PH2130

Questions for contemplation

Week 1

Why *differential* equations?

Week 2

Why usually *linear* diff eq^{n}s?

Week 3

Why usually *2nd* order?

Aims of Wk 3 Lect 1

- **Recognise diffusion eq^{n} and wave eq^{n}.**
- **Know the type of phenomena they describe**
- **Know the meaning and use of the ∇^2 symbol**
- **Understand the physical meaning of the laplacian operator**

2.3.1 One dimension: *x* and *t* independent variables

$$\frac{\partial^2 \Psi}{\partial x^2} - \frac{1}{D}\frac{\partial \Psi}{\partial t} = 0$$ **Diffusion eq$^{\mathbf{n}}$**

describes diffusion, heat flow etc.
D is the *diffusion coefficient*.

$$\frac{\partial^2 \Psi}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2} = 0$$ **Wave eq$^{\mathbf{n}}$**

describes vibrating string.
v is the *speed of propagation*.

Note different orders of time

Connection with relativity.

2.3.2 Two dimensions: *x, y* and *t* independent variables

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}-\frac{1}{D}\frac{\partial\Psi}{\partial t}=0 \quad \textbf{Diff}^{\textbf{n}}\ \textbf{eq}^{\textbf{n}}$$

describes diffusion, heat flow etc. in two dimensions

D is the *diffusion coefficient*.

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}-\frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}=0 \quad \textbf{Wave eq}^{\textbf{n}}$$

describes vibrating sheet -- a drum for example.

v is the *speed of propagation*.

2.3.3 Three dimensions: *x, y, z* and *t* independent variables

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+\frac{\partial^2\Psi}{\partial z^2}-\frac{1}{D}\frac{\partial\Psi}{\partial t}=0$$ **Diff^{n} eq^{n}**

describes diffusion, heat flow etc. in three dimensions

D is the *diffusion coefficient*.

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+\frac{\partial^2\Psi}{\partial z^2}-\frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}=0$$ **Wave eq^{n}**

describes vibrations in 3d -- sound waves for example.

v is the *speed of propagation*.

2.3.4 The laplacian

Have seen

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

before.

⌈Recall vector calculus in PH1120 and the formula div grad = ∇^2 ⌋

The *laplacian* operator, denoted by ∇^2, is given (in cartesian coordinates) by

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}.$$

⌈Some books denote ∇^2 by Δ; we don't⌋

Ubiquity of the laplacian

The laplacian appears in many differential equations:

Diffusion equation

$$\nabla^2\Psi - \frac{1}{D}\frac{\partial\Psi}{\partial t}.$$

Wave equation

$$\nabla^2\Psi - \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}.$$

Even the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

⌈recall from PH1530⌋

Why is ∇^2 so common?

2.3.5 Physical meaning of ∇^2

The laplacian gives the 'smoothness' of a function. It measures the difference between the value of Ψ at a point and its mean value at surrounding points.

A little to the left of *x*

$$\Psi(x-a)=\Psi(x)-a\frac{\partial\Psi}{\partial x}+\frac{a^2}{2}\frac{\partial^2\Psi}{\partial x^2}+\ldots$$

while a little to the right

$$\Psi(x-a)=\Psi(x)+a\frac{\partial\Psi}{\partial x}+\frac{a^2}{2}\frac{\partial^2\Psi}{\partial x^2}+\ldots$$

On taking the average

$$\overline{\Psi} = \frac{1}{2}\left[\Psi(x-a) + \Psi(x+a)\right]$$

$$= \Psi(x) + \frac{a^2}{2}\frac{\partial^2\Psi}{\partial x^2}$$

or

$$\overline{\Psi} - \Psi(x) = \frac{a^2}{2}\frac{\partial^2\Psi}{\partial x^2}$$

The argument can be extended to 2d and 3d. Thus we conclude:

The deviation from the value of Ψ at a point and its mean value in the surrounding region is proportional to $\nabla^2\Psi$.

In the Schrödinger equation bending Ψ costs kinetic energy.

2.3.6 Laplace's equation

In the *steady state* i.e. $\partial/\partial t$, $\partial^2/\partial t^2$ etc. = 0. Then both the wave equation and the diffusion equation reduce to
(another equation to spot)

$\nabla^2\Psi = 0$. **Laplace's equation**

⌈Will see this in Electromagnetism PH2420.⌋

Physical interpretation of ∇^2 implies:

In a region where Laplace's eq^n holds, there can be no maxima or minima in Ψ.

2.3.7 The d’alembertian

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Aims of Wk 3 Lect 2

- Understand separation of variables method for solving PDEs
- Use separation of variables to convert PDEs into ODEs
- Boundary conds and Initial conds in solving real problems
- Solve simple (2 indep. vars) PDEs, given BCs and ICs

3 Separation of Variables

Look for solutions of PDEs which are a product of the independent variables.

Converts PDEs into a number of ODEs.

- So in 1d case : x, t indep. vars., look for solutions like

$$\Psi(x,t) = X(x)T(t)$$

3.1 1-d wave equation

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \Psi}{\partial t^2}$$

Writing $\Psi(x,t) = X(x)T(t)$

Then

$$\frac{\partial^2 \Psi(x,t)}{\partial x^2} = \frac{\mathrm{d}^2 X(x)}{\mathrm{d}x^2} T(t)$$

and

$$\frac{\partial^2 \Psi(x,t)}{\partial t^2} = X(x)\frac{\mathrm{d}^2 T(t)}{\mathrm{dt}^2}$$

has total derivatives.

