Orthogonal Functions

Motivation – Analogy with vectors

You are probably familiar with the concept of orthogonality from vectors; two vectors are orthogonal when they make an angle of 90º with each other. We will explore the utility of this property of (basis) vectors, before applying the ideas by analogy to functions.

A vector **v** in 3d space can be expressed in terms of its components

$$\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_x\hat{\mathbf{k}} .$$

Here $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are unit vectors (basis vectors) in the *x*, *y*, and *z* directions, and v_x, v_y, and v_z are the components of the vector in these directions.

The physical vector **v** is specified when we have a) the basis set of unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ and b) the coefficients v_x, v_y, and v_z. You can change the basis set (changing the orientation of the coordinate frame). Then the coefficients v_x, v_y, and v_z will change, but they describe the same physical vector.

The orthogonality of $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ is expressed in terms of the dot product relations

$$\begin{aligned}\hat{\mathbf{i}}.\hat{\mathbf{j}} = \hat{\mathbf{j}}.\hat{\mathbf{k}} = \hat{\mathbf{k}}.\hat{\mathbf{i}} = 0\\ \hat{\mathbf{i}}.\hat{\mathbf{i}} = \hat{\mathbf{j}}.\hat{\mathbf{j}} = \hat{\mathbf{k}}.\hat{\mathbf{k}} = 1\end{aligned} .$$

⌈It is an important property of the dot product that it takes two vectors and gives a scalar from them.⌋

Determining the coefficients

Given a vector **v** and a basis set $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$, the orthogonality of the basis set allows us to determine the coefficients $\{v_x, v_y, v_z\}$.

To find the $\{v_x, v_y, v_z\}$ we take the dot product of **v** with the $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$. Recall that the dot product of **v** with a vector **i** gives the component of **v** along **i**: $\mathbf{v}.\mathbf{i} = vi\cos\theta$.

Taking the dot product we find

$$\begin{aligned}\mathbf{v}.\hat{\mathbf{i}} &= \left(v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_x\hat{\mathbf{k}}\right).\hat{\mathbf{i}}\\ &= v_x\hat{\mathbf{i}}.\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}.\hat{\mathbf{i}} + v_x\hat{\mathbf{k}}.\hat{\mathbf{i}}\end{aligned} .$$

Orthogonality causes the last two terms to vanish, and $\mathbf{i}.\mathbf{i} = 1$, so that

$$\mathbf{v}.\hat{\mathbf{i}} = v_x .$$

Applying the same procedure to the *y* and the *z* components we obtain

$$\begin{aligned} v_x &= \mathbf{v}.\hat{\mathbf{i}} \\ v_y &= \mathbf{v}.\hat{\mathbf{j}} \;. \\ v_z &= \mathbf{v}.\hat{\mathbf{k}} \end{aligned}$$

More compact notation

We can simplify the equations by using the symbol α (or any other Greek character) to denote the directions *x, y, z*. Thus if we denote

$$v_x, v_y, v_z \;\text{ by }\; v_\alpha \;\text{ where }\; \alpha = x, y, z$$

$$\hat{\mathbf{i}},\; \hat{\mathbf{j}},\; \hat{\mathbf{k}} \quad\text{by}\; \hat{\mathbf{i}}_\alpha \;\text{ where }\; \alpha = x, y, z$$

then we may express **v** in terms of its components as

$$\mathbf{v} = \sum_{\alpha = x,y,z} v_\alpha \hat{\mathbf{i}}_\alpha \;.$$

Here

v is the vector

$\{v_\alpha\}$ represent the coefficients

$\{\hat{\mathbf{i}}_\alpha\}$ represent the basis vectors.

The orthogonality of the basis vectors is expressed by

$$\begin{aligned} \hat{\mathbf{i}}_\alpha.\hat{\mathbf{i}}_\beta &= 0 \quad\text{if}\quad \alpha \neq \beta \\ &= 1 \quad\text{if}\quad \alpha = \beta \end{aligned}.$$

And this may be written in an even more compact form by the use of the Kroneker delta symbol $\delta_{\alpha\beta}$ which has the defining property

$$\begin{aligned} \delta_{\alpha\beta} &= 0 \quad\text{if}\quad \alpha \neq \beta \\ &= 1 \quad\text{if}\quad \alpha = \beta \end{aligned}.$$

In terms of the Kroneker delta symbol the orthogonality of the basis vectors is simply

$$\hat{\mathbf{i}}_\alpha.\hat{\mathbf{i}}_\beta = \delta_{\alpha\beta} \,.$$

Determining the coefficients (again)

Using the more compact notation together with the Kroneker delta symbol, we shall see how to obtain the vector's coefficients in a more straightforward manner.

