PH4211 2012

Question 1

a)

$$E_0 = E_1 + E_2$$

$\Omega_1(E_1 V_1 N_1) = $ N° of microstates in macrostate $(E_1 V_1 N_1)$
$\Omega_2(E_2 V_2 N_2) = $ " " " " " " $(E_2 V_2 N_2)$

[2]

N° of microstates in composite system is

$$\Omega = \Omega_1 \Omega_2$$

Equilibrium state is found by maximizing Ω by allowing the energy to vary in 2 systems, subject to the constraint $E_1 + E_2 = E_0$.

Alternatively, since entropy S is defined as $S = k \ln \Omega$ one may maximize [2] $S = S_1 + S_2$ similarly.

Write energy of system 1 is E, then energy of system 2 is $E_0 - E$

Since $N_1 V_1 N_2 V_2$ don't change, we may ignore them.

So maximize $\Omega = \Omega_1(E) \Omega_2(E_0 - E)$
or $S = S_1(E) + S_2(E_0 - E)$
— i.e. set derivative to zero: [2]

$$\frac{dS}{dE} = \frac{dS_1}{dE} - \frac{dS_2}{dE} = 0$$

i.e. Condition is

$$\frac{dS_1}{dE} = \frac{dS_2}{dE}$$

Identify $\frac{1}{T} = \frac{dS}{dE}$

<u>either</u> from firm law   [2]

<u>or</u> definition of statistical entropy.

So we have shown that $T_1 = T_2$ when systems have reached equilibrium.

<u>Equilibrium</u> : Experimentally — this is when no further change happens. Theoretically — this is when the system [2] is in the macrostate corresponding to the largest no. of microstates — the most possible macrostate.

⑩

b) Entropy maximum require

$$\frac{d^2S}{dE^2} < 0 \qquad\qquad [1]$$

Now $\frac{d^2S}{dE^2} = \frac{d}{dE}\frac{1}{T}$
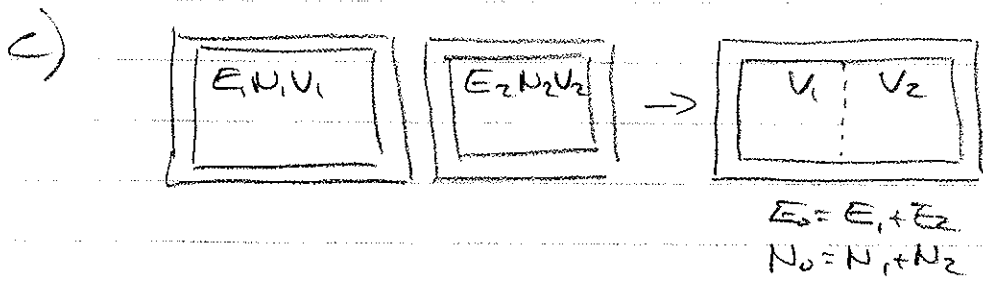
$$= -\frac{1}{T^2}\frac{dT}{dE}$$

$$= -\frac{1}{T^2 C_V} \qquad\qquad [5]$$

So since $\frac{d^2S}{dE^2} < 0$, and $T^2 > 0$ it follows that

$$\underline{\underline{C_V > 0}} \qquad\qquad [4]$$

⑩

c)

$$\boxed{E_1 N_1 V_1} \quad \boxed{E_2 N_2 V_2} \quad \rightarrow \quad \boxed{V_1 \mid V_2}$$

$$E_0 = E_1 + E_2$$
$$N_0 = N_1 + N_2$$

Now must maximize $\Omega = \Omega_1 \Omega_2$ or $S = S_1 + S_2$
by varying E, N subject to two
constraints $E_0 = E_1 + E_2$, $N_0 = N_1 + N_2$

Write energy of system 1 $\rightarrow E$
No. of particles " " " $\rightarrow N$

So maximize $\quad S = S_1(E,N) + S_2(E_0 - E, N_0 - N)$
by varying E, N. This require

$$\left.\frac{\partial S}{\partial E}\right|_N = \left.\frac{\partial S_1}{\partial E}\right|_N - \left.\frac{\partial S_2}{\partial E}\right|_N = 0$$

and $\quad \left.\frac{\partial S}{\partial N}\right|_E = \left.\frac{\partial S_1}{\partial N}\right|_E - \left.\frac{\partial S_2}{\partial N}\right|_E = 0$ [4]

