UNIVERSITY OF LONDON

MSc/MSci EXAMINATION 2019

For Students of the
University of London

DO NOT TURN OVER UNTIL TOLD TO BEGIN

PH4211: STATISTICAL MECHANICS
PH5211: STATISTICAL MECHANICS
PH5911: STATISTICAL MECHANICS

Time Allowed: **TWO AND A HALF hours**

Answer **THREE** questions
No credit will be given for attempting any further questions

Approximate part-marks for questions are given in the right-hand margin

The total available marks add up to 120

All College-approved Calculators are permitted

GENERAL PHYSICAL CONSTANTS

Permeability of vacuum	μ_0	$=$	4π	$\times$	10^{-7}	H m^{-1}
Permittivity of vacuum	ε_0	$=$	8.85	$\times$	10^{-12}	F m^{-1}
	$1/4\pi\varepsilon_0$	$=$	9.0	$\times$	10^{9}	m F^{-1}
Speed of light in vacuum	c	$=$	3.00	$\times$	10^{8}	m s^{-1}
Elementary charge	e	$=$	1.60	$\times$	10^{-19}	C
Electron (rest) mass	m_e	$=$	9.11	$\times$	10^{-31}	kg
Unified atomic mass constant	m_u	$=$	1.66	$\times$	10^{-27}	kg
Proton rest mass	m_p	$=$	1.67	$\times$	10^{-27}	kg
Neutron rest mass	m_n	$=$	1.67	$\times$	10^{-27}	kg
Ratio of electronic charge to mass	e/m_e	$=$	1.76	$\times$	10^{11}	C kg^{-1}
Planck constant	h	$=$	6.63	$\times$	10^{-34}	J s
	$\hbar = h/2\pi$	$=$	1.05	$\times$	10^{-34}	J s
Boltzmann constant	k	$=$	1.38	$\times$	10^{-23}	J K^{-1}
Stefan-Boltzmann constant	σ	$=$	5.67	$\times$	10^{-8}	W m^{-2} K^{-4}
Gas constant	R	$=$	8.31			J mol^{-1} K^{-1}
Avogadro constant	N_A	$=$	6.02	$\times$	10^{23}	mol $^{-1}$
Gravitational constant	G	$=$	6.67	$\times$	10^{-11}	N m^2 kg^{-2}
Acceleration due to gravity	g	$=$	9.81			m s^{-2}
Volume of one mole of an ideal gas at STP		$=$	2.24	$\times$	10^{-2}	m^3
One standard atmosphere	P_0	$=$	1.01	$\times$	10^{5}	N m^{-2}

MATHEMATICAL CONSTANTS

$$e \cong 2.718 \qquad \pi \cong 3.142 \qquad \log_e 10 \cong 2.303$$

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1. (a) The partition function for a single particle of mass m at temperature T, in a box of volume V is given by

$$z = V\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} = \frac{V}{\Lambda^3}.$$

Why is the quantity Λ referred to as the thermal de Broglie wavelength? [6]

(b) Explain why the partition function for a collection of N similar but *distinguishable* objects, each with partition function z, is expressed as

$$Z = z^N.$$

[10]

(c) On the assumption that $Z = z^N$ is correct for a gas of particles, show that the Helmholtz free energy for such a gas would be given by

$$F = -NkT\ln\left\{V\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2}\right\}.$$

[6]

(d) This expression gives a Helmholtz free energy that is *not extensive*. Explain what this means, and explain why this is a problem. [8]

(e) Discuss in detail how this problem may be resolved and give the resultant expression for F. [8]

(f) The resolution relies on the *thermodynamic limit*; explain. [2]

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2. (a) In the Weiss model of ferromagnetism it is assumed that the magnetic moments are subject to an additional "mean" magnetic field $\mathbf{b}$

$$\mathbf{b} = \lambda \mathbf{M}$$

where $\mathbf{M}$ is the magnetization and λ is a constant.
Explain briefly the origin of this field. [8]

(b) The magnetization of a collection of N non-interacting spin $\frac{1}{2}$ magnetic moments μ at a temperature T is given by

$$M = M_0 \tanh\left(\frac{\mu B}{kT}\right)$$

where M_0 is the saturation magnetization $M_0 = N\mu$ and the directions of $\mathbf{M}$ and the applied magnetic field $\mathbf{B}$ are parallel.
Show that the Weiss model leads to a spontaneous magnetization, in the absence of an external magnetic field, given by

$$\frac{M}{M_0} = \tanh\left(\frac{M}{M_0}\frac{T_\mathrm{c}}{T}\right)$$

where $T_\mathrm{c} = \lambda M_0^2/Nk$. [8]