The vector is written, in terms of its coefficients and basis vectors as

$$\mathbf{v} = \sum_{\alpha = x,y,z} v_\alpha \hat{\mathbf{i}}_\alpha \;.$$

We take the dot product of this with a basis vector $\hat{\mathbf{i}}_\beta$. (Since α was used as the dummy variable in the sum, we use a different variable β here.) Taking the dot product gives

$$\mathbf{v}.\hat{\mathbf{i}}_\beta = \sum_{\alpha = x,y,z} v_\alpha \hat{\mathbf{i}}_\alpha.\hat{\mathbf{i}}_\beta \,.$$

But

$$\hat{\mathbf{i}}_\alpha.\hat{\mathbf{i}}_\beta = \delta_{\alpha\beta}$$

so that

$$\mathbf{v}.\hat{\mathbf{i}}_\beta = \sum_{\alpha=x,y,z} v_\alpha \delta_{\alpha\beta} .$$

Now α ranges over x, y, z. But $\delta_{\alpha\beta}$ makes each term zero *except* when $\alpha = \beta$. I.e. the effect of the Kroneker delta is to pick out just the β term from the sum over α. Thus we conclude that

$$\mathbf{v}.\hat{\mathbf{i}}_\beta = v_\beta .$$

So this has determined the coefficients

$$v_\alpha = \mathbf{v}.\hat{\mathbf{i}}_\alpha .$$

⌈Remember that α is a dummy variable, = x, y, z.⌋

The important concepts of this section are:

- Orthogonality
- Basis set of unit vectors
- Coefficients/coordinates
- v_α, $\hat{\mathbf{i}}_\alpha$ notation, where α ranges over x, y, z.
- Kroneker delta symbol
- Determine coefficients, using orthogonality of basis vectors, by taking dot product with each basis vector.

Orthogonal functions – Fourier series

Remember the Fourier series (on the interval $-L \le x \le L$):

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left\{ a_m \cos\left(\frac{m\pi}{L}x\right) + b_m \sin\left(\frac{m\pi}{L}x\right) \right\}.$$

⌈Outside the interval $-L \le x \le L$, $f(x)$ repeats itself – an infinite, spatially periodic function. We might exploit this, or we might just be considering x within the specified interval. ⌋

The $a_0/2$ term is really the $m = 0$ term of the cos series. The factor ½ is convenient as then the same formula (when we find it) holds for all the a_m.

The Fourier series is a bit like writing a vector **v** as a linear sum of basis vectors with appropriate coefficients. Here we have a *function* $f(x)$ written as a linear sum of basis *functions* $\cos(m\pi x/L)$, $\sin(m\pi x/L)$ (integer m) with appropriate coefficients a_m, b_m.

The question is: How to find the coefficients a_m, b_m?

> a_m is the "amount" of $\cos(m\pi x/L)$ in $f(x)$
> b_m is the "amount" of $\sin(m\pi x/L)$ in $f(x)$.

We could do with something like a dot product and the concept of orthogonality.

We will introduce the idea of an "inner product" as the generalization of the dot product. This will take two functions and give a scalar from them.

Consider the following integrals for positive integers m, n.

$$\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\mathrm{d}x = 0 \ \text{ unless } \ m = n$$
$$\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\mathrm{d}x = 0 \ \text{ unless } \ m = n .$$
$$\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\mathrm{d}x = 0 \ \text{ even if } \ m = n$$

We can regard the integrals as giving the inner products of the basis functions $\cos(m\pi x/L)$ and $\sin(m\pi x/L)$.

When $m = n$ the integrals are

$$\int_{-L}^{L} \cos^2\left(\frac{n\pi}{L}x\right)\mathrm{d}x = L$$
$$\int_{-L}^{L} \sin^2\left(\frac{n\pi}{L}x\right)\mathrm{d}x = L .$$

This gives us the orthogonality relations

$$\int_{-L}^{L}\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\mathrm{d}x = L\delta_{mn}$$
$$\int_{-L}^{L}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\mathrm{d}x = L\delta_{mn}\ .$$
$$\int_{-L}^{L}\cos\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\mathrm{d}x = 0$$

⌈Note that this aspect of *orthogonality* has nothing to do with angles; it relates solely to the Kroneker delta properties of the inner products of the basis functions – just as the orthogonality of vectors relates to the Kroneker delta properties of the basis unit vectors⌋

We can use the orthogonality properties of the basis functions $\cos(m\pi x/L)$ and $\sin(m\pi x/L)$ to find the Fourier components a_m and b_m. We use the basic rule:

RULE Take the inner product of the function $f(x)$ and one of the basis functions. In other words, multiply $f(x)$ by one of the basis functions, say $\cos(m\pi x/L)$, and integrate over the interval $-L \le x \le L$.

