Series Solutions of ODEs – 1b
the simple case – further examples

The J_0 Bessel function

The equation for J_0 Bessel is the zeroth order Bessel equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+x\frac{\mathrm{d}y}{\mathrm{d}x}+x^2y=0.$$

The "standard form" of differential equations is often specified as having the coefficient of the highest order derivative cancelled through. Thus in standard form the equation would be written

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x}+y=0.$$

Our procedure for the series solution of this equation is to take the assumed series

$$\begin{aligned}y(x)&=a_0+a_1x+a_2x^2+a_3x^3+\ldots\\&=\sum_{n=0}^{\infty}a_nx^n\end{aligned}$$

and substitute it into the equation. This involves using the first and second derivatives

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}&=a_1+2a_2x+3a_3x^2+4a_4x^3+\ldots\\&=\sum_{n=1}^{\infty}a_nnx^{n-1}\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}&=2a_2+3.2a_3x+4.3a_4x^2+5.4a_5x^3+\ldots\\&=\sum_{n=2}^{\infty}a_nn(n-1)x^{n-2}\\&=\sum_{n=0}^{\infty}a_{n+2}(n+2)(n+1)x^n.\end{aligned}$$

In the expression for the second derivative we have (as in the sine/cosine case above) shifted the dummy summation variable by 2 so that the sum expression contains x^n explicitly.

So far we have left the sum for the first derivative unchanged. The point here is that what the differential equation contains is $\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x}$, and the expression for *this* must be written as a sum in x^n. For this reason we shift the dummy variable in the series for the first derivative by 2:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sum_{n=-1}^{\infty} a_{n+2}(n+2)x^{n+1}$$

so that

$$\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} = \sum_{n=-1}^{\infty} a_{n+2}(n+2)x^{n} .$$

We now substitute the series expressions into the original differential equation:

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=-1}^{\infty} a_{n+2}(n+2)x^n + \sum_{n=0}^{\infty} a_n x^n = 0 .$$

The $n = -1$ term in the second sum may be treated separately. In that case everything else falls within a sum over *n* from 0 to ∞.

$$\sum_{n=0}^{\infty}\left\{a_{n+2}(n+2)(n+1) + a_{n+2}(n+2) + a_n\right\}x^n + a_1 x^{-1} = 0 .$$

As we argued in the previous example, this expression is valid for all values of the independent variable *x*, so that each power of *x* must vanish separately.

Look at the –1 term first. The requirement that this term vanish means that

$$a_1 = 0 .$$

Now look at the general case:

$$a_{n+2}(n+2)(n+1) + a_{n+2}(n+2) + a_n = 0 .$$

This may be tidied into

$$a_{n+2}(n+2)^2 + a_n = 0 ,$$

which gives a recurrence relation for the coefficients:

$$a_{n+2} = \frac{-1}{(n+2)^2} a_n .$$

We may now build up the coefficients from the $n = 0$ term. Starting from $n = 0$ we find

$$a_2 = \frac{-1}{2^2} a_0 .$$

Now putting $n = 2$ gives

$$\begin{aligned} a_4 &= \frac{-1}{4^2} a_2 \\ &= \frac{-1}{4^2} \times \frac{-1}{2^2} a_0 \\ &= \frac{1}{4^2 2^2} a_0 \end{aligned}$$

and so on. We see that the coefficients of all the *even* powers of *x* are given in terms of a_0 and we obtain the solution to the ODE as

$$y(x) = a_0\left(1 - \frac{x^2}{2^2} + \frac{x^4}{4^2 2^2} - \frac{x^6}{6^2 4^2 2^2} + \ldots\right).$$

The series specifies the J_0 Bessel function:

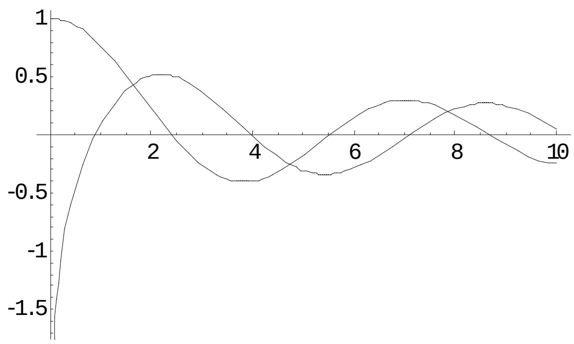
$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{4^2 2^2} - \frac{x^6}{6^2 4^2 2^2} + \ldots .$$

