UNIVERSITY OF LONDON

MSci EXAMINATION 2009

For Students of the
University of London

DO NOT TURN OVER UNTIL TOLD TO BEGIN

PH4211A: STATISTICAL MECHANICS

Time Allowed: **TWO AND A HALF** hours

Answer **THREE** questions only

No credit will be given for attempting any further questions.

GENERAL PHYSICAL CONSTANTS

Permeability of vacuum	μ_0	=	$4\pi \times 10^{-7}$	$\mathrm{H\ m^{-1}}$
Permittivity of vacuum	ε_0	=	8.85×10^{-12}	$\mathrm{F\ m^{-1}}$
	$1/4\pi\varepsilon_0$	=	9.0×10^{9}	$\mathrm{m\ F^{-1}}$
Speed of light in vacuum	c	=	3.00×10^{8}	$\mathrm{m\ s^{-1}}$
Elementary charge	e	=	1.60×10^{-19}	C
Electron (rest) mass	m_e	=	9.11×10^{-31}	kg
Unified atomic mass constant	m_u	=	1.66×10^{-27}	kg
Proton rest mass	m_p	=	1.67×10^{-27}	kg
Neutron rest mass	m_n	=	1.67×10^{-27}	kg
Ratio of electronic charge to mass	e/m_e	=	1.76×10^{11}	$\mathrm{C\ kg^{-1}}$
Planck constant	h	=	6.63×10^{-34}	J s
	$\hbar = h/2\pi$	=	1.05×10^{-34}	J s
Boltzmann constant	k	=	1.38×10^{-23}	$\mathrm{J\ K^{-1}}$
Stefan-Boltzmann constant	σ	=	5.67×10^{-8}	$\mathrm{W\ m^{-2}\ K^{-4}}$
Gas constant	R	=	8.31	$\mathrm{J\ mol^{-1}\ K^{-1}}$
Avogadro constant	N_A	=	6.02×10^{23}	$\mathrm{mol^{-1}}$
Gravitational constant	G	=	6.67×10^{-11}	$\mathrm{N\ m^{2}\ kg^{-2}}$
Acceleration due to gravity	g	=	9.81	$\mathrm{m\ s^{-2}}$
Volume of one mole of an ideal gas at STP		=	2.24×10^{-2}	$\mathrm{m^{3}}$
One standard atmosphere	P_0	=	1.01×10^{5}	$\mathrm{N\ m^{-2}}$

MATHEMATICAL CONSTANTS

$e \cong 2.718$ $\qquad \pi \cong 3.142$ $\qquad \log_e 10 \cong 2.303$

PART MARKS

1. (a) The Boltzmann expression for entropy **[4]**

$$S = k \ln \Omega$$

makes a connection between the microscopic and the macroscopic descriptions of matter. Explain the meaning of the terms in this expression and discuss how it is related to the law of entropy increase.

(b) Show that when two isolated systems are brought into thermal contact, they end up in the thermodynamic state for which **[4]**

$$\frac{\mathrm{d}\ln\Omega_1}{\mathrm{d}E} = \frac{\mathrm{d}\ln\Omega_2}{\mathrm{d}E},$$

defining the terms Ω_1, Ω_2 and E in this expression. How does this relate to temperature?

(c) Now consider a small sub-system of a large isolated system. The total energy of the isolated system is E_t. The sub-system can exchange thermal energy with the large system. Show that when the sub-system is in a microstate of energy E the entropy of the combined system may be expressed as **[4]**

$$S = S\left(E_\mathrm{t}\right) - E\frac{\mathrm{d}S}{\mathrm{d}E} + \frac{E^2}{2}\frac{\mathrm{d}^2S}{\mathrm{d}E^2} - ....$$

(d) Show how the above result leads to the Boltzmann distribution function (otherwise known as the Boltzmann factor or canonical distribution function). Include a clear justification for the truncation of the above series at the first-order term. **[4]**

(e) Instead of expanding S in powers of E one might expand Ω. Discuss why, in this case, the series *cannot* be truncated. **[4]**

2. Write an essay on the breaking of symmetry in phase transitions. You should give at least two different examples of symmetry-breaking transitions, indicating the order parameter and the symmetry broken for each. Include a discussion of Goldstone's theorem and identify the Goldstone Bosons for your examples. The Bosons of Goldstone's theorem are said to have zero mass. You should discuss the meaning of this statement and explain how such Bosons may acquire mass through the Higgs mechanism. **[20]**