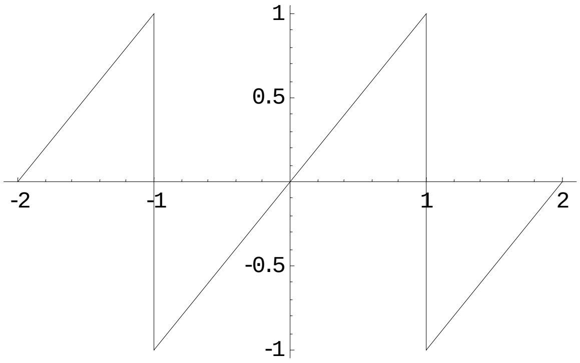
Since
$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left\{a_m\cos\left(\frac{m\pi}{L}x\right) + b_m\sin\left(\frac{m\pi}{L}x\right)\right\}$$
we then have
$$\begin{aligned}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x = &\int_{-L}^{L}\frac{a_0}{2}\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x\\ &+\sum_{m=1}^{\infty}a_m\int_{-L}^{L}\cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x\ .\\ &+\sum_{m=1}^{\infty}b_m\int_{-L}^{L}\sin\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x\end{aligned}$$
So long as $n \neq 0$, the first line is zero. The sin cos integral ensures the third line vanishes. The second line is a sum over integer m and the orthogonality relation picks out the $m = n$ term only. Thus the expression reduces to
$$\begin{aligned}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x &= a_n\int_{-L}^{L}\cos^2\left(\frac{n\pi}{L}x\right)\mathrm{d}x\\ &= a_n L\end{aligned}\quad .$$
This gives the expression for the cos coefficients of the Fourier series. We can perform the same process using the sin basis functions. That is, multiplying $f(x)$ by $\sin(m\pi x/L)$ and integrating over the interval $-L \le x \le L$. This gives, in exactly the same way, the sin coefficients of the Fourier series. Thus we find
$$\begin{aligned}a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x\\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x\end{aligned}\ .$$

These are known as the Euler formulae for the Fourier coefficients. You can see that the formula for a_n gives the correct result for the $n = 0$ term of the series:

$$\frac{a_0}{2} = \frac{1}{2L}\int_{-L}^{L} f(x)\mathrm{d}x.$$

This is the *mean value* of $f(x)$ over the interval.

Example 1 sawtooth wave



Sawtooth wave

This function is specified by $f(x) = \frac{x}{L}, \quad -L \le x \le L$.

The Euler formulae for the Fourier coefficients are

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x.$$

So in this case we have

$$a_n = \frac{1}{L^2}\int_{-L}^{L} x\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x$$
$$b_n = \frac{1}{L^2}\int_{-L}^{L} x\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x.$$

The integral for a_n vanishes. This is because $f(x)$ is an *odd* function. The expression for a_n indicates that an odd function has no cos terms.

If we evaluate the b_n integral using *Mathematica* then it gives the result

$$b_n = \frac{2}{n^2\pi^2}\left(\sin n\pi - n\pi\cos n\pi\right).$$

This is not in its simplest form since *Mathematica* does not know that we are only interested in *integral n*.

Treating quantities like sin$n\pi$

The easiest way of working with quantities like sin$n\pi$ and cos$n\pi$ when n is restricted to integer values is to write these quantities as the imaginary part and the real part of $e^{in\pi}$, and to use de Moivre's theorem.

$$\begin{aligned}\cos n\pi + i\sin n\pi &= e^{in\pi}\\ &= \left(e^{i\pi}\right)^n.\end{aligned}$$

Now $e^{i\pi} = -1$, so that

$$\cos n\pi + i\sin n\pi = (-1)^n$$

and upon taking the real and imaginary parts of this we obtain, for integer n

$$\begin{aligned}\cos n\pi &= (-1)^n\\ \sin n\pi &= 0\end{aligned}.$$

The sin Fourier component of the sawtooth wave

$$b_n = \frac{2}{n^2\pi^2}\left(\sin n\pi - n\pi\cos n\pi\right)$$

is then simply

$$b_n = \frac{-2}{n\pi}(-1)^n.$$

The Fourier series for the sawtooth function

$$f(x) = b_1\sin\left(\frac{\pi}{L}x\right) + b_2\sin\left(\frac{2\pi}{L}x\right) + b_3\sin\left(\frac{3\pi}{L}x\right) + \dots$$

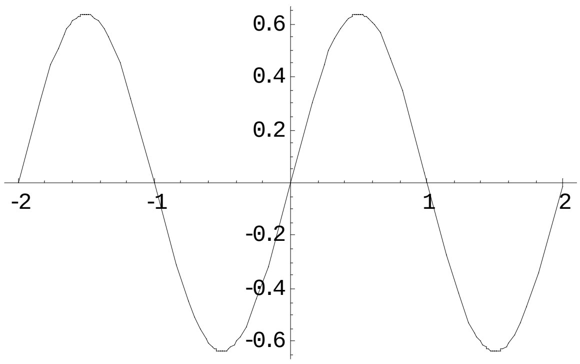
is thus given by

$$f(x) = \frac{2}{\pi}\sin\left(\frac{\pi}{L}x\right) - \frac{1}{\pi}\sin\left(\frac{2\pi}{L}x\right) + \frac{2}{3\pi}\sin\left(\frac{3\pi}{L}x\right) - \frac{1}{2\pi}\sin\left(\frac{4\pi}{L}x\right) + \dots$$

Recall that there are no cos terms because $f(x)$ is an odd function of x.

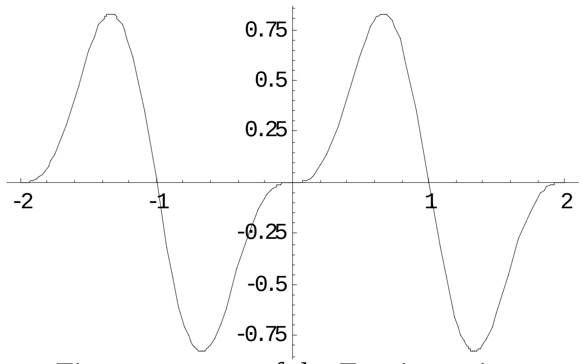
Convergence of the Fourier series

Using just the first term of the series gives the "fundamental" component of the series.



