Series Solutions of ODEs – 2
the Frobenius method

Introduction to the Methodology

The simple series expansion method works for differential equations whose solutions are well-behaved at the expansion point $x = 0$. The method works well for many functions, but there are some whose behaviour precludes the simple series method. The Bessel Y_0 function is one such example. And clearly any functions involving negative or fractional powers would not be amenable to a simple power series expansion.

The Frobenius method extends the simple power series method to include negative and fractional powers, and it also allows a natural extension involving logarithm terms.

The basic idea of the Frobenius method is to look for solutions of the form

$$\begin{aligned} y(x) &= a_0 x^c + a_1 x^{c+1} + a_2 x^{c+2} + a_3 x^{c+3} + \ldots \\ &= x^c\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\right) \\ &= x^c \sum_{i=0}^{\infty} a_i x^i . \end{aligned}$$

The extension of the simple power series method is all in the factor x^c. The power c must now be determined, as well as the coefficients a_i. Since c may be negative, positive, and possibly non-integral, this extends considerably the range of functions which may be treated. Note that a_0 is the lowest non-zero coefficient, so by definition it cannot be zero.

A simple example

We can demonstrate, with the following equation, how the Frobenius method works in practice

$$4x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0.$$

We would divide by $4x$ to get the equation into standard form:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{1}{2x}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{4x}y = 0$$

but we will work directly with the original equation.

The trial solution is given by

$$y(x) = \sum_{i=0}^{\infty} a_i x^{i+c}$$

so differentiating this gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sum_{i=0}^{\infty} a_i (i+c) x^{i+c-1}$$

and differentiating again:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \sum_{i=0}^{\infty} a_i (i+c)(i+c-1) x^{i+c-2}.$$

We substitute these expressions into the differential equation:

$$4x\sum_{i=0}^{\infty} a_i (i+c)(i+c-1) x^{i+c-2} + 2\sum_{i=0}^{\infty} a_i (i+c) x^{i+c-1} + \sum_{i=0}^{\infty} a_i x^{i+c} = 0 .$$

Next we incorporate the x factor in the first term, giving a slight simplification

$$4\sum_{i=0}^{\infty} a_i (i+c)(i+c-1) x^{i+c-1} + 2\sum_{i=0}^{\infty} a_i (i+c) x^{i+c-1} + \sum_{i=0}^{\infty} a_i x^{i+c} = 0$$

and, as in the simple series case, we alter the summation indices to obtain a common power of x for all three terms. In this case this means increasing i by 1 in the first two terms. The differential equation then becomes

$$\sum \left\{ 4a_{i+1}(i+c+1)(i+c) + 2a_{i+1}(i+c+1) + a_i \right\} x^{i+c} = 0 .$$

For the expression to be zero, each power of x must vanish, thus

$$4a_{i+1}(i+c+1)(i+c) + 2a_{i+1}(i+c+1) + a_i = 0$$

for all (allowed) i.

This equation gives a recurrence relation for the coefficients, but before examining that we must determine the values of c which are allowed.

We know that a_0 is the lowest order non-vanishing coefficient; $a_{-1} = 0$ by definition. So putting $i = -1$ into the above expression gives

$$4a_0 c(c-1) + 2a_0 c = 0$$

or

$$2a_0 c(2c-1) = 0 .$$

Now we know that a_0 cannot be zero, so to satisfy this equation either

$$c = 0$$

$$\text{or } \; c = 1/2.$$

We have found two possible values for the index c. For this reason the equation obtained by setting the lowest power of x equal to zero is called the *indicial equation*.

This is a quadratic equation. It thus gives two possible values for c that suggests there are two series that satisfy the equation. This seems to be correct for a second order equation.

Now let us return to the recurrence relation. The equation relating general coefficients may be written

$$2a_{i+1}(i+c+1)(2i+2c+1) + a_i = 0$$

or

$$a_{i+1} = \frac{-a_i}{2(i+c+1)(2i+2c+1)} .$$

We have two cases now to consider, corresponding to the two different values taken by the index c. The recurrence relation is different for these two cases.

First we consider the case $c = 0$. The recurrence relation is then

$$a_{i+1} = \frac{-a_i}{2(i+1)(2i+1)}.$$

Set $i = 0$:

$$a_1 = \frac{-1}{2} a_0 .$$

Now set $i = 1$:

$$\begin{aligned} a_2 &= \frac{-1}{2.2.3} a_1 \\ &= \frac{-1}{2.2.3} \frac{-1}{2} a_0 \\ &= \frac{1}{4!} a_0. \end{aligned}$$

Terms may be built up in this way. In general the i^{th} term may be written

$$a_i = \frac{(-1)^i}{(2i)!} a_0.$$

Consequently we have obtained the solution

$$\begin{aligned} y(x) &= a_0 \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!} x^i \\ &= a_0 \left\{ 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{x!} + \ldots \right\}. \end{aligned}$$

