IV ELECTROSTATICS II

4.1 Summary of properties of E

In Chapter 2 we derived a number of properties of the **E** field, which ultimately we expressed in terms of the vector calculus operators div, grad and curl. To recap, recall that Gauss's law, in Equation (2.9):

$$\oiint_{\substack{\text{closed}\\\text{surface}}} \mathbf{E}.\mathrm{d}\mathbf{a} = \sum_i Q_i/\varepsilon_0$$

gave us the div relation for **E** expressed in Equation (3.16):

$$\operatorname{div}\mathbf{E} = \rho/\varepsilon_0 \,. \tag{4.1}$$

It is important to realise that in understanding the physical content of this equation, the **E** field is that due to *all* charges and not only the charge at the particular point, while ρ is the charge density at the given point.

The second property of **E** was found from consideration of the work done in moving a charge in an electric field. This led us to the line integral property of **E** expressed in Equation (2.13):

$$\oint_{\text{closed loop}} \mathbf{E}.\mathrm{d}\mathbf{r} = 0$$

which we were able to write as the curl property:

$$\operatorname{curl}\mathbf{E} = 0 \,. \tag{4.2}$$

Finally, we also saw that the **E** field could be found from the electric potential V by taking the gradient, in Equation (2.19):

$$\mathbf{E} = -\operatorname{grad} V \,. \tag{4.3}$$

These properties are not independent. Obviously not; they are three aspects distilled from Coulomb's law. Thus far, that has been our only physical input. In fact Equation (4.3) leads directly to Equation (4.2). Equation (4.3) tells us that **E** may be expressed as the gradient of a scalar function. But we have an important identity of vector calculus, written in Equation (3.31), namely that

$$\operatorname{curl}\operatorname{grad} = 0.$$

Therefore, since

$$\operatorname{curl}\mathbf{E} = -\operatorname{curl}\operatorname{grad} V,$$

this must be zero. And so curl **E** *has* to be zero.

Recall that this is the property of **E** that we said ceased to be strictly true when time variation is considered. In that case we will see that **E** can no longer be expressed entirely as the gradient of the scalar potential. And in that case the curl of **E** is not required to vanish. The phenomenon responsible for this is *electromagnetic induction*, as we shall see.

4.2 The equations of Poisson and Laplace

In a region that may have a distribution of electric charge, there will be an electric field that can be described in terms of the electric potential. How does the charge distribution determine the potential? This is a fundamental question, which will lead to what is probably the most important equation of (classical) physics.

As our starting point we shall take the two equations which, by now, should be very familiar:

$$\mathrm{div}\,\mathbf{E} = \rho/\varepsilon_0, \qquad \mathbf{E} = -\,\mathrm{grad}\,V\,.$$

Combining these two equations we have:

$$\mathrm{div}\,\mathrm{grad}\,V = -\,\rho/\varepsilon_0\,.$$

But now we recall one of the important identities of vector calculus, namely that

$$\mathrm{div}\,\mathrm{grad} = \nabla^2$$

so that we obtain, finally,

$$\nabla^2 V = -\,\rho/\varepsilon_0 \qquad (4.4)$$

This is known as *Poisson's Equation*, which in rectangular Cartesian co-ordinates is:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\,\rho/\varepsilon_0$$

Poisson's equation is very important. It is telling us that the *second* spatial derivative of the electric potential is proportional to the source strength, which here is the electric charge density. In the next section we shall consider the physical interpretation of the ∇^2 operator, but here we must examine one special case of Poisson's equation.

In regions of space where there is no electric charge there still can be an electric field, and therefore an electric potential – in the *vicinity* of an electric charge, for example. If there is no charge in a region, then ρ in this region is zero, and then Poisson's equation reduces to

$$\nabla^2 V = 0\,. \qquad (4.5)$$

This is known as *Laplace's Equation*. Poisson's equation is observed to be *inhomogeneous*, while Laplace's equation is seen to be *homogeneous*. They are both *second order*, *linear*, *partial* differential equations.

Recall that there are two stages in solving partial differential equations such as these. Firstly one must find solutions to the mathematical equation, and then one must ensure that the actual solution also satisfies the *boundary conditions*. A number of mathematical results come to our assistance here.

Both Poisson's equation and Laplace's equation, are subject to *the Uniqueness theorem*: If a function V is found which is a solution of $\nabla^2 V = -\rho/\varepsilon_0$, (or the special case $\nabla^2 V = 0$) and if the solution also satisfies the boundary conditions, then it is the only solution.

Solutions of Laplace's equation are known as *harmonic functions*. The general procedure for solving Laplace's equation is to construct a linear combination of harmonic functions so as to satisfy the boundary conditions of the given problem.