So the solution to the ODE which we have discovered is a constant times the J_0 Bessel function

$$y(x) = a_0 J_0(x) .$$

Thus far this is quite good; we have discovered a new function which solves the above differential equation. But it is a second order differential equation and therefore, as with the previous SHO equation, there should be two independent solutions. Where is the other solution?

When we examined the solution of the wave equation for a drumhead we found the separated radial equation took the form of the zeroth order Bessel equation. And at that stage we simply noted that *Mathematica* gave, as independent solutions to that equation, the two zeroth order Bessel functions $J_0(x)$ and $Y_0(x)$. We plotted the functions and the behaviour of the functions in the vicinity of $x = 0$ gives us an important clue about the "other" solution.



$J_0(x)$ and $Y_0(x)$ Bessel functions

The $J_0(x)$ function goes to 1 as x goes to 0. This we see on the plot and we have discovered this in the series solution. The $Y_0(x)$ function, on the other hand, looks as if it is heading for minus infinity as x goes to 0. That is the problem.

Recall the point made when we introduced the power series method. A series

$$\begin{aligned} y(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \\ &= \sum_{n=0}^{\infty} a_n x^n \end{aligned}$$

will only work when the function $y(x)$ is "well behaved". This is OK for $J_0(x)$, but going off to infinity is an example of "bad behaviour"; then a simple power series won't work. We will see how to overcome this in a later section.

The important concepts of this section are:

- The simple power series method works only for "well-behaved" functions; it cannot cope with "badly-behaved" functions.

- The basic idea is to substitute the power series into the differential equation.

- With a slight juggling of the dummy summation variables the equation is cast into the form $\sum_n \{\ldots\} x^n = 0$. Special care must be taken with the first derivative term.

- Each power of x must equate to zero.

- The term in x^{-1} must be treated separately; this tells us that there is no simple series in *odd* powers of x. The series method is only going to give us one solution to the ODE, which is *even* in x.

- Equating the general term in x^n to zero gives a recurrence relation for the coefficients.

- We build up *one* solution to the ODE from the a_0 coefficient.

- A 2$^{\text{nd}}$ order ODE has two independent solutions; where is the other? We recognise that the simple power series method can't cope with it as it is "badly-behaved".

- Properties of the J_0 function are obtained from the series solutions and the original ODE.

Legendre's equation

Legendre's equation follows from separating the laplacian in spherical polar coordinates. This equation arises from the separated equation in the polar angle θ. Legendre's equation is

$$\left(1-x^2\right)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}-2x\frac{\mathrm{d}y}{\mathrm{d}x}+n(n-1)y=0.$$

In this equation n is often a positive integer; we will explore this a little later.

As before, we start with a power series expression for the function $y(x)$

$$\begin{aligned} y(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \\ &= \sum_{n=0}^{\infty} a_n x^n \end{aligned}$$

Now, however, we will use *Mathematica* to obtain the recurrence relation for the coefficients. This is outlined in the *Mathematica* Notebook "Legendre". The recurrence relation is

$$a_{s+2} = \frac{(-n+s)(1+n+s)}{(1+s)(2+s)} a_s .$$

A series in even powers of x will be built up from a_0 and a series in odd powers of x will be built up from odd powers of x. The general solution to the Legendre equation is thus

$$y(x) = a_0 f_{\text{even}}(x) + a_1 f_{\text{odd}}(x)$$

where

$$\begin{aligned} f_{\text{even}}(x) &= 1-\frac{n(1+n)}{2}x^2 \\ &\quad -\frac{(2-n)n(1+n)(3+n)}{4!}x^4-\frac{(2-n)(4-n)n(1+n)(3+n)(5+n)}{6!}x^6-\ldots \\ f_{\text{odd}}(x) &= x-\frac{(1-n)(2+n)}{3!}x^3+\frac{(1-n)(3-n)(2+n)(4+n)}{5!}x^5 \\ &\quad +\frac{(1-n)(3-n)(5-n)(2+n)(4+n)(6+n)}{7!}x^7+\ldots \end{aligned}$$

