Mean field theory, the Ginzburg criterion, and marginal dimensionality of phase transitions*

J. Als-Nielsen† and R. J. Birgeneau
Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
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By applying a real space version of the Ginzburg criterion, the role of fluctuations and thence the self-consistency of mean field theory are assessed in a simple fashion for a variety of phase transitions. It is shown that in using this approach the concept of "marginal dimensionality" emerges in a natural way. For example, it is shown that for many homogeneous structural transformations the marginal dimensionality is two, so that mean field theory will be valid for real three-dimensional systems. It is suggested that this simple self-consistent approach to Landau theory should be incorporated in the teaching of elementary phase transition phenomena.

I. INTRODUCTION

The most common technique for teaching the subject of phase transitions is to use the mean field theory[1-3] or, equivalently, the Landau theory.[4,5] The reason is obvious: using a minimum of mathematics, many of the salient physical features of phase transition phenomena are elucidated, at least in a qualitative sense. However, as is well known, mean field theory fails to describe many real experimental systems; also, for certain mathematically tractable Hamiltonians such as the two-dimensional Ising model, the discrepancies are very substantial.[3] It is probably well known to the reader that an important breakthrough has occurred within the last few years in the theory of phase transitions with Wilson's renormalization group (RG) theory.[6,7] The starting point for this approach is the Landau theory. Through a rather complicated mathematical procedure the RG theory, among other results, introduces the concept of *marginal dimensionality*, d^*.[8] When the dimensionality d for a system is larger than the corresponding d^*, the system exhibits mean field or "classical" behavior, whereas "critical" behavior occurs for $d < d^*$. When $d = d^*$, Landau behavior is modified by additional "weak" singular behavior such as, for example, logarithmic corrections. In nature, systems exist with marginal dimensionality of $d^* = 2$, 3, 4, and 6 and indeed many other values. When $d = d^*$, the RG equations are exact and one then makes the so-called ϵ expansion, $\epsilon = d^* - d$, to estimate the critical behavior for $d < d^*$.[6]

It is the purpose of the present paper to introduce the concept of marginal dimensionality in a simple fashion and to find d^* for a number of systems without using the complexity of RG theory, but rather to use a real space version of the so-called Ginzburg criterion[9] to assess the validity of mean field theory. One example has previously been published for tricritical beahvior[10]; here we apply the same criterion for the dipolar-coupled, uniaxial ferromagnet ($d^* = 3$), and a structural phase transition driven by the softening of an acoustic phonon ($d^* = 2$). The results for d^* for these systems are known from rather complicated RG theory,[11,12] so this paper does not contain new results. It is our hope, however, that teachers might find our approach simple enough to incorporate it in their course and thereby introduce the students to concepts which are essential for a proper understanding of critical phenomena. Or, stated more strongly, we believe that in teaching Landau theory with a modern perspective it is essential that the limits of the theory be addressed directly. As we shall show, this automatically introduces the marginal dimensionality concept.

II. GENERAL FEATURES

All second-order phase transitions have the same qualitative features. In order to describe these it is convenient, however, to use the nomenclature of one particular type of system, and we have therefore chosen the language of *magnetic* systems. It is emphasized that simply by changing a few words the following description may be adapted to describe any second-order phase transition; for example, in the case of the gas–liquid transition around the critical point insert "density" for "magnetization," "pressure" for "magnetic field," and "compressibility" for "susceptibility."

The origin of magnetization is the atomic spin S. To be more specific, the magnetization $M_{\mathbf{r}}$ around $\mathbf{r}$ is the thermal average of $S_{\mathbf{r}}$ times $g\mu_B$, that is, $M_{\mathbf{r}} \equiv g\mu_B \langle S_{\mathbf{r}} \rangle$. For simplicity we set $g\mu_B = 1$ in the following. At high temperatures disorder prevails and the spatial correlation between the spins is only of short range. As the temperature is lowered the size of correlated regions of the spins grows and grows, and eventually at the critical temperature spontaneous ordering sets in—that is, $M \neq 0$.

Let H be the field conjugate to the order parameter M. For a ferromagnet this is just an ordinary uniform field, for an antiferromagnet it it is the staggered field corresponding to M being the staggered magnetization. The response to the field is linear at small fields.

$$M = M^{H=0} + \chi H. \quad (1)$$

The correlation function $\langle S_0 S_r \rangle$ is decomposed into two parts describing short-range correlations and long-range order; that is, by definition

$$\langle S_0 S_r \rangle \equiv g(\mathbf{r}) + \langle S \rangle^2 = g(\mathbf{r}) + M^2. \quad (2)$$

From elementary statistical mechanics[3,4,13] one can show that χ and $g(\mathbf{r})$ are related by

$$\chi = \sum_{\mathbf{r}} g(\mathbf{r}). \quad (3)$$

This relation expresses the fact that one obtains a large response to a small field if the system is "cooperative," that is, when $g(\mathbf{r})$ is of long range. It is useful to generalize Eqs.

