Series Solutions of ODEs – 1
the simple case

Introduction to the Methodology

Series expansions are powerful tools of the quantitative scientist. You will be familiar with Taylor expansions (about a point x_0) and Maclaurin expansions (about the point $x_0 = 0$). You will have used the Maclaurin series on functions about which you know. This enables a function to be expressed as a power series in its argument. We will now turn this around. We have discovered, and we will continue to discover, new ODEs where we don't know what the solution is. In particular, we are concerned with ODEs whose solutions can't be expressed in terms of the familiar "usual" functions (sin, cos, exp, log, etc.). In this case we will see how to express the solutions of the ODEs as power series and how the properties of the solutions may be found from the series expansions. In this way we will discover a range of new functions, often called "special" functions of mathematical physics. In many respects these functions are *defined* by their series expansion, although we must not forget the differential equation which underlies this.

Sines and cosines

The solution of the Simple Harmonic Oscillator equation

$$\frac{d^2 y}{dx^2} + y = 0$$

may be expressed in terms of sines and cosines.

$$y(x) = A\cos x + B\sin x.$$

⌈This may be checked by differentiation and substitution:

$$\frac{dy}{dx} = -A\sin x + B\cos x$$

$$\frac{d^2 y}{dx^2} = -A\cos x - B\sin x$$

$$= -y$$

so that

$$\frac{d^2 y}{dx^2} + y = 0$$

and the equation is satisfied.   ⌋

But try and imagine that you don't know about the trigonometrical functions. You want to solve the SHO equation, and you wonder what to do. Under such circumstances, particularly if you know about Maclaurin expansions, you might be inclined to investigate solutions in the form of a power series. You would write the function $y(x)$ as

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$
$$= \sum_{n=0}^{\infty} a_n x^n$$

and see if you could determine the coefficients $a_0, a_1, a_2, a_3, \ldots$ by substituting the series into the differential equation. At the back of your mind you would know that so long as $y(x)$ is a "well-behaved" function, it does have a power series expansion, so the method should work.

Let's have a go. We need the second derivative of $y(x)$. Differentiating once we obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots$$
$$= \sum_{n=1}^{\infty} a_n n x^{n-1}$$

and differentiating again then gives

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2a_2 + 3.2a_3 x + 4.3a_4 x^2 + 5.4a_5 x^3 + \ldots$$
$$= \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

Since *n* in the summation expression here is a dummy variable we can shift it up by 2 and, correspondingly, perform the sum from zero to infinity. In other words we write the second derivative as

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1) x^n .$$

Substituting the power series for *y* and the second derivative into the original differential equation gives

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1) x^n + \sum_{n=0}^{\infty} a_n x^n = 0 .$$

Here we see the benefit of having shifted the summation variable for $\mathrm{d}^2 y / \mathrm{d}x^2$; both parts of the sum are written in terms of the same power of *x*. So we may write this as

$$\sum_{n=0}^{\infty} \{ a_{n+2}(n+2)(n+1) + a_n \} x^n = 0 .$$

But this equation must be true for all values of the independent variable *x*. This can only be so if all the coefficients of all the powers of *x* are zero. Thus we are saying

$$a_{n+2}(n+2)(n+1) + a_n = 0 \quad \text{for all} \quad n .$$

If we write this equation as

$$a_{n+2} = \frac{-1}{(n+2)(n+1)} a_n$$

it has the form of a *recurrence relation*; if we know the coefficient a_n then we can obtain the coefficient a_{n+2} .

Starting from $n = 0$ we find

$$a_2 = \frac{-1}{2} a_0 .$$

Now putting $n = 2$ gives

$$a_4 = \frac{-1}{4.3} a_2$$

$$= \frac{1}{4.3.2} a_0$$

and so on. We see that the coefficients of all the *even* powers of x are given in terms of a_0.

If we start from $n = 1$ we find

$$a_3 = \frac{-1}{3.2} a_1$$

Now putting $n = 3$ gives

$$a_5 = \frac{-1}{5.4} a_3$$

$$= \frac{1}{5.4.3.2} a_1$$

and so on. We see that the coefficients of all the *odd* powers of x are given in terms of a_1.

The solution to the ODE is then

$$y(x) = a_0(\text{series in even powers of } x) + a_1(\text{series in odd powers of } x).$$

In the expressions for the coefficients you should recognize the factorial functions in the denominator. Then the solution $y(x)$ may be written as

$$y(x) = a_0\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right)$$

$$+ a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right).$$

At this stage the constants a_0 and a_1 are undetermined. This is not surprising since we are dealing with a *second* order differential equation.

We have discovered that the general solution to the SHO equation may be written as a linear combination of the two bracketed functions. We should give these names. We can define

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

and in terms of these "special" functions the solution to the SHO equation is
$$y(x) = A\cos x + B\sin x .$$

Having discovered two new functions, the solutions of the SHO equation, we would want to investigate their properties. The first thing to do would be to make plots of the functions. We know that increasing the number of terms in a power series will improve its approximation to the original function. The figure shows the series for our $\cos x$ up to terms in x^{10}, x^{20}, x^{30}, x^{40}, x^{50}, and x^{60}.
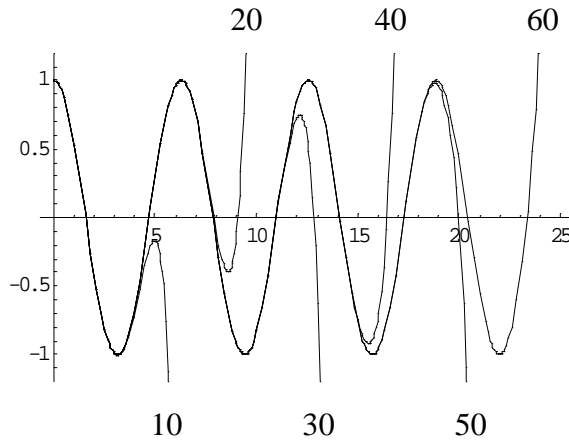
Series approximations to $\cos x$ with 10, 20, 30, 40, and 50 terms

We can see the periodicity property of the function emerging. *We* know that
$$\sin(x + 2\pi) = \sin x .$$
(I am not sure if there is a simple way of deriving this from the differential equation and/or the series solutions.)

The differential properties of the sine and cosine follow directly from the series solutions:
$$\frac{d}{dx}\cos x = \frac{d}{dx}\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...\right)$$
$$= -\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...\right)$$
$$= -\sin x$$

so we conclude
$$\frac{d}{dx}\cos x = -\sin x$$
$$\frac{d}{dx}\sin x = \cos x$$
.

The important concepts of this section are:

- This simple power series method works only for "well-behaved" functions.

- The basic idea is to substitute the power series into the differential equation.

- With a slight juggling of the dummy summation variables the equation is cast into the form $\sum_n \{...\} x^n = 0$.

- Each power of x must equate to zero.

- This gives a recurrence relation for the coefficients.

- A 2$^{\text{nd}}$ order ODE has two independent solutions, one built up from the a_0 coefficient and one from the a_1 coefficient.

- Properties of the functions are obtained from the series solutions and the original ODE.