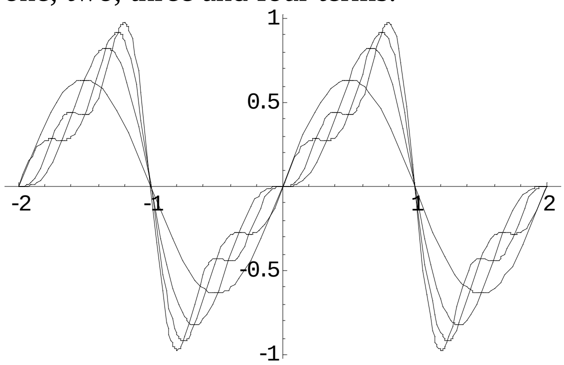
First term of the Fourier series

This reflects the periodicity of the sawtooth function. Adding the second term of the series gives a slight improvement (does it?)



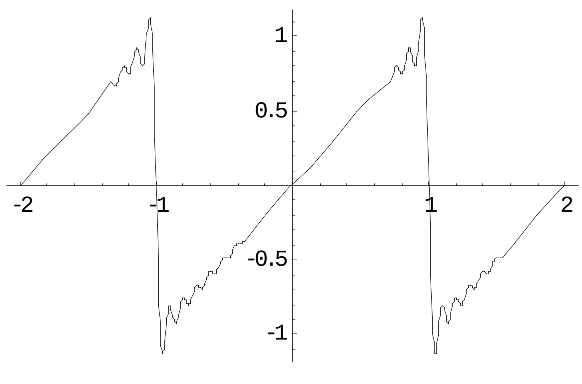
First two terms of the Fourier series

We can see how the terms gradually build up to the required function by looking at the partial sums of the first one, two, three and four terms.



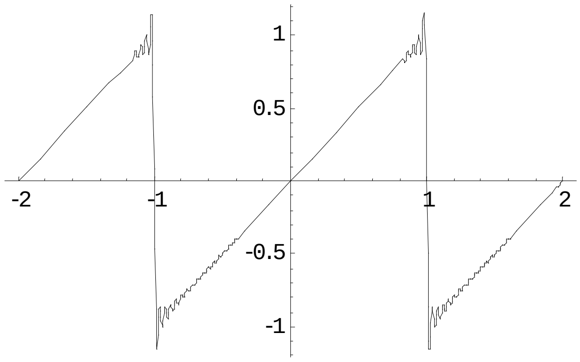
Sums of one, two, three and four terms

For twenty terms the sawtooth is looking pretty realistic. Observe the "wiggles" in the vicinity of the sharp corners. This is known as the *Gibbs phenomenon* and it is a feature of Fourier series when there are discontinuities in the function.



The first 20 terms of the Fourier series

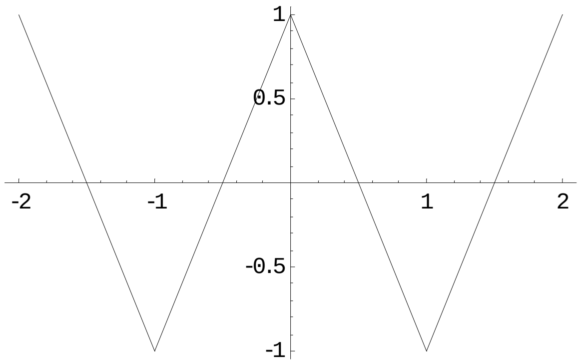
Adding more terms gives an improvement to the problematic parts:



The first 50 terms of the Fourier series

The Gibbs phenomenon is still there, but on a finer scale.

Example 2 triangular wave



Triangular wave

This function must be specified in a piecewise fashion. The function is defined on the interval $-L \le x \le L$. And the triangular profile may be expressed as

$$\begin{aligned} f(x) &= \frac{2}{L}\left(\frac{L}{2}+x\right) \quad \text{for} \quad -L \le x \le 0 \\ &= \frac{2}{L}\left(\frac{L}{2}-x\right) \quad \text{for} \quad 0 \le x \le L \end{aligned}.$$

We want to find the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left\{a_m \cos\left(\frac{m\pi}{L}x\right) + b_m \sin\left(\frac{m\pi}{L}x\right)\right\}$$

for this function, and we have the Euler expression for the coefficients:

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x \end{aligned}.$$

The integrals must be done in two parts since the expression for $f(x)$ is different for positive and for negative x