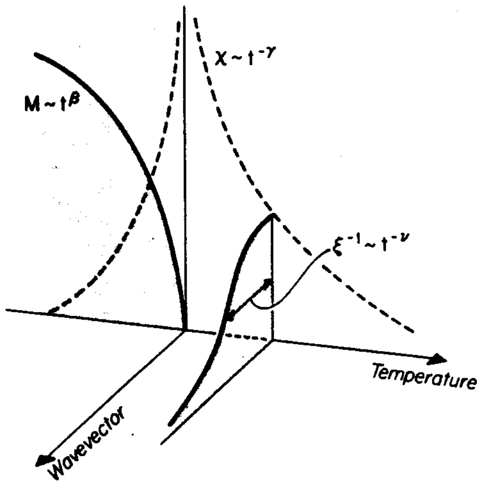

Fig. 1. Order parameter M as well as the wave vector-dependent susceptibility $\chi_{\mathbf{q}}$ are directly determined in a scattering experiment. The width of $\chi_{\mathbf{q}}$ is the inverse correlation range.

(1) and (3) in the following manner[13]: Suppose that the field varies sinusoidally in space with a wave vector $\mathbf{q}$, $H_{\mathbf{r}} = H_{\mathbf{q}} \exp(i\mathbf{q}\cdot\mathbf{r})$. The response $M_{\mathbf{r}}$ will then also vary sinusoidally with the same wave vector, $M_{\mathbf{r}} = M_{\mathbf{q}} \exp(i\mathbf{q}\cdot\mathbf{r})$ with $M_{\mathbf{q}}$ and $H_{\mathbf{q}}$ being related by the wave vector-dependent susceptibility $\chi_{\mathbf{q}}$:

$$M_{\mathbf{q}} = M_{\mathbf{q}}^{H=0} + \chi_{\mathbf{q}} H_{\mathbf{q}}. \qquad (1')$$

It should be noted that by definition the largest response occurs for $q = 0$ and $M_{q\neq 0}^{H=0} = 0$. The generalization of Eq. (3) is then

$$\chi_{\mathbf{q}} = \sum_{\mathbf{r}} g(\mathbf{r}) \exp(i\mathbf{q}\cdot\mathbf{r}). \qquad (3')$$

This relationship is exact for classical spins and in general is a good approximation for quantum-mechanical systems near a phase transition. The peak of $\chi_{\mathbf{q}}$ around $q = 0$ has a half-width, ξ^{-1}, and from Eq. (3′) we identify ξ as the correlation range of the short-range order. Often it is easiest to derive $\chi_{\mathbf{q}}$ theoretically and then to find $g(\mathbf{r})$ by Fourier inversion, but more importantly $\chi_{\mathbf{q}}$ is the quantity which is directly measured in a scattering experiment when the radiation couples to the order parameter. Why? Because the phase difference between the waves scattered from an element around 0 and an element around $\mathbf{r}$ is $\exp(i\mathbf{q}\cdot\mathbf{r})$ with $\mathbf{q} = \mathbf{k}_i - \mathbf{k}_f$, the difference between incident and scattered wave vectors. The wave amplitude from element 0 is proportional to M_0, whereas the wave amplitude from element $\mathbf{r}$ is proportional to $M_{\mathbf{r}}$, so the scattering cross section is simply proportional to $\chi_{\mathbf{q}}$ when $q \neq 0$. At $\mathbf{q} = 0$ the scattering is a superposition of the $\chi_{\mathbf{q}=0}$ and Bragg scattering originating from the $M^{H=0}$ term in Eq. (1); the latter is proportional to M^2. We have summarized this discussion in Fig. 1, showing the wave vector- and temperature-dependent susceptibility diverging as $T \to T_c$, $q \to 0$ and the onset of spontaneous long-range order. The behavior near T_c is often given by power laws as indicated in Fig. 1 with t being the reduced temperature $t = |T - T_c|/T_c$.

III. MEAN FIELD THEORY

We shall now present by far the simplest approximate calculation of $\chi_{\mathbf{q}}$, that is, the mean field theory; we shall show how this relates to the conventional Landau description and we shall then discuss the limitations of the mean field approach using a real space version of the Ginzburg criterion. Let us first assume that the spins of the system do not interact at all. In that case there are no correlations and the response function, denoted χ^0, does not depend on q. In an applied conjugate field H_i^{app} the order parameter at position $\mathbf{r}_i$ would be $M_i = \chi^0 H_i^{\text{app}}$. For simplicity we now consider temperatures above T_c.