i.e. the conditions

$$\left.\frac{\partial S_1}{\partial E}\right|_N = \left.\frac{\partial S_2}{\partial E}\right|_N \quad \text{and} \quad \left.\frac{\partial S_1}{\partial N}\right|_E = \left.\frac{\partial S_2}{\partial N}\right|_E$$
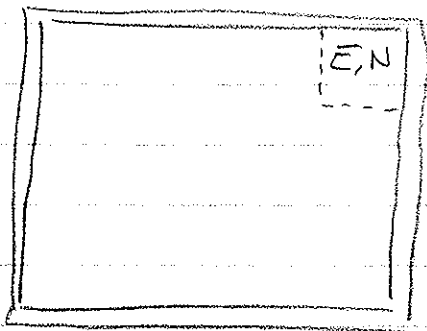
$$\left.\frac{\partial S}{\partial E}\right|_N = \frac{1}{T} \quad \rightarrow \quad \text{equality of temperature}$$

$$\left.\frac{\partial S}{\partial N}\right|_E = -\frac{\mu}{T} \quad \text{and Since } T \text{ is equalized}$$
$$\rightarrow \text{equality of } \mu. \qquad [4]$$

$$\Delta S = \left(\frac{\partial S_1}{\partial N} - \frac{\partial S_2}{\partial N}\right) \Delta N > 0$$

$$= \frac{1}{T}(\mu_2 - \mu_1) \Delta N$$

[2]

so flow is from high μ to low μ.

d)

E,N

Total energy $= E_0$
Total no. of particles $= N_0$

$$S = S_{reservoir} + S_{subsystem}$$

Subsystem is in a given microstate,
so $S_{subsystem} = 0$

$$S = S_{reservoir} = S(E_0 - E, N_0 - N) \qquad [4]$$

but $E \ll E_0$, $N \ll N_0$ so 1st
term in Taylor exp^n is adequate:

$$S = S_0 \quad - E \frac{\partial S}{\partial E} - N \frac{\partial S}{\partial N}$$

$\uparrow$
$S(E_0, N_0)$

but $\frac{\partial S}{\partial E} = \frac{1}{T}$ $\qquad \frac{\partial S}{\partial N} = -\frac{\mu}{T}$

i.e. $\quad S = S_0 - \frac{E}{T} + \frac{\mu N}{T} \qquad [4]$

Probability $\propto e^{S/k}$

so $P(E,N) \propto e^{S_0/k} e^{-(E - \mu N)/kT}$

$\lfloor$ const.

so $P \propto e^{-(E - \mu N)/kT} \qquad [2]$

Question 2.

a) Phase space is the space of the
generalized coordinates and generalized
momenta of a system (Classical   [2]
description).

In the Gibbs picture an N-particle
system will have 3N p's & 3N q's
so the phase space will have 6N
dimensions — very large indeed.
And then the MICRO state of a system
is described by a single point   [3]
in the phase space.

In the Boltzmann picture each
particle will be represented by a
point in a 6-dimensional space.
So then a system of N particles
is represented by a 'cloud' of N
points in the 6-dimensional space   [3]

(8)