PART MARKS

3. (a) The partition function for a single particle moving freely in a box of volume V may be written as

$$z = V\left(\frac{2\pi mkT}{h^2}\right)^{3/2} = \frac{V}{\Lambda^3},$$

where the symbols have their usual meaning.

i) The quantity Λ is known as the *thermal de Broglie wavelength*. Explain the meaning of this quantity. **[3]**

ii) Explain why the partition function Z for a gas of N classical indistinguishable particles is related to the partition function z of a single particle by **[4]**

$$Z = \frac{1}{N!} z^N .$$

Show that the pressure of a gas of N indistinguishable such particles is related to z by

$$p = NkT \left.\frac{\partial \ln z}{\partial V}\right|_T .$$

iii) Evaluate the pressure for this system, hence deriving the equation of state of an ideal gas. **[2]**

(b) In the van der Waals approach to an *interacting* gas the single particle partition function may be approximated by

$$z = \frac{V - V_{\text{ex}}}{\Lambda^3} e^{-\langle E \rangle / kT} .$$

i) Sketch the expected variation with separation of the inter-particle interaction potential. Discuss how the various features of the interaction are incorporated into the above expression for z through the quantities V_{ex} and $\langle E \rangle$. **[5]**

ii) Sketch the way the Helmholtz free energy of the van der Waals gas varies with volume. Indicate regions of the curve that are not stable and explain how the equilibrium state of the system is determined using the *double tangent* construction. **[3]**

iii) Explain the connection between the double tangent construction and the equal areas construction of Maxwell. **[3]**

PART MARKS

4. (a) What is *Brownian motion*? **[2]**

(b) The force on a Brownian particle may be written as **[4]**

$$F(t) = f(t) - \frac{1}{\mu}v$$

where $f(t)$ is a randomly fluctuating force, v is the velocity and μ the mobility of the particle. Discuss the separation of the force into these two parts.

(c) Show that the equation of motion for the Brownian particle may be written as **[3]**

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} + \gamma v(t) = A(t)$$

and identify the terms.

(d) The solution to the equation of motion may be written **[3]**

$$v(t) = v(0)e^{-\gamma t} + \int_0^t e^{\gamma(u-t)} A(u)\mathrm{d}u \, .$$

Describe how this solution arises and explain its implications.

(e) The autocorrelation function for the random force per unit mass is defined by the average **[3]**

$$\langle A(t)A(t+\tau)\rangle \, .$$

Discuss the physical meaning of this expression and explain why it is independent of the time t.

(f) A galvanometer is an instrument for measuring very small electric currents. It consists of a coil attached to a mirror and suspended by a torsion fibre between the poles of a magnet. Angular deflections of the coil, caused by flowing current are measured by the deflection of a light beam reflected by the mirror. **[5]**

The mirror may be regarded as a Brownian particle, obeying equipartition. What determines the mean square magnitude of the Brownian fluctuations?

Such fluctuations are understood to arise from impacts between air molecules and the mirror. What happens to the Brownian fluctuations of the mirror if the air is pumped out of the galvanometer while the mirror's temperature remains constant?

PART MARKS

5. When the mean field theory of phase transitions is applied to the ferromagnetic transition in the absence of a magnetic field, the free energy is given approximately, in the vicinity of the transition, by

$$F = \frac{Nk}{2}\left(T - T_{\mathrm{c}}\right)m^2 + \frac{Nk}{12}T_{\mathrm{c}}m^4$$

where the symbols have their usual meaning.

(a) Explain the structure of this expression and indicate what the various quantities are. Why are there no higher powers of *m*? **[4]**

(b) By sketching the form of this expression, explain how the paramagnetic to ferromagnetic transition occurs as the temperature is lowered through T_{c} and discuss the magnitude of the fluctuations at T_{c}. **[4]**

(c) Using the above expression show that in the vicinity of the transition **[5]**

$$m = \pm\sqrt{\frac{3\left(T_{\mathrm{c}} - T\right)}{T_{\mathrm{c}}}}$$

and sketch this behaviour.

(d) What is the order of this transition? Explain your reasoning. **[2]**

(e) Show that below the transition there is a contribution to the entropy given by **[5]**

$$\frac{3Nk}{2}\left(T - T_{\mathrm{c}}\right)/T_{\mathrm{c}}$$

and that this leads to a jump in the thermal capacity on cooling through the transition of

$$\Delta C = \frac{3Nk}{2}.$$

END