We assume, and this is the essential approximation, that the interaction between S_i and its neighbors is equivalent to a molecular field, $M_i = \chi^0\{H_i^{\text{app}} + H_i^{\text{mf}}\}$, and that the molecular field H_i^{mf} is of the form $H_i^{\text{mf}} = \Sigma_j f(\mathbf{r}_{ij})\langle S_j\rangle$. That is, only the *average* interaction of the j neighbors is taken into account; fluctuations are neglected. Let H_i^{app} be sinusoidal in space, that is, $H_i^{\text{app}} = H_{\mathbf{q}} \exp(i\mathbf{q}\cdot\mathbf{r}_i)$, and correspondingly $M_i = M_{\mathbf{q}} \exp(i\mathbf{q}\cdot\mathbf{r}_i)$. Recalling that $\langle S_j\rangle \equiv M_j$, we find above T_c

$$M_{\mathbf{q}} \exp(i\mathbf{q}\cdot\mathbf{r}_i) = \chi^0[H_{\mathbf{q}} \exp(i\mathbf{q}\cdot\mathbf{r}_i) + \sum_j f(\mathbf{r}_{ij}) M_{\mathbf{q}} \exp(i\mathbf{q}\cdot\mathbf{r}_j)] \qquad (4)$$

and thereby

$$\chi_{\mathbf{q}} = \chi^0[1 - \alpha_{\mathbf{q}}\chi^0]^{-1} \qquad (5)$$

with

$$\alpha_{\mathbf{q}} = \sum_{\mathbf{r}} f(\mathbf{r}) \exp(i\mathbf{q}\cdot\mathbf{r}). \qquad (6)$$

For the case of a simple ferromagnet $\alpha_{\mathbf{q}}$ has its maximum for $q = 0$ and expansion around $q = 0$ yields

$$\alpha_{\mathbf{q}} = \alpha_0[1 - a^2(\mathbf{q})], \quad \text{with } a^2(0) = 0. \qquad (7)$$

From Eq. (5) we find the critical point by noting $\chi_{\mathbf{q}=0} \to \infty$ when $\alpha_0\chi^0 \to 1$. For a degenerate ground state, χ_0 varies as $1/T$ and thence one obtains $\alpha_0\chi^0 = T_c/T$; thus Eq. (5) can be rewritten in the form

$$\chi_{\mathbf{q}} \sim [t + a^2(\mathbf{q})]^{-1}, \quad t = (T - T_c)/T_c. \qquad (8)$$

If the interaction function $f(\mathbf{r})$ is isotropic, the long-wavelength form of $a^2(\mathbf{q})$ becomes particularly simple, $a^2(\mathbf{q}) = \xi_0^2 q^2$, and in that case $\chi_{\mathbf{q}}$ has a Lorentzian line shape.

$$\chi_{\mathbf{q}} \sim [\xi^{-2} + q^2]^{-1}, \quad \xi = \xi_0 t^{-1/2}. \qquad (9)$$

From Eqs. (8) and (9) we find the mean field values of the critical exponents: $\gamma = 1$ and $\nu = \frac{1}{2}$. A similar analysis below T_c yields the primed exponents[14] $\gamma' = \gamma = 1$, $\nu' = \nu = \frac{1}{2}$.

IV. LANDAU EXPANSION

So far we have only discussed the susceptibility. We could equally well have considered the free energy using the molecular field *Ansatz*. This assumption basically reduces the many-body problem to a single-site problem in which a magnetic moment m at any site can be either parallel or antiparallel to the total magnetic field H^{tot} composed of the applied field and the molecular field. The single-site partition function is then $\cosh(mH^{\text{tot}}/kT)$, and one can immediately write down the entropy S, the internal energy U, and thereby the generalized Gibbs free energy[5] $G(\tilde{M},T,H) = U - TS - g\mu_B\tilde{M}H$. Here $\tilde{M}$ denotes the nonequilibrium magnetization.[5] The equilibrium magnetization M is given by the condition $0 = (\partial G/\partial \tilde{M})_{\tilde{M}=M}$. The magnetization is small near T_c, so that if one expands $G(\tilde{M},T,H)$ in powers of $\tilde{M}$, one finds for the dimensionless quantity $g(\tilde{M},t,h) = G(\tilde{M},T,H)/kT$ in terms of the dimensionless arguments $\tilde{M}$, $t = (T - t_c)/T_c$, and $h = mH/kT$:

$$g(\tilde{M},t,h,) = g_0(t) - h\tilde{M} + at\tilde{M}^2/2 + b\tilde{M}^4/4 + c\tilde{M}^6/6 + \cdots \quad (10)$$

with $a = 1$, $b = \frac{1}{2}$, etc.

The equation of state, $(\partial g/\partial \tilde{M})_{\tilde{M}=M} = 0$, is explicitly $h = atM + bM^3$. The magnetization in zero field, M_0, is given by $M_0\,(at + bM_0^2) = 0$. Above T_c the solution is $M_0 = 0$, whereas below T_c, $M_0^2 = (a/b)t$, that is, the order parameter critical exponent $\beta = \frac{1}{2}$. The susceptibility is found by differentiating the equation of state, $\chi^{-1} = (\partial h/\partial M)_{h=0} = at + 3bM_0^2$. Thus above T_c $\chi^{-1} = at$, whereas below T_c, $\chi^{-1} = -2at$; that is, $\gamma = \gamma' = 1$, as expected.

The expansion of Eq. (10) is known as the Landau expansion. In this we have not included spatial fluctuations; to lowest order this is done by adding a term $D(\nabla\tilde{M})^2$.[10] With this term included one can derive the Lorentzian wave vector-dependent susceptibility of Eq. (9). Equation (10) does not involve odd powers of $\tilde{M}$ since g has to be invariant under time reversal. In constructing the Landau expansion for a general physical system, one must use symmetry arguments to decide whether or not odd powers of the order parameter are allowed. A cubic term would, for instance, imply a first-order transition.

