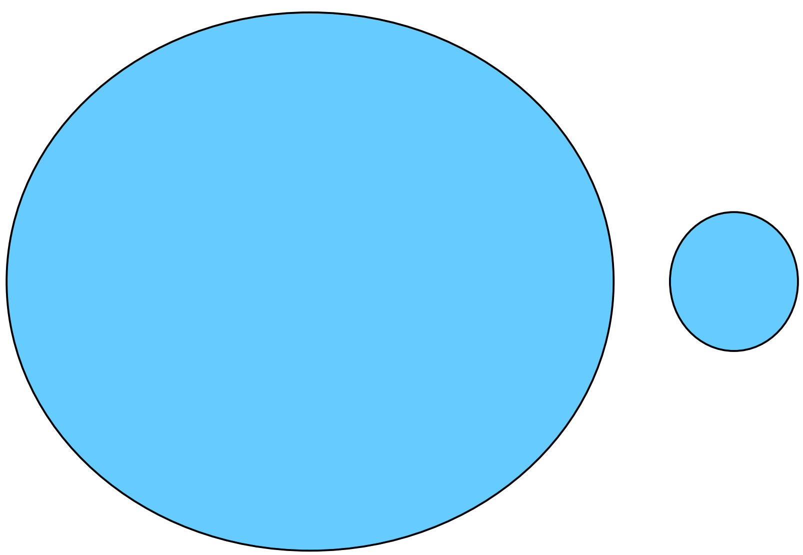
vertices by a cyclic triangle.

What about the symmetric tournaments ?

Here we can combine results from group theory about the structure of large subgroups of the symmetric group with the connection between efficient adjacency labelling schemes and universal tournaments [Kannan, Naor and Rudich (92)] to prove

Theorem 2: there is an (explicit) tournament with $o(2^{k/2})$ vertices that contains all symmetric k-vertex tournaments as induced subgraphs.

Theorems 1 and 2 establish Theorem A: the existence of a universal tournament T for T(k) with $\mathbf{n=(1+o(1))2^{(k-1)/2}}$ vertices, where T consists of two vertex disjoint parts, a large random piece containing the asymmetric tournaments and a much smaller one containing the symmetric ones.



V Open

A better estimate of the additive error term in the minimum possible number of vertices of a k-universal tournament

$$2^{\frac{k-1}{2}} + \frac{\mathbf{k-1}}{2} \le f(k) \le 2^{\frac{k-1}{2}} \left(1 + O\left(\frac{(\log k)^{\frac{3}{2}}}{\sqrt{k}}\right)\right)$$

Hitting time questions for the random tournament process

Happy Birthday, Zvi !