You might observe that this can be expressed in terms of the cosine function

$$y(x) = a_0 \cos\sqrt{x} .$$

Now we consider the case $c = 1/2$. The recurrence relation is then

$$a_{i+1} = \frac{-a_i}{2(2i+2)(i+3/2)}.$$

Set $i = 0$:

$$\begin{aligned} a_1 &= \frac{-1}{2.2.3/2} a_0 \\ &= \frac{-1}{3!} a_0. \end{aligned}$$

Now set $i = 1$:

$$\begin{aligned} a_2 &= \frac{-1}{2.4.5/2} a_1 \\ &= \frac{-1}{2.4.5/2} \frac{-1}{3!} a_0 \\ &= \frac{1}{5!} a_0. \end{aligned}$$

Terms may be built up in this way. In general the i^{th} term may be written

$$a_i = \frac{(-1)^i}{(2i+1)!} a_0 .$$

So the other solution is

$$\begin{aligned} y(x) &= a_0 \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} x^{i+1/2} \\ &= a_0 \left\{ x^{1/2} - \frac{x^{3/2}}{3!} + \frac{x^{5/2}}{5!} - \ldots \right\}. \end{aligned}$$

You might observe that this can be expressed in terms of the sine function

$$y(x) = a_0 \sin\sqrt{x} \, .$$

The role of the a_0 in each case is that of an arbitrary constant and it is not fixed by the differential equation. The general solution of the ODE is thus

$$y(x) = A\cos\sqrt{x} + B\sin\sqrt{x}.$$

In general one cannot sum the series in terms of standard functions.

The important concepts of this section are:

- The general expression for a solution is x^c multiplied by a simple power series.
- The index c may be non-integral, positive or negative.
- The series is substituted into the differential equation and a slight juggling of the summation index casts the equation into the form $\sum_i \{\ldots\} x^{i+c} = 0$.
- Each power of x must equate to zero.
- By definition $a_0 \neq 0$, while a_i for negative i vanish.
- The equation obtained by setting the lowest power of x equal to zero is the *indicial equation*. This (for a 2$^{\text{nd}}$ order ODE) is a quadratic equation; it gives two values for the index c.
- Each value of c gives a difference recurrence relation and a different power series solution to the ODE.

Convergence and existence of solutions

The Frobenius method extends the range of equations for which a solution may be expressed in terms of power series (by extending/generalising what we mean by a power series). We will give without proof a theorem which tells us something about the validity of the Frobenius method. First we need some definitions. These will refer to the general second order homogeneous differential equation expressed in standard form:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = 0 .$$

Definitions

The point x_0 is called an *ordinary point* of the equation if $p(x)$ and $q(x)$ both have Taylor expansions about x_0 in powers of $(x - x_0)$. Thus for well behaved functions $p(x)$ and $q(x)$ all points in the range will be ordinary points; in this case the ODE can be solved with the simple series method.

A point which is not ordinary is called a *singular point* or a *singularity*. Singularities are bad behaviour; but there is bad and really bad!

If x_0 is a singular point but $(x - x_0)p(x)$ and $(x - x_0)^2 q(x)$ both have Taylor series about x_0, then x_0 is called a *regular singular point* or a *regular singularity*. Here the behaviour is not *so* bad, and at least one series solution about x_0 (Frobenius type) is possible.

Theorem

If $xp(x)$ and $x^2 q(x)$ can be expressed as a power series in *x* then the Frobenius series solutions to

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}y}{\mathrm{d}x} + q(x)y = 0$$

obtained from a root *c* of the indicial equation converges for $|x| < R$ where *R* is the minimum radius of convergence of $xp(x)$ and $x^2 q(x)$.

Classification of equation types

If the origin is a regular singular point then we can see that the indicial equation will be a quadratic in *c*. This, in general, has two roots but if these differ by an integer then failure occurs; further consideration is needed to find the second solution. We can classify the equations to be solved by the Frobenius method into four types.

I Roots of indicial equation unequal and *not* differing by an integer.
In this case we get two independent solutions by substituting the two values of the index into the series for $y(x)$.

II Roots of indicial equation zero.
In this case we get two independent solutions by substituting the value for c into the series for $y(x)$ and $\partial y / \partial c$.

III Roots of indicial equation differing by an integer, making a coefficient infinite

IV Roots of indicial equation differing by an integer making a coefficient indeterminate.

The important concepts of this section are:

- For well behaved functions $p(x)$ and $q(x)$ the expansion point (here $x = 0$) will be an ordinary point; in this case the ODE can be solved with the simple series method.

- For expansions about a regular singular point the Frobenius method may be used.

- In this case the indicial equation will be a quadratic in the index c; in general this will give the two solutions of the ODE. There are, however, special cases.