

## 2.11 Problems for Chapter 2

2.1 In Section 2.1 we saw that the density of free-particle states for a three dimensional volume  $V$  was shown to be

$$g(\varepsilon) = \frac{1}{4} \frac{V}{\pi^2 \hbar^3} (2m)^{3/2} \varepsilon^{1/2};$$

this followed from counting the number of states in the octant of radius

$$R = \sqrt{n_x^2 + n_y^2 + n_z^2}.$$

(a) By similar arguments show that in two dimensions, by counting the number of states in the quadrant of radius

$$R = \sqrt{n_x^2 + n_y^2},$$

the density of states is given by

$$g(\varepsilon) = \frac{mA}{2\pi\hbar^2}$$

where  $A$  is the area. Note in two dimensions the density of states is independent of energy.

(b) And similarly, show that in one dimension the density of states is

$$g(\varepsilon) = \frac{L}{\pi\hbar} \left(\frac{m}{2}\right)^{1/2} \varepsilon^{-1/2}.$$

2.2 Show, using arguments similar to those in Section 2.1.3, that the energy levels of an ultra-relativistic or a massless particle with energy-momentum relation  $E = cp$  are given by

$$\varepsilon = \frac{c\pi\hbar}{V^{1/3}} (n_x^2 + n_y^2 + n_z^2)^{1/2}.$$

Hence show that the pressure of a gas of such particles is one third of the (internal) energy density.

2.3 In Sections 2.3.1 and 2.3.2 the ideal gas partition function was calculated quantum-mechanically and classically. Although the calculations were quite different, they both resulted in (different) gaussian integrals. By writing the gaussian integral of the classical case as

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz e^{-(x^2+y^2+z^2)}$$

and transforming to spherical polar coordinates, you can perform the integration over  $\theta$  and  $\phi$  trivially. Show that the remaining integral can be reduced to that of the quantum case.

2.4 The Sackur-Tetrode equation, discussed in Section 2.3.3, and written as

$$S = Nk \ln V - Nk \ln N + \frac{3}{2}Nk \ln T + Nks_0$$

is often interpreted as indicating different contributions to the entropy: the volume contribution is in the first term, the number contribution in the second term and the temperature contribution in the third term. Show that such an identification is fallacious, by demonstrating that the various contributions depend on the choice of units adopted – even though the total sum is independent. Discuss the origin of the fallacy.

2.5 The Gibbs-Duhem relation tells us that the intensive variables  $T, p, \mu$  are not independent so that, for example,  $\mu$  depends on  $T$  and  $p$ . Demonstrate this by showing that

$$\mu = -kT \ln \left[ \left( \frac{m}{2\pi\hbar^2} \right)^{3/2} \frac{(kT)^{5/2}}{p} \right]$$

for an ideal gas.

2.6 Show that the Fermi energy for a two-dimensional gas of Fermions is

$$\varepsilon_F = \frac{2\pi\hbar^2 N}{\alpha m A}$$

where  $A$  is the area of the system.

2.7 Show that the chemical potential of a two-dimensional gas of fermions may be expressed analytically as

$$\mu = kT \ln \{ e^{\varepsilon_F/kT} - 1 \}.$$

2.8 Calculate the low temperature chemical potential of a two-dimensional gas of fermions by the Sommerfeld expansion method of Section 2.5.3. Observe that the temperature series expansion terminates. Compare this result with the exact result of the previous question. Discuss the difference between the two results.

- 2.9 Obtain expressions for the chemical potential  $\mu$ , the internal energy  $E$  and the heat capacity  $C_V$  for a system of fermions with general density of states  $g(\varepsilon)$ . That is, show that these are given in terms of the behaviour of the density of states at the Fermi surface.
- 2.10 Evaluate the Fermi temperature for liquid  $^3\text{He}$ , assuming it to be a Fermi “gas”. Its molar volume is  $36\text{ cm}^3$ . Calculate the thermal deBroglie wavelength at  $T = T_F$  and show that it is comparable with the interparticle spacing as expected.
- 2.11 In Problem 2.1 we found the expression for the energy density of states  $g(\varepsilon)$  for a gas of fermions confined to two dimensions and we saw that it was independent of energy. What surface density of electrons is necessary in order that  $T_F = 100\text{ mK}$ ? Show that, for a given area, the heat capacity is independent of the number of electrons.
- 2.12 Use the Sommerfeld expansion method of Section 2.5.3 to show that the Fermi-Dirac distribution function may be approximated, at low temperatures, by

$$\frac{1}{e^{(\varepsilon-\mu)/kT} + 1} \sim \Theta(\mu - \varepsilon) - \frac{\pi^2}{6}(kT)^2 \delta'(\varepsilon - \mu) + \dots$$

where  $\Theta$  is the unit step function and  $\delta'$  is the first derivative of the Dirac delta function.