Turning now to Poisson's equation, once we have *any* solution of the equation, then other solutions (including the one which obeys the boundary conditions) can be obtained by adding to it solutions to the corresponding Laplace equation. The procedure for finding the correct solution to Poisson's equation is thus to obtain an initial solution to the equation, which will most likely not satisfy the boundary conditions. Next one adds to this solutions of the corresponding Laplace equation until the final result does satisfy the boundary conditions.

A formal solution to Poisson's equation for a localised distribution of charge may be found in the following way. The potential at a point P due to a charge q_i is given by

$$V = \frac{q_i}{4\pi\varepsilon_0 r_i}$$

where r_i is the displacement from the charge to P. If there is a continuous distribution of charge of density ρ then the charge in the volume dv is ρdv so that the potential due to this volume element is

$$\mathrm{d}V = \frac{\rho_i \mathrm{d}v}{4\pi\varepsilon_0 r_i}.$$

Integrating this over the volume of the charge distribution gives the potential as

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r})\mathrm{d}v}{r} \tag{4.6}$$

where $\rho(\mathbf{r})$ is the charge density at the point $\mathbf{r}$.

4.3 Physical meaning of ∇^2

Laplace's equation and Poisson's equation appear in so many areas of physics that it can't be simple accident. There is something very special contained in the laplacian operator ∇^2; it is even there in Schrödinger's equation. We shall see that the requirement that a function obey Laplace's equation is equivalent to saying that the function must be as smooth as possible.

The laplacian is closely related to the difference between the value of a function at the given position and its mean value at neighbouring points. Let us examine this for the one-dimensional case. At coordinate x the value of V is $V(x)$. A little to the left of this we may express the value of V in terms of a Taylor expansion:

$$V(x-a) = V(x) - a\frac{\partial V}{\partial x} + \frac{a^2}{2}\frac{\partial^2 V}{\partial x^2} - \ldots$$

while correspondingly to the right of the given point V is given by

$$V(x+a) = V(x) + a\frac{\partial V}{\partial x} + \frac{a^2}{2}\frac{\partial^2 V}{\partial x^2} + \dots$$

On taking the average of these two we see that the first derivative term cancels; the average value of V surrounding the point is

$$\langle V \rangle = V(x) + \frac{a^2}{2}\frac{\partial^2 V}{\partial x^2} - \dots$$

and thus the *deviation* from the value of V at a point and its mean value in the neighbourhood is proportional to the *second* spatial derivative. Extending this argument to three dimensions, we see that there the deviation from the value of V at a point and its mean value in the surrounding region is proportional to the laplacian $\nabla^2 V$. In other words, the laplacian gives a measure of the *distortion* of the field.

This is equivalent to the requirement that a function obeying Laplace's equation must have the *minimum curvature*. From this we see that a harmonic function can not have a maximum or a minimum in free space. This is called the *extreme value theorem* or Earnshaw's theorem. The extreme values must occur on the boundaries – very important practically for fusion reactors!

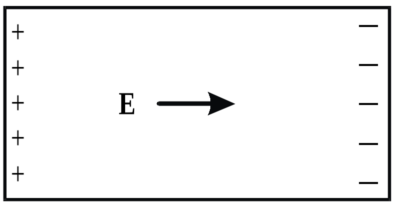
Proof of Earnshaw's theorem: If V has a maximum (minimum) then at that point $\partial V/\partial x$, $\partial V/\partial y$ and $\partial V/\partial z$ will be zero and $\partial^2 V/\partial x^2$, $\partial^2 V/\partial y^2$ and $\partial^2 V/\partial z^2$ must be negative (positive). However if the second derivatives are all negative (positive) then their sum cannot be zero as asserted by Laplace's equation. Thus extrema in V cannot occur in a region where Laplace's equation applies.

4.4 Properties of conductors

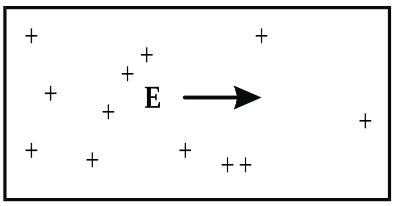
An electric *conductor* is an object that contains many charges which can move freely around within the body, but which cannot leave the surface. Positive and negative charges cancel in a *neutral* conductor. In the following subsections we shall discuss a number of properties of conductors which apply when *no currents are flowing*.

4.4.1 Property 1: There is no **E** field inside a conductor

Let us assume that there is an **E** field inside the conductor. This could happen in one of two ways.



