5.6 Problems for Chapter 5

5.1 A random quantity has an exponential autocorrelation function $G(t) = G(0)\, e^{-\gamma t}$. What are the dimensions of γ? Calculate the correlation time of $G(t)$ using the usual definition.

Calculate the correlation time for a gaussian autocorrelation function $G(t) = G(0)e^{-\gamma^2 t^2/2}$

5.2 Show that the autocorrelation function of a periodically varying quantity $m = m \cos \omega t$ is given by

$$G = \frac{m^2}{2} \cos \omega t.$$

Show that the autocorrelation function is independent of the *phase* of $m(t)$. In other words, show that if $m = m \cos(\omega t + \varphi)$, then $G(t)$ is independent of φ.

5.3 The mean square displacement of a Brownian particle at long times was given by Eq. (5.2.12)

$$\langle x^2 \rangle = 2t \int_0^\infty G_v(\tau) \mathrm{d}\tau - 2 \int_0^\infty \tau G_v(\tau) \mathrm{d}\tau \,.$$

It was stated in the text that the second term was negligible, compared with the first, and so could be ignored.

(a) In that case, show that the mean square displacement may be expressed

$$\langle x^2 \rangle = 2G_v(0)\tau_v\, t \,,$$

where τ_v is the correlation time associated with $G_v(\tau)$.

(b) Show that the second integral above may be expressed, approximately, as

$$\int_0^\infty \tau G_v(\tau) \mathrm{d}\tau \approx G_v(0)\tau_v^2 \,.$$

There is a choice of ways for demonstrating this. You might approximate $G_v(\tau)$ by a decaying exponential $G_v(0)e^{-\tau/\tau_v}$ before doing the integral.

(c) Using this approximate expression, show that without neglecting the second term, the expression for $\langle x^2 \rangle$ becomes

$$\langle x^2 \rangle = 2G_v(0)\tau_v\, (t - \tau_v) \,.$$

Hence justify the neglect of the second term.

5.4 A small mirror is suspended from a quartz fibre whose torsion constant is κ. When the mirror is rotated an angle θ the torque exerted by the fibre is $\Gamma = -\kappa\theta$. The moment of inertia of the mirror about the suspension axis is I. The mirror reflects a beam of light so that the angular fluctuations caused by the impact of surrounding molecules can be read on a suitable scale. The position of the equilibrium is $\langle\theta\rangle = 0$. The average value $\langle\theta^2\rangle$ is observed. From this the goal is to find the Avogadro constant (or, what is the same thing since the gas constant R is known, to determine the Boltzmann constant).

The following are the data: At $T = 287\,\mathrm{K}$, for a fibre with $\kappa = 9.43 \times 10^{-16}\,\mathrm{Nm}$ it was found that $\langle\theta^2\rangle = 4.20 \times 10^{-6}$.

Calculate the Avogadro constant.

5.5 The assembly of the previous question is placed in a chamber from which the air may be evacuated. Can the amplitude of these fluctuations be reduced by reducing air density? Describe the change in the behaviour of the fluctuations as the air pressure is lowered. In particular, discuss the behaviour as the pressure goes to zero.

5.6 In Section 5.3.9 we considered an electrical analogue of the Langevin Equation based on a circuit comprising an inductor and a resistor in series. In this problem we shall examine a different analogue: a circuit of a capacitor and a resistor in parallel. Show that the equation analogous to the Langevin equation, in this case, is

$$C\frac{\mathrm{d}V}{\mathrm{d}t} + \frac{1}{R}V = I.$$

Hence show that the fluctuation-dissipation result relates the resistance to the current fluctuations through

$$\frac{1}{R} = \frac{1}{2kT}\int_{-\infty}^{\infty} \langle I\,(0)\,I\rangle\,\mathrm{d}t.$$

5.7 The dynamical response function $X(t)$ must vanish at zero times, as shown in Fig. 5.13. What is the physical explanation of this? What is the consequence for the step response function $\Phi(t)$? Is this compatible with an exponentially decaying $\Phi(t)$?

5.8 We saw in Section 5.4.2 that $\chi'(\omega)$ and $\chi''(\omega)$ may be regarded as the cosine and sine transforms of the dynamical susceptibility $X(t)$. Now these real Fourier transforms may be inverted; $X(t)$ may be found equivalently from either $\chi'(\omega)$ or $\chi''(\omega)$. The point about this is that $\chi'(\omega)$ and $\chi''(\omega)$ are not independent; they both come (invertibly) from $X(t)$. So, in particular, $\chi'(\omega)$ may be found from $\chi''(\omega)$ or *vice versa* (the Kramers-Kronig relations).

(a) In this way derive the following expressions

$$\chi'(\omega) = -\frac{2}{\pi}\int_0^\infty \frac{\omega'\chi''(\omega')}{\omega'^2-\omega^2}\,\mathrm{d}\omega'$$

$$\chi''(\omega) = \frac{2}{\pi}\int_0^\infty \frac{\omega\chi'(\omega')}{\omega'^2-\omega^2}\,\mathrm{d}\omega'.$$

(b) These are not *quite* the same as those in Eq. (5.4.61). Why is this?

(c) Where, exactly, does the requirement of causality enter into this derivation?

5.9 Show that for the Debye susceptibility, the relation

$$\chi_0 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\chi''(\omega)}{\omega}\mathrm{d}\omega$$

holds. Demonstrate that χ'' vanishes sufficiently fast at $\omega = 0$ so there is no pole in the integral and there is thus no need to take the principal part of the integral in the Kramers-Kronig relations.

5.10 In Section 5.4.11 we examined the form of the dynamical susceptibility $\chi(\omega)$ that followed from the assumption that the step response function $\Phi(t)$ decayed exponentially. In this question consider a step response function that decays with a gaussian profile: $\Phi = \chi_0 e^{-t^2/2\tau^2}$. Evaluate the real and imaginary parts of the dynamical susceptibility and plot them as a function of frequency. The real part of the susceptibility is difficult to evaluate without a symbolic mathematics system such as *Mathematica*. Compare and discuss the differences and similarities between this susceptibility and that deduced from the exponential step response function (Debye susceptibility).

5.11 The Debye form for the dynamical susceptibility, Eq. (5.4.84), is

$$\chi'(\omega) = \chi_0 \frac{1}{1+\omega^2\tau^2}$$
$$\chi''(\omega) = \chi_0 \frac{\omega\tau}{1+\omega^2\tau^2}$$

Plot the imaginary part against the real part and show that the figure corresponds to a semicircle. This pictorial representation is known as a Cole-Cole plot.

5.12 Plot the Cole-Cole plot (Problem 5.11) for the dynamical susceptibility considered in Problem 5.10. How does it differ from that of the Debye susceptibility?

5.13 The full quantum-mechanical calculation of the Johnson noise of a resistor gives

$$\left\langle v^2 \right\rangle_{\Delta f} = 4R \frac{hf}{e^{hf/kT} - 1} \Delta f.$$

Show that this reduces to the classical Nyquist expression at low frequencies. At what frequency will there start to be serious deviations from the Nyquist value? Estimate the value of this frequency.