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x \\ &= \frac{1}{L}\int_{-L}^{0} f(x)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x + \frac{1}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x \\ &= \frac{2}{L^2}\int_{-L}^{0}\left(\frac{L}{2}+x\right)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x + \frac{2}{L2}\int_{0}^{L}\left(\frac{L}{2}-x\right)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x \end{aligned}$$

with a similar expression for the b_n:

$$\begin{aligned} a_n &= \frac{2}{L^2}\int_{-L}^{0}\left(\frac{L}{2}+x\right)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x + \frac{2}{L2}\int_{0}^{L}\left(\frac{L}{2}-x\right)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x \\ b_n &= \frac{2}{L^2}\int_{-L}^{0}\left(\frac{L}{2}+x\right)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x + \frac{2}{L2}\int_{0}^{L}\left(\frac{L}{2}-x\right)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x \end{aligned}$$

These integrals may be evaluated (using *Mathematica* or by hand), giving

$$\begin{aligned} a_n &= \frac{4}{n^2\pi^2}(1-\cos n\pi) \\ b_n &= 0 \end{aligned}.$$

In this case there are no sin terms, because this function is even in x. You can see this general property from the Euler expression for the b_n coefficients. We can simplify the $\cos n\pi$ term as we did in the previous case. There we saw that $\cos n\pi = (-1)^n$, so that

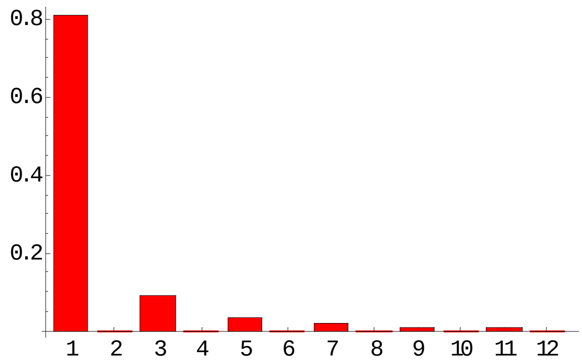
$$a_n = \frac{4}{n^2\pi^2}\left(1-(-1)^n\right).$$

Observe that this is telling us that all the *even* terms vanish (including he constant $n = 0$ term).

The values of a_n are

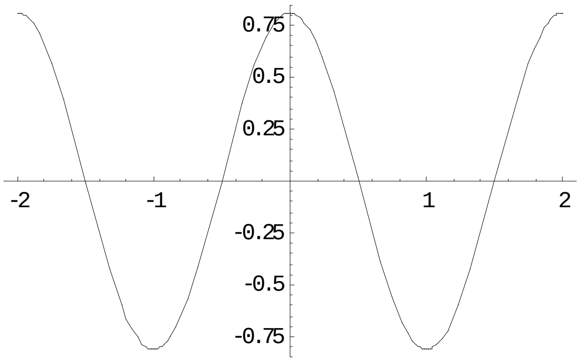
$$a_n = \frac{8}{\pi^2}\left\{1, 0, \frac{1}{3^2}, 0, \frac{1}{5^2}, 0, \frac{1}{7^2}, 0, \frac{1}{9^2}, 0, \frac{1}{11^2}, 0, \ldots\right\}$$

and these may be represented in the bar chart:



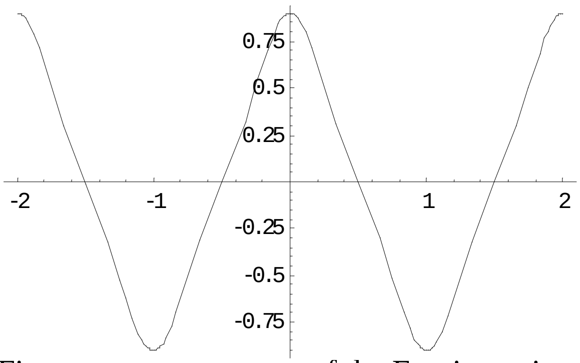
Fourier cos coefficients of triangular wave

Convergence of the Fourier series
Using just the first term of the series gives the "fundamental" component of the series



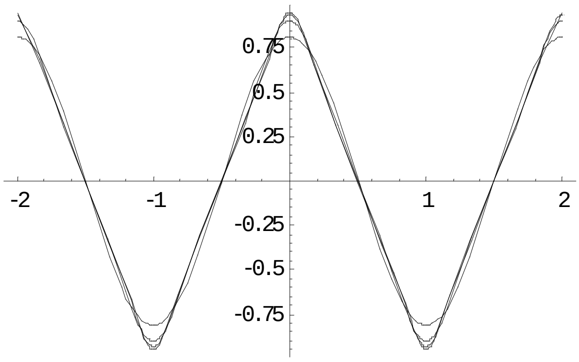
First term of the Fourier series

This reflects the periodicity of the triangular function. Adding the next term of the series gives a significant improvement.



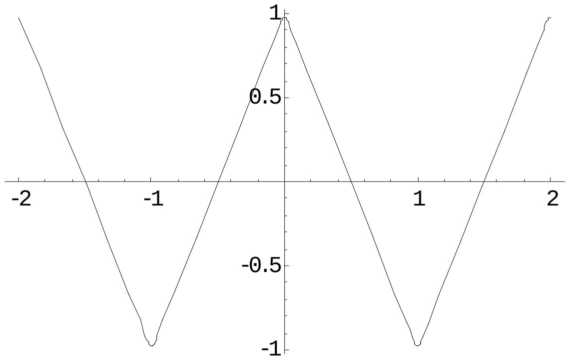
First two nonzero terms of the Fourier series

We can see how the terms gradually build up to the required function by looking at the partial sums of the first, third, fifth and seventh.