V. GINZBURG CRITERION

It is clear that the essential approximation above is the neglect of fluctuations in the molecular field acting on a given site. On the other hand, the fluctuations become more and more pronounced as the temperature approaches T_c. In order to obtain a self-consistent picture, Ginzburg stated that *below* T_c [9] the fluctuations of M averaged over a suitable region Ω (to be specified below) must be small compared to the value of M itself, that is,

$$(\delta M)_\Omega^2 \ll M_\Omega^2.$$

Let us assume that we have divided Ω into N identical lattice cells. One then has for the mean square fluctuation amplitude

$$(\delta M)_\Omega^2 = \left\langle \left[\sum_\Omega (S_i - \langle S\rangle)\right]^2 \right\rangle = \left\langle \sum_{\substack{i\in\Omega\\ j\in\Omega}} (S_i - \langle S\rangle)(S_j - \langle S\rangle) \right\rangle = N\sum_\Omega [\langle S_0S_i\rangle - \langle S\rangle^2].$$

If the sum had been over the entire crystal, then by Eq. (3) it would simply be equal to $\chi_{\mathbf{q}=0}$.

In order to assess the importance of fluctuations in the Landau theory, the region Ω must be chosen appropriately. On physical grounds it is clear that fluctuations are important over linear dimensions of the order of ξ, the correlation range. Indeed, it is an essential feature of our present understanding of phase transitions that ξ is the only length in the problem. Therefore, we take $\Omega = \Omega_\xi$, the region of correlated spins.[10] As $T \to T_c$, $\Omega_\xi \to \infty$, but the sum becomes a constant fraction of $\chi_{\mathbf{q}=0}$, that is,

$$(\delta M^2)_{\Omega_\xi} = FN(\Omega_\xi)\chi_{\mathbf{q}=0}(t), \quad (11)$$

with F independent of ξ or temperature. This may be easily justified within the context of Landau theory. If $\chi_\mathbf{q} = A/(\kappa^2 + q^2)$ with $\kappa = \xi^{-1}$ then by Fourier inversion in d dimensions one obtains $g(\kappa,r) = \kappa^{d-2}v(\kappa r)$; that is, $g(\kappa,r)$ is a homogeneous function of κ and r. More generally, this is simply the scaling *Ansatz*. Thus

$$\sum_{\Omega_\xi} [\langle S_0S_i\rangle - \langle S\rangle^2] = c\int_0^\xi dr\, r^{d-1}\kappa^{d-2}v(\kappa r) = c\kappa^{-2} \times \int_0^\xi d(\kappa r)\,(\kappa r)^{d-1}v(\kappa r) = c\kappa^{-2}\int_0^1 dx\, x^{d-1}v(x).$$

Hence F of Eq. (11) is given by

$$F = \left(\int_0^1 dx\, x^{d-1}v(x)\right)\left(\int_0^\infty dx\, x^{d-1}v(x)\right)^{-1},$$

independent of ξ.

For the mean square order parameter one has

$$M_{\Omega_\xi}{}^2 = \left\langle \sum_{\Omega_\xi} S_i \right\rangle^2 = N^2\langle S\rangle^2 = N^2M^2.$$

Thus the Ginzburg criterion may be rewritten as

$$F\chi(t) \ll N(\Omega_\xi)M^2(t). \quad (12)$$

The number of spins $N(\Omega_\xi)$ within a correlated region Ω_ξ is, of course, proportional to the volume of Ω_ξ.

Quite generally, we write $\Omega_\xi = \xi^{d+m}$, where d is the dimensionality of the lattice and, for the cases considered here, $m \geq 0$. As we shall discuss below, for spatially isotropic systems $m = 0$, but in a variety of important physical cases, in fact, one has $m > 0$. As we noted in Sec. I, the correlation range diverges as $t^{-\nu'}$, t being the reduced temperature $|T - T_c|/T_c$, so that $\xi = \xi_0 t^{-\nu'}$. Thus the real space Ginzburg criterion may be rewritten as

$$(\text{const})t^{-\gamma'} \ll \xi_0{}^{d+m}\, t^{-(d+m)\nu'+2\beta}$$

with the constant being of the order of unity.[14] In order for this relation to be fulfilled for arbitrarily small t, it is required that $\gamma' - (d+m)\nu' + 2\beta < 0$. Thus we arrive at a marginal dimensionality

$$d^* = (\gamma' + 2\beta)/\nu' - m. \quad (13)$$

When $d > d^*$, the mean field theory gives a self-consistent picture of the phase transition at least insofar as the critical exponents are concerned. However, when $d < d^*$, the Landau theory will break down at some distance from the critical point. This distance *may* be very small if the basic interaction range, ξ_0, is very large, as is the case for superconductors, and it is in this context that the Ginzburg criterion is most well known.

Finally, when $d = d^*$, Landau theory almost works. In this case renormalization group theory gives logarithmic corrections to the mean field behavior.