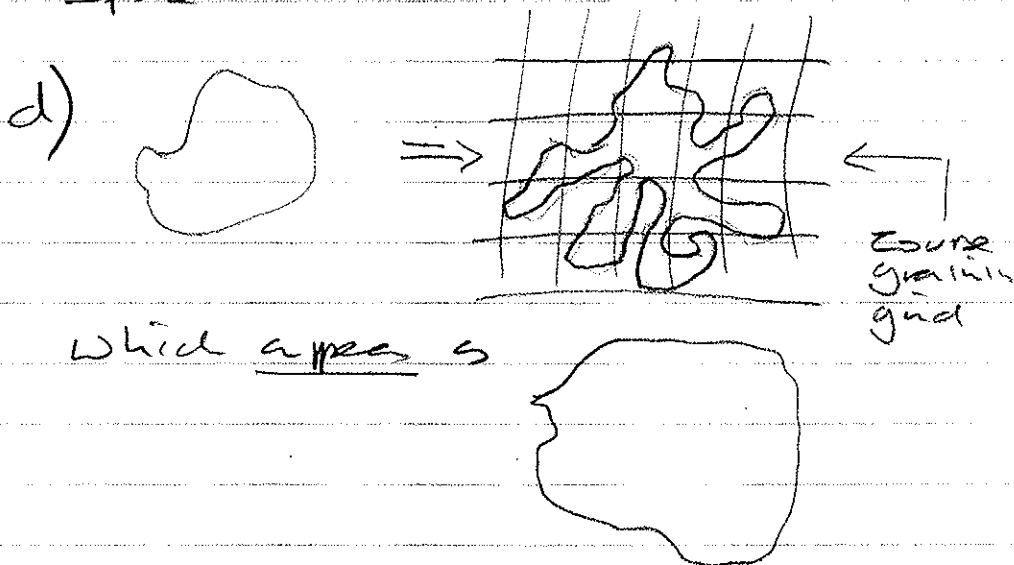b) Conventional expression is

$$ S = k \ln p $$

(But this hides the question of the
'size' of the 'cells' in phase space
that will need Planck's constant
in order to get the correct classical-
quantum correspondence)

(8)

c) Liouville's theorem $\Rightarrow$ $\rho = $ const
2$^{\text{nd}}$ law again ρ get smaller. ⑧

Explanation is in the nature of the
flow of the cloud of points in phase
space.

d)

which appears

Coarse graining is based on the idea
that there is a volume in phase
space 'finer' than which you
cannot resolve

— This can be regarded simply as a ⑧
statement of the __Uncertainty principle__.

e) Third law relates to the uniqueness of
the quantum ground state (microstate at
$T=0$). So $S = k \ln \Omega = 0$
(simplistic argument). If classical
mechanics were to apply then at $T=0$
there would be a single point in
phase space — ρ would be a δ function ⑧
& then S would be $= -\infty$

Question 3

a) i   Order parameter magnetization   2
    ii   Order parameter not conserved   2
    iii Rotational Symmetry (broken time rev.)   2
    iv   Continuous   2
    v   Second order.   2

                                                              [10]

b) Put $B \to B_0 + \lambda M$

    into $\dfrac{M}{M_0} = \tanh\left(\dfrac{M_0}{N}\dfrac{B}{kT}\right)$
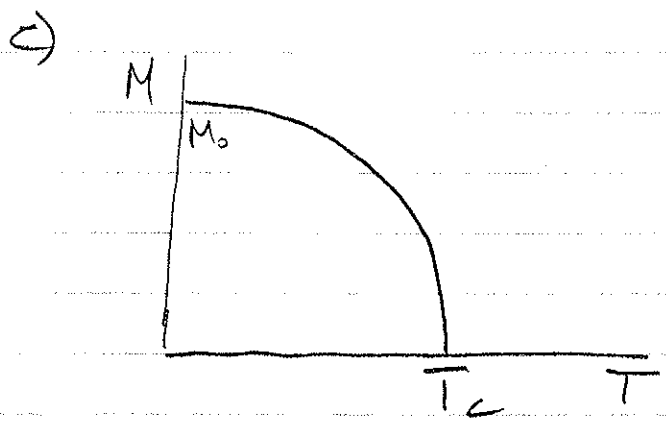
    to get $\dfrac{M}{M_0} = \tanh\left[\dfrac{M_0}{N}\dfrac{(B_0+\lambda M)}{kT}\right]$

    Spontaneous magnetization is
    when $B_0 = 0$

    i.e. $\dfrac{M}{M_0} = \tanh \dfrac{M_0 \lambda M}{NkT}$

          $= \tanh \dfrac{M_0}{M}\dfrac{T_c}{T}$

    if $T_c = \lambda M_0^2 / Nk$.

    T_c is the "Critical temperature"
    - the temperature at which the
    spontaneous magnetization   [18]
    appears.

c)

Since M goes to zero continuously
as $T \to T_c$ (from below), this
means it's a continuous transition
or a 2nd order. [4]

d) $b = \lambda M$

w/ $\lambda = \dfrac{kT_c}{\mu^2 N}$ w/ $M = \mu N$

so $b = \dfrac{kT_c}{\mu}$

$\sim 10^3$ Tesla.

— Too high for a dipole field
for electrons.

— Exchange interaction [8]

Question 5.

a) <u>Order Parameter</u> : quantity that
measures the order in the ordered
phase ~ will be zero in
the disordered phase. (Although
there is some freedom in this
choice, this is limited from
considering the irreducible representation
of the broken symmetry group)   [6]

b) When the order parameter small
the series may be truncated.
But if one is interested in the
<u>generic</u> properties of a transition
then one uses only sufficient
terms to exhibit the phenomenon
of the transition — the extra
terms give no qualitative change
to the behavior.   [7]

c) Need an even-order term with
positive coefficient to ensure
that the order parameter is
<u>bounded</u> . This is an absolute
<u>requirement</u>. The higher-order term
cannot change sign.

d) Put $F = F_2 \phi^2 + F_4 \phi^4$

Symmetry implies no odd-order
terms (and no external field)

Equilib. state - minimize F

$$\frac{dF}{d\phi} = 2F_2\phi + 4F_4\phi^3$$

So either $\phi = 0$ or $\phi = \pm\sqrt{\frac{-F_2}{2F_4}}$

F_4 has to be positive (argue above).