Can you write down the general term of the series?

- 2.13 Liquid  $^4\text{He}$  has a molar volume of  $27\text{ cm}^3$  at saturated vapour pressure. Treating the liquid as an ideal gas of bosons, find the temperature at which Bose-Einstein condensation will occur. How will this temperature change as the pressure on the fluid is increased?

The superfluid transition temperature of liquid helium decreases with increasing pressure; very approximately  $\partial T_c/\partial p \sim -0.015\text{ K bar}^{-1}$ . How does this compare with the behaviour predicted for ideal gas Bose-Einstein condensation?

- 2.14 Consider the Bose gas at low temperatures. You saw in Section 2.6.3 and 2.6.4 that when the occupation of the ground state is appreciable then the chemical potential  $\mu$  is very small and it may be ignored, compared with  $\varepsilon$  in the integral for the number of excited states.

- (a) Show that when the ground state occupation  $N_0$  is appreciable then  $\mu$  may be approximated by

$$\mu \sim -kT/N_0.$$

- (b) Now consider the requirement that  $\mu$  may be neglected in comparison with  $\varepsilon$  in the integral for the number of excited states. This will be satisfied if  $|\mu|$  is much less than the energy  $\varepsilon_1$  of the first excited state. The expression for  $\varepsilon_1$  is

$$\varepsilon_1 = \frac{\pi^2 \hbar^2}{2mV^{2/3}}.$$

Where does this expression come from?

- (c) Show that the condition  $|\mu| \ll \varepsilon_1$  is satisfied right up to  $T_B$  in the thermodynamic limit. Hint: demonstrate that  $|\mu|/\varepsilon_1 \sim N^{-1/3}$ .
- (d) For a finite system comprising  $1 \text{ cm}^3$  of  ${}^4\text{He}$  (molar volume  $27 \text{ cm}^3$ ) show that the expression

$$N_0 = N \left\{ 1 - \left( \frac{T}{T_B} \right)^{3/2} \right\}$$

is valid to temperatures below  $T_B$  right up to within  $\sim 10^{-8}T_B$  of the Bose temperature.

- 2.15 (a) Show that the pressure of a Bose gas below  $T_B$  depends *only* on temperature; that is, it is independent of other thermodynamic variables such as  $N$  and  $V$ .
- (b) Comment on this result in the context of the Gibbs-Duhem relation (asserting that the intensive variables  $T, p, \mu$  are not independent).
- 2.16 (a) Show that below the transition temperature the entropy of a Bose gas is given by

$$S = \frac{5}{2} Nk \frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})} \left( \frac{T}{T_B} \right)^{3/2}.$$

- (b) Since the number of excited particles is given by

$$N_{\text{ex}} = N \left( \frac{T}{T_B} \right)^{3/2},$$

show that the entropy per excited particle is given by

$$\frac{S}{N_{\text{ex}}} = \frac{5}{2} \frac{\zeta(\frac{5}{2})}{\zeta(\frac{3}{2})} k \approx 1.28 k.$$

- (c) Discuss the connection between this result and the two fluid model of superfluid  $^4\text{He}$ .

- 2.17 Show that the chemical potential of a two-dimensional gas of bosons may be expressed analytically as

$$\mu = kT \ln \{1 - e^{-\varepsilon_q/kT}\}$$

where  $\varepsilon_q$  is the two dimensional quantum energy, the analogue of the 2d Fermi energy in Problem 2.6

This problem is the Bose analogue of the Fermi case treated in Problem 2.7.

- 2.18 Show that the Bose-Einstein transition temperature of a gas of bosons and the Fermi temperature for a gas of “similar” fermions are of comparable magnitude. Discuss why this should be.