VI. CASES

Quite generally, it is convenient to subdivide the various possibilities into cases where $m = 0$ and where $m > 0$.

A. $m = 0, \Omega_\xi = \xi^d$

1. Short-range interactions

If the interaction $f(\mathbf{r})$ is of short range, for example, between nearest neighbors only, the Fourier transform of $f(\mathbf{r})$ will, in the long-wavelength limit, be of the form $\alpha_\mathbf{q} \simeq \alpha_0(1 - \xi_0^2 q^{*2})$, where q^{*2} is of the form $q^{*2} \equiv \Sigma_{i=1}^d p_i q_i^2$. The region of critical fluctuations in q space therefore scales as ξ^{-1} in all d directions, and by Fourier inversion it follows that $\Omega_\xi = \xi^d$.

We have already found that the mean field critical exponents in this case are $\beta = \frac{1}{2}, \gamma' = 1, \nu' = \frac{1}{2}$, and Eq. (13) gives $d^* = 4$. Consequently, the mean field theory is not self-consistent for, say, the three-dimensional Ising model, and indeed one finds experimentally (β-brass[15]) as well as theoretically, using series expansion techniques,[16] that $\gamma = 1.25$, $\nu = 0.64$, and $\beta = 0.30$. When d is even further away from d^*, for example, $d = 2$, we would expect even larger deviations from the mean field exponents. Indeed, experiments on the $2d$ antiferromagnet K_2CoF_4,[17] which is a model system of the $2d$ Ising model, as well as Onsager's exact solution of the $2d$ Ising model,[3] give exponents very far from mean field behavior: $\gamma = \frac{7}{4}, \nu = 1, \beta = \frac{1}{8}$.

2. Tricritical points

In certain physical systems, of which ^3He–^4He mixtures and the metamagnet $FeCl_2$ in a magnetic field are perhaps the best known examples, one has a line of second-order transitions which terminates at a tricritical point.[18] The mean field theory for such systems is rather more complicated since one must consider two order parameters, the so-called ordering and nonordering densities and their conjugate fields. To be explicit, in the antiferromagnet $FeCl_2$ the order parameter is the sublattice magnetization M_S and its conjugate field is a staggered field H_S. The nonordering density is simply the magnetization M with a nonordering conjugate field, the applied field H. One may then write down a Landau expansion in M_S identical to Eq. (10). As a function of applied field H, $b[H,T_c(H)]$ decreases continuously until at some point $b[H = H_t, T_c(H_t) = T_t] = 0$. For $b < 0$ the phase transition becomes first order. The point $b(H_t, T_t) = 0$ then is the tricritical point. It is immediately apparent that because of this vanishing of the fourth-order term in the expansion the Landau theory will yield $\beta_t = \frac{1}{4}$ as opposed to $\beta = \frac{1}{2}$ for a conventional second-order transition. One finds in addition $\gamma_t' = 1$, $\nu_t' = \frac{1}{2}$. The critical fluctuations are again isotropic in space, so that $m = 0$. Thus for a tricritical point $d^* = (1 + \frac{1}{2})/\frac{1}{2} = 3$. Hence, in conventional three-dimensional systems one expects Landau exponents with logarithmic correction terms.

Experimentally, the phase diagrams of ^3He–^4He mixtures[19,20] and of the metamagnet $FeCl_2$[21-23] are well described by Landau theory with marginal dimensionality corrections. The critical exponents, albeit only roughly determined because of the difficulty of the experiments, all seem to be in agreement with the Landau theory. For $FeCl_2$ and certain other systems β_t seems to be slightly less than $\frac{1}{4}$, perhaps due to the presence of logarithmic corrections.

3. Percolation

Percolation has been of interest for many decades although it is only recently that it has been cast in a lattice-gas phase transition language. We do not want to enter into an extensive discussion of percolation here.[24] Instead, we limit our description to the following. Consider a hypercubic (i.e., square in $2d$, cubic in $3d$, etc.) lattice with nearest neighbor bonds only. If we now remove atoms continuously, then below some critical concentration $p = p_c$ the system will break up into finite clusters. This transition at p_c is closely related to a thermodynamic transition with the analogies: for $p > p_c$, order parameter → percentage of atoms in the infinite cluster $\sim(p - p_c)^\beta$; susceptibility → mean square finite cluster size $\sim(p - p_c)^{-\gamma'}$; correlation length → $(1/e)$ × (decay length) for the probability that two atoms belong to the some cluster $\sim(p - p_c)^{-\nu'}$. In this case the mean field result is not that given by a simple Landau expansion [Eq. (10)] but rather is the solution of the percolation problem on the Bethe lattice.

Fisher and Essam showed that for a Bethe lattice,[25] which is essentially an infinite dimensional system, $\beta = 1$, $\gamma' = 1$, $\nu' = \frac{1}{2}$. The Ginzburg criterion with $m = 0$ still applies to this problem. Thus we have for percolation $d^* = (1 + 2)/\frac{1}{2} = 6$. This has been demonstrated "experimentally" by Kirkpatrick in a very beautiful d-dimensional computer experiment.[26]

B. $m \neq 0, \Omega_\xi = \xi^{d+m}$

We shall now give examples of physical systems where the linear dimensions of the correlated region of spins may scale as ξ^2 rather than ξ in one or more directions as T approaches T_c.