F_2 can vary with temperature.

Put simpler : $F_2 = a(T - T_c)$

then $\phi = \sqrt{\frac{a(T_c - T)}{F_4}}$ $T \leq T_c$

this goes to zero as $[T_c - T]^{1/2}$
— i.e. continuously —

So Transition is 2nd order
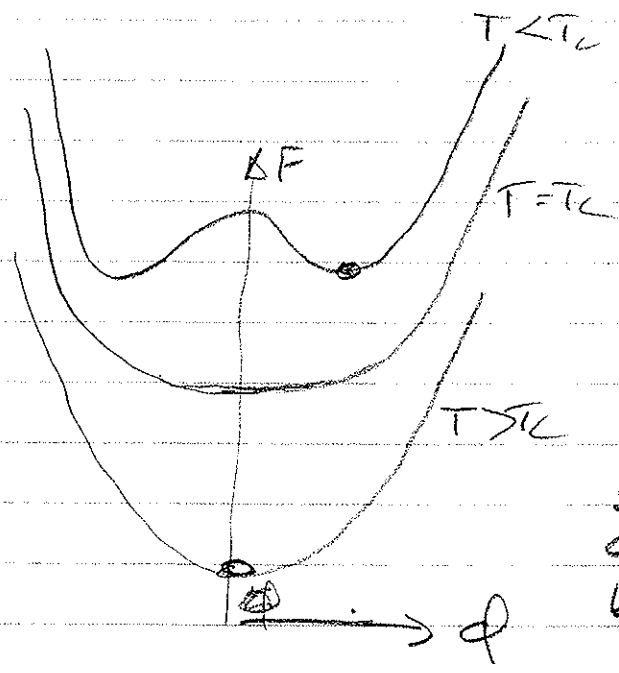
as $\beta = \frac{1}{2}$
    coeff of $(T_c - T)$ in ϕ

e) The calculated β value is a
   <u>mean field</u> result.

Mean fields neglects (vital)
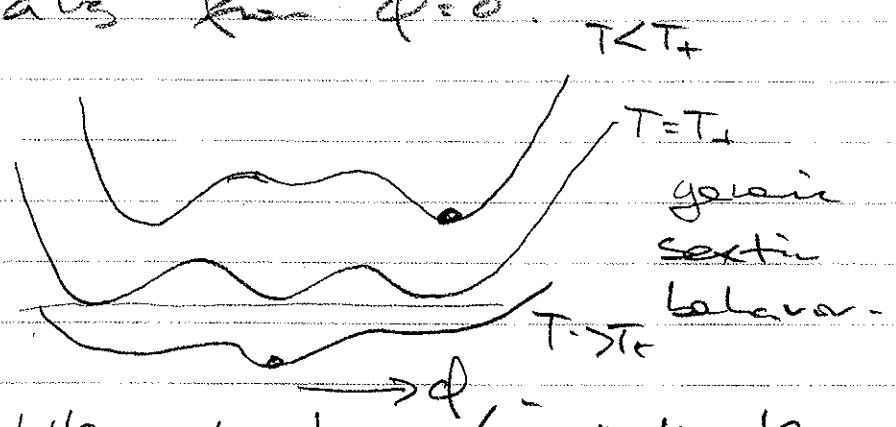fluctuations at T_c. It is
this which leads to $\beta = 1/3$.

[6]

— i.e. problem is limitation of
theory in the <u>vicinity</u> of T_c.

f)

$T < T_c$

ΔF

$T = T_c$

$T > T_c$

generic
quartic
behavior.

$\rightarrow \phi$

This is a 2ⁿᵈ-order transition. At T_c the extra minima develop and the minima move continuously away from $\phi = 0$.

$T < T_+$

$T = T_+$

generic
sextic
behavior.

$T > T_+$

$\rightarrow \phi$

When you have 6ᵗʰ order term this is the generic curve and you see that at the order parameter jumps at the transition temperature.

[8]