(i) If the conductor is electrically neutral, then the **E** field exists because of an uneven distribution of charges. In this case, since we are dealing with a conductor, the charges will flow until the inhomogeneity is eliminated and the **E** field will become zero.



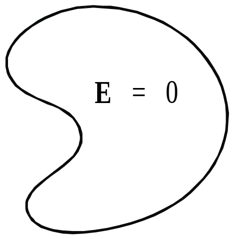
(ii) If there is a distribution of, say, positive charge in the conductor then there will be an **E** field inside, following from $\mathrm{div}\,\mathbf{E} = \rho/\varepsilon_0$. But the **E** field will cause the charge to flow. The equilibrium state, when no current is flowing, must be when there is no **E** field inside; all the charge must therefore be on the surface.

Either way, we see that inside a conductor the electric field must be zero.

4.4.2 Property 2: Excess charges reside only on the surface

This follows from the above discussion.

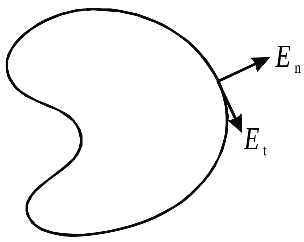
4.4.3 Property 3: The potential *V* is constant in a conductor



From Property 1 we know that the **E** field inside the conductor is zero. Thus $\mathrm{grad}V = 0$, from which it follows immediately that

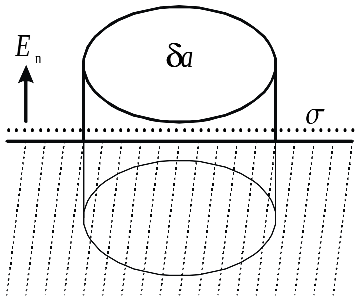
$$V = \text{constant.}$$

4.4.4 Property 4: The external **E** field is normal to the surface



Suppose that there is a component of the electric field tangential to the surface. Then charge would flow along the surface. The equilibrium state is when no current flows; thus there is no tangential component and **E** must therefore be normal to the surface.

4.4.5 Property 5: At the surface $E_n = \sigma/\varepsilon_0$.



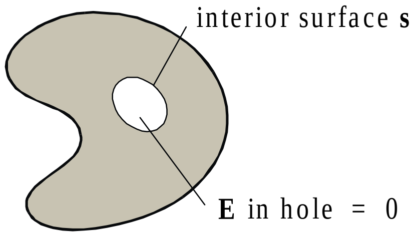
This follows from considering the Gaussian "pill-box". The total charge contained in the pill-box is $\sigma\delta a$. So applying Gauss's law we have

$$E_n \delta a = \sigma\,\delta a/\varepsilon_0$$

And from this we conclude that the electric field normal to the surface is related to the surface charge density σ by:

$$E_\mathrm{n} = \sigma/\varepsilon_0 \tag{4.7}$$

4.4.6 Property 6: In a hole the electric field is zero

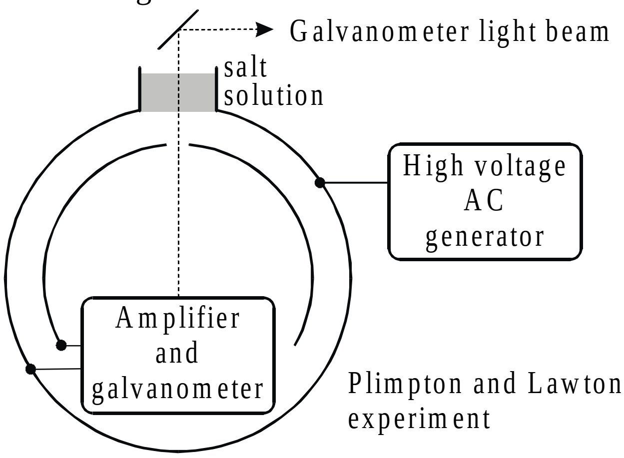


By a *hole* we mean a totally enclosed empty chamber within the body of the conductor.

We know that the entire conductor is at a uniform potential – Property 3. Thus, in particular, *V* is constant on the interior surface. Now in the hole *V* is given by the solution of Laplace's equation subject to the boundary conditions. And the boundary condition is that *V* is constant on the closed surface. But by the extreme value theorem we know that *V* can have neither maxima nor minima in the volume – only at the surface. So *V* must be constant within the volume of the hole, and then **E** must be zero in the hole.

4.5 Validity of the inverse square law

The experiments of Coulomb investigated the validity of the inverse square law by *direct* measurement. Such experiments are not capable of very high precision (compare the difficulty in measuring G accurately). Whenever possible, a null experiment has a much greater sensitivity. Now the vanishing of the **E** field in a cavity within a conductor is a direct consequence of the inverse square law. Thus *testing* for an **E** field in a hole provides a means of checking Coulomb's law.