1. Dipolar-coupled, uniaxial ferromagnet or ferroelectric

Let us consider an Ising ferromagnet (ferroelectric) where the magnetic (electric) dipole moments are only coupled by the dipolar interaction, that is, the interaction between two spins pointing in the z direction and situated at the origin and at $\mathbf{r} = (x,y,z)$ is $f(\mathbf{r}) = (3z^2 - r^2)/r^5$. In the long-wavelength limit the Fourier transform of $f(\mathbf{r})$ is of the form

$$\alpha_\mathbf{q} = \alpha_0[1 - a_1 q^2 - a_2(q_z/q)^2 + a_3 q_z^2]. \quad (14)$$

This form of $\alpha_\mathbf{q}$ is very peculiar since the limiting value of $\alpha_\mathbf{q}$ when $\mathbf{q} \to 0$ depends on the *direction* of $\mathbf{q}$: if $\mathbf{q} \to 0$ along the x axis $\alpha_\mathbf{q} \to \alpha_0$, but if $\mathbf{q} \to 0$ along the z axis the limiting value is $\alpha_0(1 - a_2)$. Any limiting value between these two extremes is obtained by choosing the appropriate direction of $\mathbf{q}$.

As $\chi_\mathbf{q}$ is of the form

$$\chi_\mathbf{q}^{-1} \sim 1 + [(\xi q)^2 + g(q_z\xi^2/q\xi)^2 - (b_3/\xi)^2(q_z\xi^2)^2], \quad (15)$$

it is evident that q_z scales as ξ^{-2}, whereas modulus q scales as ξ^{-1}. This is most clearly demonstrated pictorially in Fig. 2. Here the half-contour of $\chi_\mathbf{q}$, defined by the surface in q space where $\chi_\mathbf{q} = (\frac{1}{2})\chi_{q_x\to 0, q_y = q_z = 0}$, is sketched in the q_x-q_z plane. The entire surface is generated by rotation of this contour around the z axis. The contour intersects the q_x axis at ξ^{-1}, and if we omit the term b_3^2/ξ^2, which becomes negligible near T_c, the contour starts out from the origin with a slope of $g^{-1/2}\xi^{-1}$. The maximum dimension along the q_z axis is $g^{-1/2}\xi^{-2}$, so the contour is therefore not only

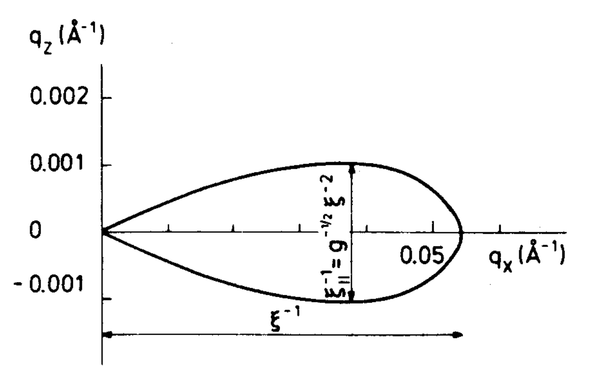

Fig. 2. Half-contour of $\chi(\mathbf{q},T)$ for the dipolar-coupled, uniaxial ferromagnet. The contour gives the points in the q_x-q_z plane for which $\chi(q_x,0,q_z,T) = (\frac{1}{2}) \lim_{q_x\to 0} \chi(q_x,0,0,T)$. Note the scale difference on the q_z and q_x axes.

shrinking as $T \to T_c$, due to the fact that $\xi \to \infty$, but it is also changing shape in becoming more and more confined to the x-y plane. We infer immediately that the correlation range along the z axis, $\xi_\|$, is superdiverging, that is $\xi_\| \sim \xi^2$. The correlated regions are long rods along the Ising axis of length $\xi_\|$ and with a diameter of ξ. We thence have $\Omega_\xi = \xi\cdot\xi\cdot\xi_\| \sim \xi^4$, so that $m = 1$.[27] The mean field critical exponents are $\gamma' = 1$, $\nu' = \frac{1}{2}$, $\beta = \frac{1}{2}$; thus the marginal dimensionality from Eq. (13) is $d^* = 3$. The three-dimensional ferromagnet or ferroelectric exhibits critical behavior of marginal dimensionality, and we expect "almost" mean field behavior. It is possible to calculate the exact critical behavior in this case using renormalization group theory, essentially because RG uses mean field theory as the "basis," and for marginal dimensionality this "basis" is already quite close to the correct solution. The results are[11]

$$\chi_{\mathbf{q}\to 0} \sim \xi^2 \sim |t|^{-1} |\ln t|^{1/3}, \tag{16}$$

$$M \sim t^{1/2} |\ln t|^{1/3}. \tag{17}$$

Furthermore, the specific heat C should vary as

$$C \sim |\ln t|^{1/3}, \tag{18}$$

and the renormalization group equations imply the relation[8]

$$\xi^2 \xi_\| C t^2 / k_B = (3/32\pi) |\ln t|. \tag{19}$$

Experiments on a model system of the uniaxial, dipolar-coupled ferromagnet, that is $LiTbF_4$,[28] have indeed confirmed the logarithmic corrections for the specific heat[29] (where the logarithmic singularity is the *leading* singularity), and neutron scattering experiments clearly showed the superdiverging longitudinal correlation range.[27] These experiments also confirmed the nontrivial relation of Eq. (19) to within an accuracy of about 2%.