Sums of the first, third, fifth and seventh terms

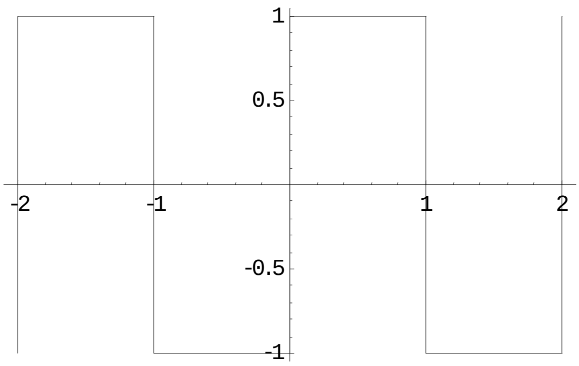
For 20 terms the triangle wave looks very good



Sum of the first twenty terms

There is no Gibbs phenomenon since the function is not discontinuous.

Example 3 square wave



Square wave

The square wave function shown is specified by

$$\begin{aligned} f(x) &= -1 \quad \text{for} \quad -L \le x \le 0 \\ &= +1 \quad \text{for} \quad 0 \le x \le L \end{aligned}.$$

We want to find the Fourier series for this function. We note that as represented here, this is an *odd* function. Thus there are no cos terms in the expansion. The coefficients if the sin terms, the b_n, are given by

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x \\ &= -\frac{1}{L}\int_{-L}^{0}\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x + \frac{1}{L}\int_{0}^{L}\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x \end{aligned}.$$

We can simplify this expression a little by substituting $x \to -x$ in the first term. This gives

$$b_n = \frac{2}{L}\int_{0}^{L}\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x$$

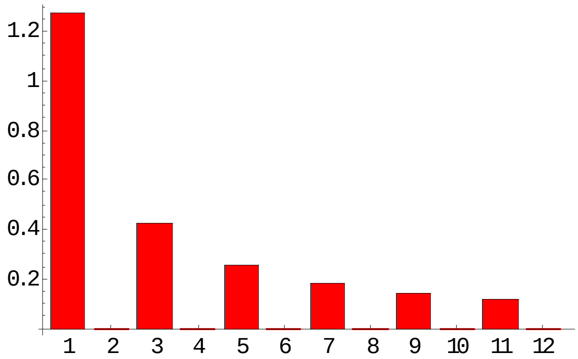
which, upon integration is

$$\begin{aligned} b_n &= \frac{2}{n\pi}(1-\cos n\pi) \\ &= \frac{2}{n\pi}\left(1-(-1)^n\right) \end{aligned}.$$

As in the case of the triangular wave, the even terms vanish. The values of b_n are

$$b_n = \frac{4}{\pi}\left\{1, 0, \frac{1}{3}, 0, \frac{1}{5}, 0, \frac{1}{7}, 0, \frac{1}{9}, 0, \frac{1}{11}, 0, \ldots\right\}$$

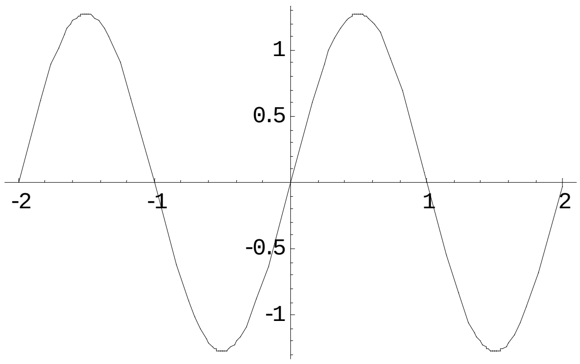
and these may be represented in the bar chart:



Fourier sin coefficients of square wave

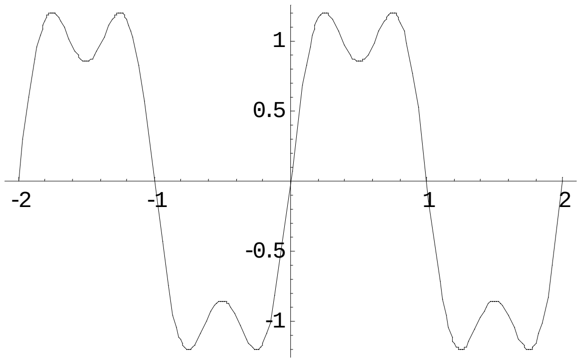
Convergence of the Fourier series

Using just the first term of the series gives the "fundamental" component of the series



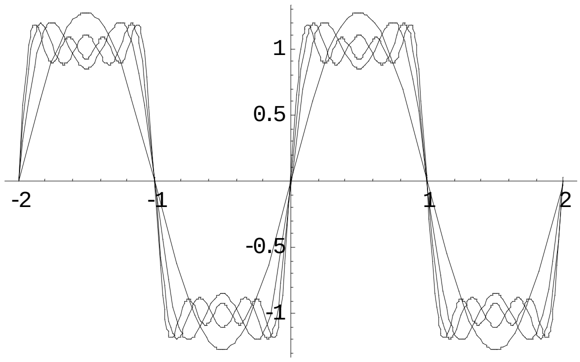
First term of the Fourier series

This reflects the periodicity of the square wave. Adding the next term looks a little better