Plimpton and Lawton's experiment

Cavendish, in 1772, looked for a static **E** field inside a spherical conductor, and in 1936 Plimpton and Lawton enhanced the sensitivity of this type of experiment by looking for an induced alternating **E** field using a resonant detector. Further refinements were made by Williams, Faller and Hill in 1971.

The basic idea of the experiment is that the high voltage generator charges and discharges the outer cylindrical conductor at a frequency that is on resonance with the mechanism of the galvanometer. No motion of the galvanometer was detected when the salt solution completed the conducting sphere at the top.

The results of such experiments may be interpreted by writing the **E** field of a point charge as

$$E \sim r^{-(2+\varepsilon)}.$$

Then the results of the various measurements may be summarised as:

Cavendish (1772)	$\lvert\varepsilon\rvert \le 0.02$
Maxwell (1870)	$\lvert\varepsilon\rvert \le 5\times10^{-5}$
Plimpton & Lawton (1936)	$\lvert\varepsilon\rvert \le 2\times10^{-9}$
Williams, Faller & Hill (1971)	$\lvert\varepsilon\rvert \le 5\times10^{-16}$

An important reason for wanting to test the inverse square law is that it can be regarded as a test that the mass of the photon is zero. The photon is the "messenger particle" which mediates the electromagnetic force. If the mass of the photon were m then the electric potential of a charge Q would vary with distance as

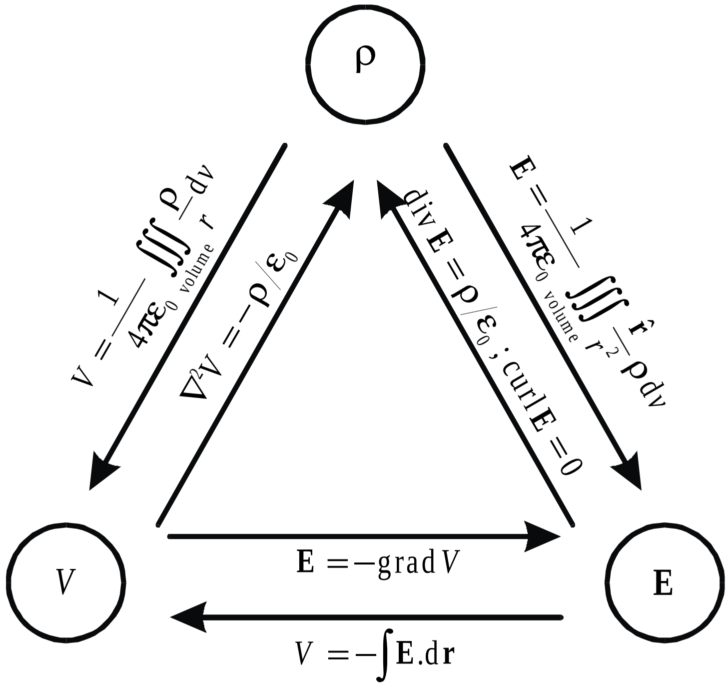
$$V(r) = \frac{Q}{4\pi\varepsilon_0 r} e^{-\frac{mc}{\hbar}r}$$

where c is the speed of light and $\hbar$ is Planck's constant. Thus only when $m = 0$ does the potential vary as $1/r$.

Looked at in this way, the measurements of Williams, Faller & Hill can be interpreted as setting the mass of the photon at be less than 1.6×10^{-50} kg.

4.6 Summary of electrostatics results

The relations between the three quantities **E**, *V* and ρ are summarised in the following diagram, borrowed from *Introduction to Electrodynamics* by D. J. Griffiths. While **E** *may* be calculated directly from ρ, it is usually easier to calculate *V* first and then to calculate **E** from *V*.



When you have completed this chapter you should:

- be able to write down the expressions for div**E** and curl**E** and know what these mean in physical terms;

- know how to calculate **E** from *V*;

- understand the meaning and usage of the Laplace and Poisson equation for *V*;

- be competent in working in terms of continuous charge distributions $\rho(\mathbf{r})$;

- appreciate that solutions of the Laplace equation obey the uniqueness theorem and the extreme value theorem, and understand the physical meaning of these;

- be familiar with the various electrostatic properties of conductors;

- understand why the electric field in a cavity within a conductor is zero and appreciate that this can be used as a test of the inverse square law;

- appreciate that deviations from the inverse square law can be interpreted in terms of mass of the photon;

- understand the various relationships between the quantities **E**, *V* and ρ.