2. *Structural phase transitions*

In this example we consider a structural phase transition driven by the softening of an acoustic phonon. It is probably easiest to visualize the distortion of a cubic unit cell. Elementary theory of elasticity[2] yields that the number of ways a cubic lattice can distort is quite limited: one can have a uniform expansion or contraction of the unit cell (the A_1 mode), an expansion along one cube axis and an equivalent contraction along another (the E mode), or a shear of the unit cell (the T_2 mode). The latter two modes are associated with softening of the elastic constants $c_{11} - c_{12}$ and c_{44}, respectively. For the T_2 mode only transverse acoustic phonons with polarization along a cube edge and wave vectors perpendicular to that direction will lead to the static deformation as $q \to 0$. This two-dimensional softening leads to a fluctuation surface in q space geometrically identical to that of the dipolar-coupled, uniaxial ferromagnet discussed above.[12] In this case, therefore, we also have $d^* = 3$. For the E mode with $c_{11} - c_{12} \to 0$ the appropriate elastic waves propagate along (1,1,0) with polarization along $(1,\bar{1},0)$, so in this case the fluctuations in q space are confined to a line [along (1,1,0)] rather than to a plane. As discussed by Cowley,[12] the corresponding response function is of the form $\{1 + \xi^2[q^2 + A(q_z/q)^2 + B(q_\perp/q)^2]\}^{-1}$, and the correlated regions in direct space are therefore sheets with the normal along (1,1,0), with a thickness of ξ, and with planar dimensions of $\xi^2 \times \xi^2$. We therefore find $\Omega_\xi = \xi^5$ and, consequently, the marginal dimensionality $d^* = 2$. Thus Landau theory gives a self-consistent solution for $d = 3$ in this case. It should be noted, however, that the transition of a cubic crystal generally is of first order because odd powers of the order parameter are allowed by symmetry in the Landau expansion of the free energy, Eq. (10). Our discussion is therefore only strictly applicable if the coefficient of the cubic term accidentally is zero (or very small), and we have only chosen the cubic case because it allows a very simple discussion without presuming knowledge of group theory. It turns out that the transition in Nb_3Sn is almost second order, and the results of a neutron scattering study of the cubic to tetragonal phase transition in this material[30] indeed is consistent with the picture presented above.

Cowley has classified the phase transitions driven by the softening of acoustic phonons in general,[12] and it turns out that there are several cases where the cubic terms are forbidden by symmetry so that a truly second-order transition may occur. As an example, we mention the orthorhombic–monoclinic phase transition at 151 K in $PrAlO_3$, which has been studied in great detail and which indeed displays Landau behavior.[31] For similar reasons random phase approximation-based theories work very well for the dynamics of such systems.[31,32]

VII. CONCLUSION

In the traditional exposition of Landau theory one argues that fluctuations are only accounted for in an average sense, but these fluctuations diverge at the critical point and the Landau theory can therefore not be expected to be valid near the critical point. A more sensible assessment of the role of the fluctuations can be obtained by using the Ginzburg criterion for self-consistency. In this way the concept of marginal dimensionality is introduced in a natural fashion. In the examples, we have emphasized the geometry of the critical fluctuations in reciprocal space and in direct space. In order to facilitate visualization, we show in Fig. 3 the salient features of the topology of the region where the critical fluctuations take place. For a two-dimensional magnet the correlated region varies as ξ^2, the Ginzburg criterion is very far from being fulfilled, and in reciprocal space the fluctuations occur within an infinitely long rod perpendicular to the two-dimensional planes, as indicated in part (a) of Fig. 3. For a conventional short-

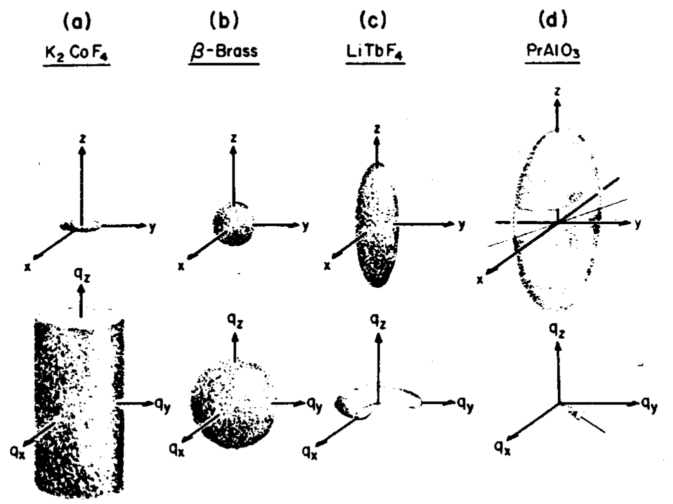

Fig. 3. Regions of critical fluctuations in direct space (top) and in reciprocal space (bottom) for the [2d] Ising model (a), the [3d] Ising model (b), the dipolar-coupled, uniaxial ferromagnet (c), and for a structural phase transition driven by the softening of an acoustic transverse phonon (d). The dimensions of the regions in direct space enter in the Ginzburg criterion.