- 2.19 The formula for the Bose integrals  $\mathcal{B}_n$  of Sec. 2.6.4 was quoted as

$$\begin{aligned} \mathcal{B}_n &= \int_0^\infty \frac{x^n}{e^x - 1} dx \\ &= \Gamma(n+1)\zeta(n+1). \end{aligned}$$

Derive this result.

**Hint:** Use the sequence of transformations below

$$\frac{1}{e^x - 1} = e^{-x} \frac{1}{1 - e^{-x}} = e^{-x} \sum_{m=0}^{\infty} e^{-mx} = \sum_{m=0}^{\infty} e^{-(m+1)x} = \sum_{m=1}^{\infty} e^{-mx}$$

and then change the variable of integration to  $y = mx$ .

- 2.20 The general formula for the Fermi integrals  $\mathcal{F}_n$  of Section 2.5.3 was quoted as

$$\begin{aligned} \mathcal{F}_n &= \int_{-\infty}^{\infty} \frac{e^x}{(e^x + 1)^2} x^n dx \\ &= 2(1 - 2^{1-n}) \zeta(n) n! \end{aligned}$$

where  $\zeta(n)$  is the Reimann zeta function. Derive this result. This is difficult; *Mathematica* v.12 can't do it!

**Hint:** The first step is to integrate by parts, essentially the reverse of Eq. (2.5.28), to obtain

$$\mathcal{F}_n = 2n \int_0^\infty \frac{x^{n-1}}{e^x + 1} dx,$$

noting that this holds for even  $n$ , while  $\mathcal{F}_n = 0$  for odd  $n$ . Next, follow a sequence of transformations *similar* to those for the Fermi calculation of Problem 2.19.

- 2.21 The Reimann zeta function and the Dirichlet eta function are defined as infinite sums:

$$\zeta(n) = \sum_{m=1}^{\infty} \frac{1}{m^n}, \quad \eta(n) = \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^n}.$$

Show that  $\zeta(n) - \eta(n) = 2^{1-n}\zeta(n)$ , and hence that  $\eta(n) = (1 - 2^{1-n})\zeta(n)$ .

- 2.22 A dilute gas of  $10^6$  rubidium atoms (relative atomic mass 86.9) is confined to an isotropic harmonic trap of frequency 150 Hz.
- Calculate the Bose-Einstein condensation (BEC) temperature  $T_B$ .
  - Estimate the width of the thermal cloud at  $T_B$ .
  - Estimate the width of the condensate cloud.

- 2.23 This is the trapped-gas analogue of Problem 2.14. Consider a Bose gas confined to an isotropic harmonic trap at low temperatures, as in the previous Problem. You know that when the occupation of the ground state is appreciable then the chemical potential  $\mu$  is very small and it may be ignored, compared with  $\varepsilon$  in the integral for the number of excited states. Moreover you know that when the ground state occupation  $N_0$  is appreciable, then  $\mu \sim -kT/N_0$ .

- The requirement that  $\mu$  may be neglected in comparison with  $\varepsilon$  in the integral for the number of excited states will be satisfied if  $|\mu|$  is much less than the energy  $\varepsilon_1$  of the first excited state. The expression for  $\varepsilon_1$  is

$$\varepsilon_1 = \hbar\omega$$

where  $\omega$  is the (angular) frequency of the trap. Where does this expression come from?

- (b) Show that the condition  $|\mu| \ll \varepsilon_1$  is satisfied right up to  $T_B$  in the thermodynamic limit. Hint: demonstrate that  $|\mu|/\varepsilon_1 \sim N^{-2/3}$ .
- (c) For a finite system, as in the previous Problem, comprising  $10^6$  atoms, show that the expression

$$N_0 = N \left\{ 1 - \left( \frac{T}{T_B} \right)^3 \right\}$$

is valid to temperatures below  $T_B$  right up to within  $\sim 10^{-5}T_B$  of the Bose temperature.

2.24 In Section 2.10 we studied a paramagnetic *solid*: a collection of essentially *distinguishable* magnetic moments. If we were to consider a (classical) gas of indistinguishable magnetic moments, how would the partition function be modified? What would be the observable consequences of this modification?

2.25 According to Curie's law, the magnetic susceptibility of a paramagnet is inversely proportional to temperature.

- (a) Show that a different behaviour is implied by the Third Law.
- (b) What is wrong with the Curie law model?