(c) i. Sketch the behaviour of the spontaneous magnetization as a function of temperature. [4]
ii. What is the interpretation of T_c? [2]
iii. Discuss the order of the transition. [2]

(d) When T is very close to T_c then M/M_0 is very small. By expanding the $\tanh$ ($\tanh x \approx x - x^3/3 + \ldots$) show that, in the vicinity of T_c, the magnetization behaves as

$$\frac{M}{M_0} \propto \left(1 - \frac{T}{T_\mathrm{c}}\right)^{1/2}.$$

[8]

(e) How does this result compare with the behaviour of real systems? Discuss the difference. [8]

3. In the Landau theory of phase transitions the free energy is expanded in powers of the order parameter to a finite number of terms.

   (a) In the context of phase transitions what is meant by the term *order parameter*? [4]

   (b) Why is the Landau expansion terminated, and what determines the highest power in the expansion? [6]

   (c) What is the distinction between a *conserved* order parameter and a *non-conserved* order parameter? Give an example of each. Describe how one determines the equilibrium state of the system in these two cases. [6]

   (d) Make a labelled sketch of the free energy for a binary alloy. Show curves for the temperature *above* and *below* the critical temperature. [6]

   (e) The free energy of mixing for the binary alloy may be written

   $$F_{\mathrm{m}} = 2NkT_{\mathrm{c}}x(1-x) + NkT\left[x\ln x + (1-x)\ln(1-x)\right].$$

   Explain the structure of this expression. [6]

   (f) The expansion of F_{m} may be written as

   $$F_{\mathrm{m}} = F_0 + 2Nk\left\{(T-T_{\mathrm{c}})\left(x-\frac{1}{2}\right)^2 + \frac{2}{3}T_{\mathrm{c}}\left(x-\frac{1}{2}\right)^4 + \frac{16}{15}T_{\mathrm{c}}\left(x-\frac{1}{2}\right)^6 + \cdots\right.$$

   Discuss the Landau truncation of this expression; in particular explain at which term the series should be terminated. Why is the expansion written in powers of $x - \frac{1}{2}$? [6]

   (g) Within this model the order parameter critical exponent β has the value $\frac{1}{2}$. Show how this follows from the Landau free energy. The value $\beta = \frac{1}{2}$ does *not* agree with experiment. Why is this? [6]

4. Write an essay on the logical foundations of Statistical Mechanics and its connection with Thermodynamics. You should include a discussion of the nature of *equilibrium states*, the emergence of the concept of *temperature*, and the importance of the *thermodynamic limit*. [40]

5. The force on a Brownian particle in one dimension may be written as $F(t) = f(t) - v/\mu$ where $f(t)$ is a randomly fluctuating force, v is the velocity and μ the mobility of the particle.

   (a) Discuss the separation of the force into the two parts. In particular, explain qualitatively how the damping force, proportional to the velocity, arises as a consequence of the random motion of the background fluid atoms. [10]

   (b) Show that the equation of motion for the Brownian particle of mass M (the Langevin equation) may be written as [4]

   $$M\frac{\mathrm{d}v(t)}{\mathrm{d}t} + \frac{1}{\mu}v(t) = f(t).$$

   (c) The solution to the Langevin equation is given by

   $$v(t) = v(0)e^{-t/M\mu} + \frac{1}{M}\int_0^t e^{(s-t)/M\mu} f(s)\,\mathrm{d}s.$$

      i. Show, using appropriate approximations, that the equilibrium mean square velocity may be expressed as $\langle v^2 \rangle = \frac{\mu}{2M}\int_{-\infty}^{\infty}\langle f(0)f(t)\rangle \mathrm{d}t$, and [8]

      ii. by invoking the equipartition theorem, show it follows that the mobility may be expressed $\frac{1}{\mu} = \frac{1}{2kT}\int_{-\infty}^{\infty}\langle f(0)f(t)\rangle\,\mathrm{d}t$. [2]

   (d) Discuss how this may be regarded as an example of the *fluctuation-dissipation theorem*. [4]

   (e) The voltage across an $L - R$ circuit is given in terms of the current $I(t)$ by

   $$L\frac{\mathrm{d}I(t)}{\mathrm{d}t} + RI(t) = V(t)$$

   where $V(t)$ may be regarded as a fluctuating voltage. By making the appropriate identifications, show that the resistance may be related to the voltage fluctuations through [8]

   $$R = \frac{1}{2kT}\int_{-\infty}^{\infty}\langle V(0)V(t)\rangle\,\mathrm{d}t.$$

   (f) How may such voltage fluctuations be observed? [4]

END