range interaction, three-dimensional system the fluctuations in reciprocal space take place within a sphere of radius ξ^{-1}, and the correlated region in direct space is a sphere with radius ξ, part (b) of Fig. 3. The Ginzburg criterion is still not fulfilled. For the uniaxial, dipolar-coupled ferromagnet in three dimensions the fluctuation region in q space is a peculiar disk, so the correlated region in **r** space takes the shape of long rods of length ξ^2 and diameter ξ, part (c) of Fig. 3. This represents the border line for the validity of the Ginzburg criterion.

The critical fluctuations are also restricted in q space for structural phase transitions driven by the softening of an acoustic phonon (elastic wave), the restriction being brought about because only those lattice vibrations which in the long-wavelength limit approach the static deformation go soft and contribute to the critical divergence. For certain structural phase transitions the restricted region in q space is identical to that of the uniaxial dipolar-coupled ferromagnet, part (c); for others the critical region in **q** space condenses around a line, part (d). In the latter case the correlated region in **r** space forms sheets of thickness ξ and of planar dimensions $\xi^2 \times \xi^2$. In that case the Ginzburg criterion *is* fulfilled in three dimensions and the phase transition is self-consistently described by mean field theory.

In order to illustrate the actual difference in critical behavior for the four examples of Fig. 3, we show in Fig. 4 experimental results for the *squared* order parameter for four corresponding real physical systems. The order parameter is in all four cases normalized to unity at low temperatures, but in order to emphasize the temperature region of critical fluctuations we have only shown the results in the temperature region $0.9 < T/T_c < 1$.

In summary, then, we see that by a simple application of the Ginzburg criterion it is possible to assess the importance of fluctuations and thence to elucidate the basic physics of the phase transitions in a surprisingly large range of real physical systems. In partucular, mean field theory works very well in homogeneous structural phase transitions simply because the fluctuations are confined in phase space.

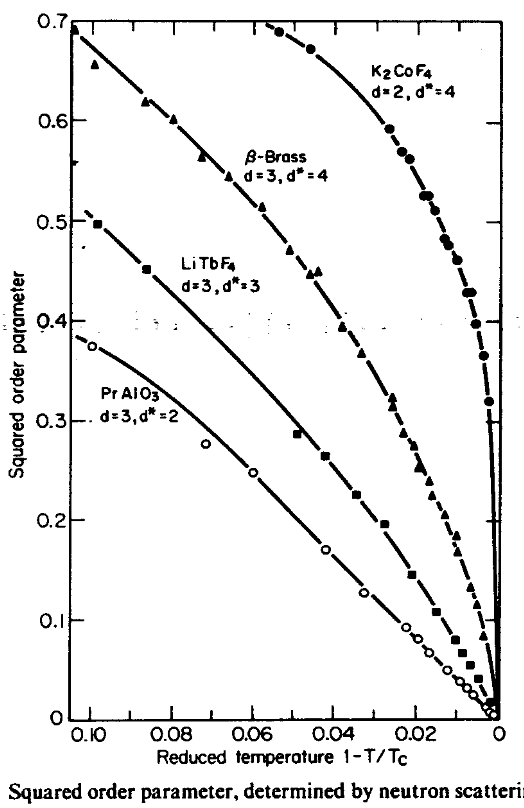

Fig. 4. Squared order parameter, determined by neutron scattering, for the four examples shown in Fig. 3. When the marginal dimensionality $d^* \leq d$, we see Landau behavior (with logarithmic corrections when $d^* = d$); for $d^* > d$, clear deviations from Landau theory are observed.

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†On leave from Research Establishment Risø, Roskilde, DK4000, Denmark.

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GLORIOUS CONTEMPLATION

... I had plenty of time at school to think about all manner of interesting things. It was utterly necessary, however, to avoid the heinous crime of "not paying attention." In my day, you inevitably suffered physical punishment for not paying attention. Stupidly doing so, by staring out of a window, at some interesting cloud formation, for instance, would inevitably earn for you a stinging series of blows about the head. To a little chap like me, such blows were not to be thought of, so I avoided staring at interesting things in the wide world outside the windows. Even this precaution was not sufficient however, because teachers would sometimes ask you to repeat what they had just said. So long as you were able to do so, it was accepted that indeed you were really "paying attention." So I learned, always at all times, to let the last few words the teacher was saying to register in my brain. Each new sentence wiped out the preceding one, thereby permitting instant recall of the new sentence. This invention won for me hour upon hour of glorious contemplation.

–Fred Hoyle, *Ten Faces of the Universe* (Freeman, San Francisco, 1977).