Put in wave equation $\Rightarrow$

$$\frac{\mathrm{d}^2 X(x)}{\mathrm{d}x^2} T(t) = \frac{1}{v^2} \frac{\mathrm{d}^2 T(t)}{\mathrm{d}t^2}$$

Divide by $X(x)T(t)$, gives

$$\frac{1}{X} \frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = \frac{1}{v^2} \frac{1}{T} \frac{\mathrm{d}^2 T}{\mathrm{d}t^2}$$

LHS depends on x only
RHS depends on t only

But x and t are independent!

So both sides must be constant

Put const = $-k^2$.
Called separation constant.

Have 2 ODEs:

$$\left.\begin{aligned} \frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + \quad k^2 X &= 0 \\ \frac{\mathrm{d}^2 T}{\mathrm{d}t^2} + v^2 k^2 T &= 0 \end{aligned}\right\}$$

- Have turned 1 PDE into 2 ODEs

- Assuming k^2 *is* positive, these are both SHO equations.

3.1.1 Boundary conditions & Initial conditions

Need some physical information to solve real problems.

E.g. Piano string, length L, where $\Psi(x,t)$ is displacement of string.

- Fixed at both ends:
  $\Psi(0,t) = \Psi(L,t) = 0$ for all t.
  Restriction on Ψ by the boundary, so called *boundary condition*.

- Initial shape: $\Psi(x,0) = f(x)$,
  Restriction on Ψ by the initial state called initial condition.

The *Boundary Condition* helps solve the X equation.

BC is $X(0) = X(L) = 0.$

Gen. Sol^{n} of

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + k^2 X = 0$$

is

$$X(x) = A\sin(kx) + B\cos(kx).$$

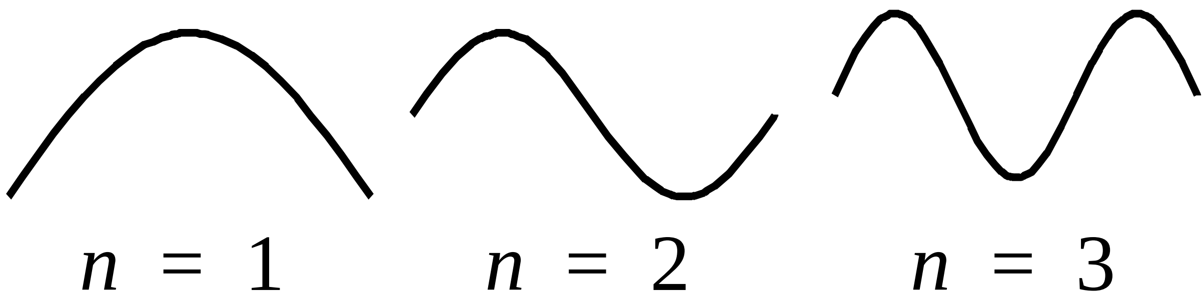
⌈Recall from PH1110⌋

BC $X(0) = 0 \quad \Rightarrow \quad B = 0$

BC $X(L) = 0$ restricts allowed values of k since $\sin kL$ must = 0; i.e $kL = n\pi$ for integer n.
(See why k^2 must be +ve now)

PICTURE of Piano string

Recall *particle in a box* in PH2530. There we saw you needed an integer n° of ½ waves to fill *L*. – Same thing.



We label the allowed values of *k*:

$$k_n = \frac{\pi}{L} n$$

Then *X* solutions are:

$$X_n(x) = A_n \sin(k_n x)$$

↑ underermined as yet

See that

Boundary conditions ⇒ Quantisation

The *Initial Condition* helps solve the T equation

$$\frac{\mathrm{d}^2 T(t)}{\mathrm{d}t^2} + k^2 v^2 T = 0$$

another SHO equation – since we know k^2 is positive.

Solution is

$$T_n(t) = P_n \cos(k_n v t) + Q_n \sin(k_n v t).$$

Solution for $\Psi(x, t)$ for given n is then

$$\begin{aligned}\Psi_n(x,t) &= X_n(x) T_n(t) \\ &= \sin k_n x \left\{ P_n \cos(k_n v t) + Q_n \sin(k_n v t) \right\}\end{aligned}$$

(have subsumed the A_n into the P_n, Q_n)

Linearity allows us to write the *general solution* as a linear superposition

$$\Psi(x,t) = \sum_n \sin k_n x \left\{ P_n \cos(k_n vt) + Q_n \sin(k_n vt) \right\}$$

again have subsumed coeffs into the P_n and Q_n.

Satisfying the *initial condition* will determine the P_n and Q_n.

$$\Psi(x,0) = f(x)$$

so

$$\sum_n P_n \sin k_n x = f(x).$$

This is a *Fourier* sine series.

⌈Remember from PH1120⌋

The Fourier components P_n are found from $f(x)$ using the *inversion formula*:

$$P_n = \frac{2}{L}\int_0^L f(x)\sin k_n x \,\mathrm{d}x$$

So:

Solution to vibrating string obeying

$$\frac{\partial^2\Psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}$$

and subject to:

BC: $\Psi(0,t) = \Psi(L,t) = 0$ for all t

(fixed at both ends)

and

IC: $\Psi(x,0) = f(x)$ (shape at $t = 0$)

Is

$$\Psi(x,t) = \sum_n P_n \sin(k_n x)\cos(k_n v t)$$

where

$$k_n = \frac{\pi}{L}n$$

and

$$P_n = \frac{2}{L}\int_0^L f(x)\sin k_n x \,\mathrm{d}x.$$

Summary of S.V method:

1 Express Ψ as a product ⇒ ODEs plus separation constant

2 Solve ODEs

3 Boundary conditions determine allowed spatial solutions, values of separation constant

4 Make linear superposition of $X_n T_n$ solutions.

5 Initial conditions allow determination of superposition coefficients.