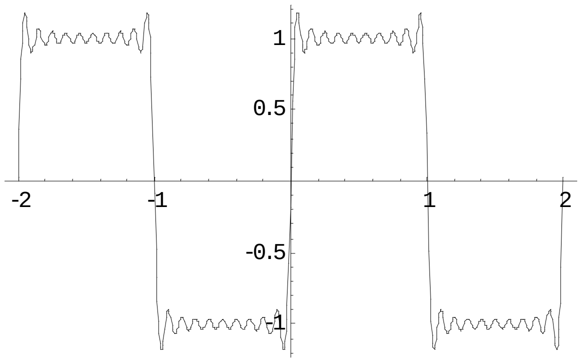
Fundamental and next (3rd) harmonic of square wave

We can see how the terms gradually build up to the required function by looking at the partial sums of the first, third, fifth and seventh



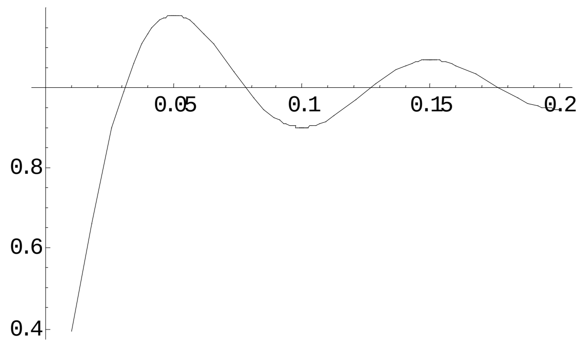
Gradual build-up of square wave

The sum up to the 20th harmonic looks quite good.



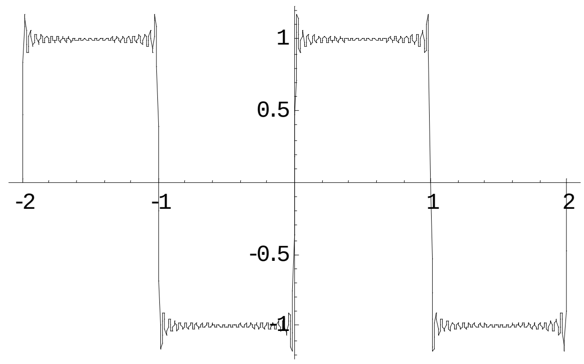
Fourier series for square wave up to 20th harmonic

The Gibbs phenomenon is apparent in the vicinity of the discontinuities. This may be seen most clearly in the expanded region



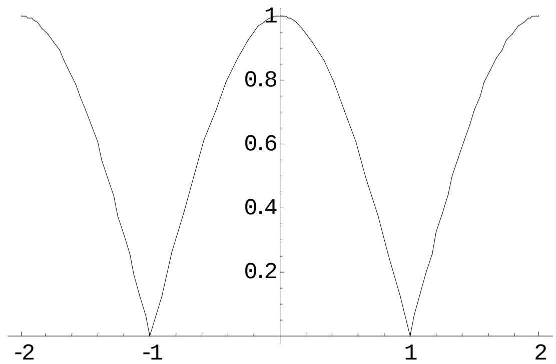
Vicinity of $x = 0$ showing the Gibbs phenomenon

The sum to the 50th harmonic looks much improved, but the Gibbs phenomenon is still present.



Sum to 50th harmonic

Example 4 Full wave rectified sine curve
In full wave rectification positive regions of a signal remain positive, and negative regions of a signal are inverted to that they, also, appear positive. We shall consider a full wave rectifies sin curve(actually a full wave rectified cos curve).



Full wave rectified (co)sine wave

The function is specified in the interval $-L \le x \le L$. In this interval $f(x)$ is given, simply, by

$$f(x) = \cos\left(\frac{\pi}{2L}x\right).$$

This is an even function so we know that there will only be cos terms in the Fourier series. The Euler formula gives the Fourier cos coefficients

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} \cos\left(\frac{\pi}{2L}x\right)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x \\ &= \frac{4}{\pi}\frac{\cos n\pi}{1-4n^2} \end{aligned}.$$

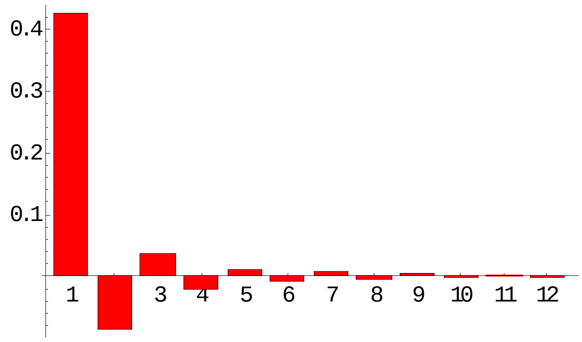
Using the trick for $\cos n\pi$ we write this as

$$a_n = \frac{4}{\pi}\frac{(-1)^n}{1-4n^2}.$$

The values of a_n are

$$a_n = \frac{4}{\pi}\left\{\frac{1}{3}, -\frac{1}{15}, \frac{1}{35}, -\frac{1}{63}, \frac{1}{99}, -\frac{1}{143}, \frac{1}{195}, -\frac{1}{255}, \ldots\right\}$$

and these may be represented in the bar chart:

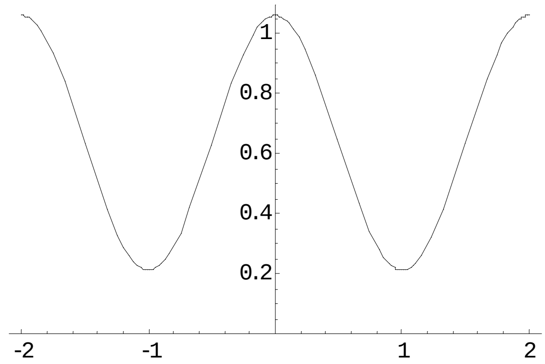


Fourier coefficients of full wave rectified cos wave

In this case there is an $n = 0$ (constant) term, since the mean of the function is not zero. This is given by $a_0 = 4/\pi$.

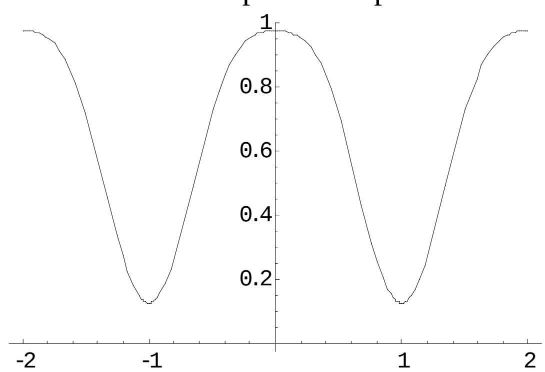
Convergence of the Fourier series

Using the constant a_0 together with just the first term of the series gives the "fundamental" component of the series



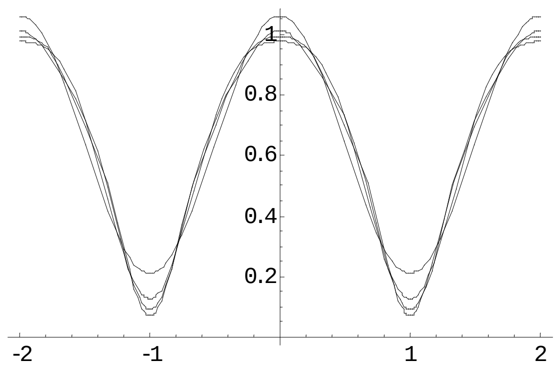
Fundamental of full rectified sine wave

Adding the second harmonic flattens the top and sharpens the bottom.



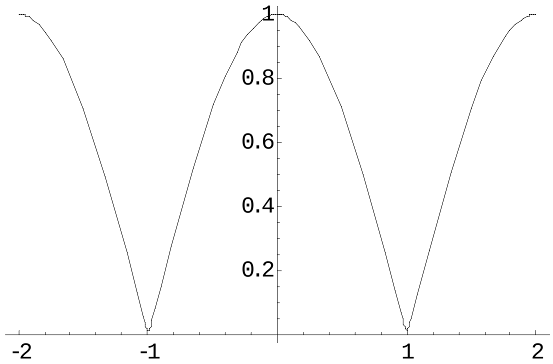
Terms up to the second harmonic

The first few partial sums show how the Fourier series approaches the function



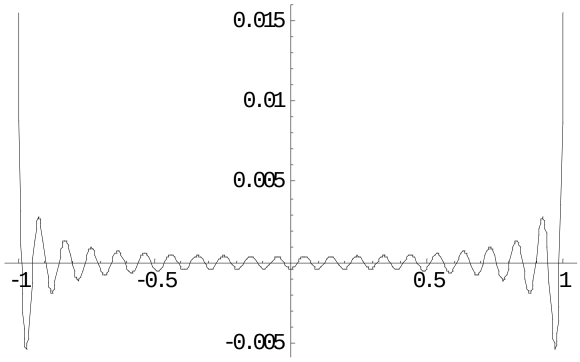
Gradual build-up of full wave rectified sine wave

With the sum to 20 terms the function is looking pretty good.



Sum to 20 terms

Finally we plot the difference between the 20 term series and the original function.



Error in 20 term Fourier series

The important concepts of this section are:

- Expression of a function as a Fourier series
- Orthogonality integrals for sines and cosines
- Euler formulae for Fourier coefficients
- For piecewise defined functions do Euler integrals in separate bits
- Odd functions use only sines; even functions use only cosines
- Constant a_0 term, needed when mean of function is not zero
- Gibbs phenomenon when function is discontinuous.
- Terminology: *fundamental*, *harmonics*.
- Small number of terms needed when the coefficients decrease rapidly