PH4211 Statistical Mechanics

Problem Sheet 4

4.1 Obtain an expression for the Helmholtz free energy for the Weiss model in zero external magnetic field, in terms of the magnetisation. Plot $F(M)$ for $T > T_{\mathrm{C}}$, $T = T_{\mathrm{C}}$ and $T < T_{\mathrm{C}}$.

4.2 Show that $F = \frac{Nk}{2}\left\{(T - T_{\mathrm{C}})m^2 + \frac{T_{\mathrm{C}}}{6}m^4 + \cdots\right\}$ for the Weiss model ferromagnet in the limit of small *m*. Explain the appearance of T_{C} in the m^4 term.

4.3 Show that $\mathrm{d}^2F/\mathrm{d}\varphi^2 > 0$ below T_{c} at the two roots $\varphi = \pm\sqrt{-F_2/2F_4}$ in the Landau model. Show that $\mathrm{d}^2F/\mathrm{d}\varphi^2 < 0$ below T_{c} and $\mathrm{d}^2F/\mathrm{d}\varphi^2 > 0$ above T_{c} at the single root $\varphi = 0$. What is the physical meaning of this?

4.4 In the Landau theory of second order transitions calculate the behaviour of the order parameter below the critical point, $\varphi(T)$, when the *sixth* order term in the free energy expansion is not discarded. What influence does this term have on the critical exponent β? Comment on this.

4.5 A ferroelectric has a free energy of the form

$$F = \alpha(T - T_{\mathrm{c}})P^2 + bP^4 + cP^6 + DxP^2 + Ex^2$$

where *P* is the electric polarisation and *x* represents the strain. Minimise the system with respect to *x*. Under what circumstances is there a first order phase transition for this system?

4.6 Consider a one-dimensional binary alloy where the concentration of A atoms varies slowly in space: $x = x(z)$. Show that the spatial variation of *x* results in an additional term in the free energy per bond of $3a^2\varepsilon(\mathrm{d}x/\mathrm{d}z)^2/2$, where *a* is the spacing between atoms and ε is the energy parameter defined in Section 4.7.3.

4.7 Show that in the vicinity of the critical point the free energy of the binary alloy may be written as

$$F_{\mathrm{m}} = F_0 + 2Nk\left\{(T - T_{\mathrm{c}})\left(x - \tfrac{1}{2}\right)^2 + \frac{2}{3}T_{\mathrm{c}}\left(x - \tfrac{1}{2}\right)^4 + \frac{16}{15}T_{\mathrm{c}}\left(x - \tfrac{1}{2}\right)^6 + \ldots\right\}$$

Discuss the Landau truncation of this expression; in particular, explain at what term the series may/should be terminated.

4.8 Plot some isotherms of the Clausius equation of state $p(V - Nb) = NkT$. How do they differ from those of an ideal gas? Does this equation of state exhibit a critical point? Explain your reasoning.

4.9 The scaling expression for the reduced free energy is given in Section 4.1.9 by

$$f\left(T,B\right)=A\left|t\right|^{2-\alpha}\,Y\left(D\frac{B}{\left|t\right|^{\Delta}}\right).$$

Show that the heat capacity is given by

$$C\sim\frac{\mathrm{d}^2 f\left(t,B\right)}{\mathrm{d}t^2}$$

and hence identify α as the heat capacity critical exponent.

4.10 Using the scaling expression for the reduced free energy in the previous section, show that the magnetisation is given by

$$M\sim\frac{\mathrm{d}f\left(t,B\right)}{\mathrm{d}B}$$

and hence show that the order parameter exponent β is given by

$$\beta=2-\alpha-\Delta\,.$$

Show that the magnetic susceptibility is given by

$$\chi\sim\frac{\mathrm{d}^2 f\left(t,B\right)}{\mathrm{d}B^2}$$

and hence show that the susceptibility exponent g is given by

$$\gamma=2-\alpha-2\Delta\,.$$

4.11 Show that the Landau free energy has the scaling form of Problem 4.9 above, with $\alpha = 0$.