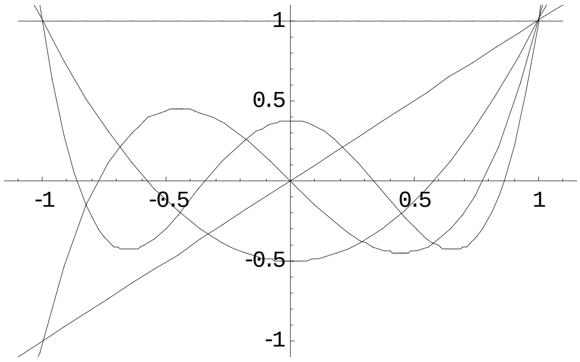
Often n is a positive integer. In that case: if n is even then the series for $f_{\text{even}}(x)$ will terminate at the x^n term, while if n is odd then the series for $f_{\text{odd}}(x)$ will terminate at the x^n term. These solutions, normalised to $\left|P_n(\pm 1)\right| = 1$, are called the Legendre *polynomials*, denoted by $P_n(x)$. The first few are given by

$$P_0(x)=1 \qquad P_1(x)=x$$

$$P_2(x)=\frac{1}{2}\left(3x^2-1\right) \qquad P_3(x)=\frac{1}{2}\left(5x^3-3x\right)$$

$$P_4(x)=\frac{1}{8}\left(35x^4-30x^2+3\right) \qquad P_4(x)=\frac{1}{8}\left(63x^5-70x^3+15x\right)$$

They are plotted in the following figure



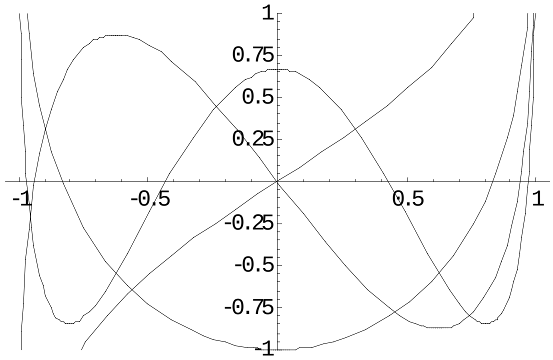
First few Legendre polynomials

When *n* is an integer one series solution of the Legendre terminates and we thus have the Legendre polynomials $P_n(x)$. The other series solution does not terminate. These are denoted by $Q_n(x)$, and they can be expressed in terms of logarithms:

$$Q_0(x)=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad Q_1(x)=-1+\frac{1}{2}x\ln\left(\frac{1+x}{1-x}\right)$$

$$Q_2(x)=-\frac{3x}{2}+\frac{3x^2-1}{4}\ln\left(\frac{1+x}{1-x}\right) \qquad Q_3(x)=\frac{2}{3}-\frac{5x^2}{2}-\frac{3}{4}x\left(1-\frac{5x^2}{3}\right)\ln\left(\frac{1+x}{1-x}\right)$$

They are plotted in the following figure



First few Legendre Q_n functions

The general solution of the Legendre equation will be

$$y(x)=AP_n(x)+BQ_n(x).$$

But note that $Q_n(x)$ has a logarithmic divergence at $x=\pm1$; the only *well-behaved* solutions of Legendre's equation for integer *n* are the Legendre polynomials $P_n(x)$.

The important concepts of this section are:

- *Mathematica* can be used to derive the coefficient recurrence relation by substituting some general terms of the power series into the differential equation.

- The recurrence relation connects every *other* coefficient.

- Therefore there are two independent solutions, one in the even powers of x and one in the odd powers.

- For the Legendre equation with integer n the coefficients of one of the series solutions will *terminate*.

- If n is even then the series for $f_{\text{even}}(x)$ will terminate at the x^n term, while if n is odd then the series for $f_{\text{odd}}(x)$ will terminate at the x^n term.

- With appropriate normalisation these are the Legendre polynomials.