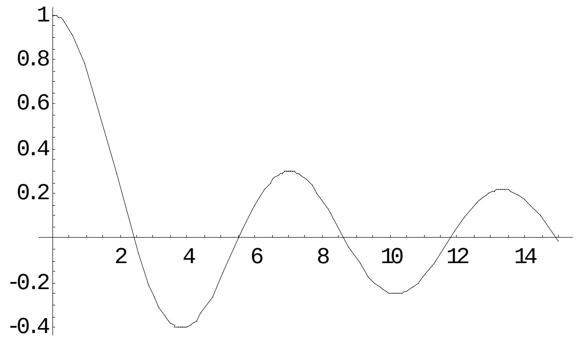
Orthogonal Functions II

Orthogonal functions – Fourier-Bessel series

Remember the J_0 Bessel function appeared as the radial part of the solution of the wave equation with cylindrical symmetry. The J_0 Bessel function looks like this.



The $J_0(r)$ Bessel function

Unlike the sin and cos functions, the zeros of the Bessel functions are not equally spaced. So if we are trying to satisfy the boundary condition of $f(R) = 0$ then only certain values of the arguments of the J_0 function are allowed since there must be a zero at the boundary. We are concerned with the interval $0 \le r \le R$.

We denote the m^{th} zero of J_0 by α_m. That is,

$$J_0(\alpha_m) = 0 .$$

Then

$$J_0\left(\frac{\alpha_m}{R} r\right) = 0 \quad \text{when} \quad r = R .$$

So if we have the boundary condition

$$f(R) = 0$$

then

$$f(r) = \sum_{m=1}^{\infty} a_m J_0\left(\frac{\alpha_m}{R} r\right)$$

will satisfy the boundary condition. This series is called the Fourier-Bessel series.

Assuming we can write $f(r)$ as a Fourier-Bessel series, the question is how to determine the coefficients a_m. This will be helped significantly if the $J_0(\alpha_m r / R)$ are an orthogonal set of functions, for positive integer m. We shall see that they are.

The orthogonality integral is

$$I = \int_0^R r\ J_0\left(\frac{\alpha_m}{R} r\right) J_0\left(\frac{\alpha_n}{R} r\right) \mathrm{d}r .$$

Observe the factor r in the integral. This is called a weight function, and it is part of the definition of the appropriate inner product for these basis functions. Mathematica can perform the integral for us, giving (at this stage α_m and α_n are just variables):

$$I = \frac{R^2}{\alpha_m^2 - \alpha_n^2}\left\{\alpha_m J_0(\alpha_n) J_1(\alpha_m) - \alpha_n J_0(\alpha_m) J_1(\alpha_n)\right\}.$$

Here we see the J_1 Bessel function has appeared. Since we know that $J_0(\alpha_m)$ and $J_0(\alpha_n)$ are zero, we can be sure that I will vanish, except possibly in the case when $m = n$. Since then both the numerator and the denominator vanish, we must look a little more closely. The best thing to do is to evaluate the integral directly for $m = n$. *Mathematica* gives the following result; here again the J_1 Bessel function appears. The normalisation factor is

$$\int_0^R r\, J_0^2\left(\frac{\alpha_m}{R} r\right) \mathrm{d}r = \frac{R^2}{2} J_1^2(\alpha_m).$$

This enables us to write the orthogonality relation for the J_0 Bessel functions as

$$\int_0^R r\, J_0\left(\frac{\alpha_m}{R} r\right) J_0\left(\frac{\alpha_n}{R} r\right) \mathrm{d}r = \frac{R^2}{2} J_1^2(\alpha_m)\, \delta_{mn}.$$

Determining the Bessel coefficients

The coefficients of the Fourier-Bessel series

$$f(r) = \sum_{m=1}^{\infty} a_m J_0\left(\frac{\alpha_m}{R} r\right)$$

may be determined from the orthogonality properties of the J_0 Bessel functions. To find the a_m we use the basic rule:

RULE Take the inner product of the function $f(r)$ and one of the basis functions. In other words, multiply $f(r)$ by r and by one of the basis functions, say $J_0(\alpha_n r / R)$ and integrate over the interval $0 \le r \le R$.

We thus have

$$\int_0^R r\, f(r) J_0\left(\frac{\alpha_n}{R} r\right) \mathrm{d}r = \sum_{m=0}^{\infty} a_m \int_0^R r\, J_0\left(\frac{\alpha_m}{R} r\right) J_0\left(\frac{\alpha_n}{R} r\right) \mathrm{d}r.$$

On the right hand side we recognize the orthogonality relation for the basis functions. This gives

$$\begin{aligned}\int_0^R r\, f(r) J_0\left(\frac{\alpha_n}{R} r\right) \mathrm{d}r &= \sum_{m=0}^{\infty} a_m \frac{R^2}{2} J_1^2(\alpha_m) \delta_{mn} \\ &= a_n \frac{R^2}{2} J_1^2(\alpha_n)\end{aligned}$$

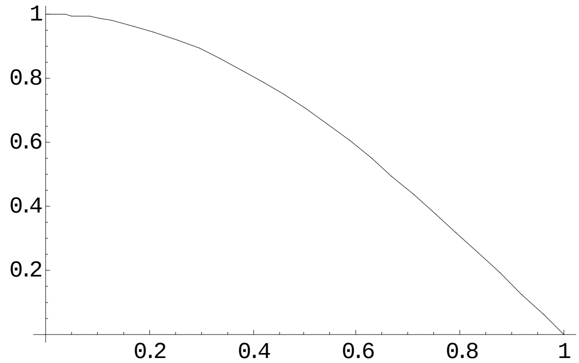
since the Kroneker delta has the effect of picking out just the n term of the sum.

Finally, this gives the expression for the Bessel coefficients as

$$a_n = \frac{2}{R^2 J_1^2(\alpha_n)} \int_0^R r\, f(r) J_0\left(\frac{\alpha_n}{R} r\right) \mathrm{d}r\ .$$

This is the analogue of the Euler formula, for the Bessel coefficients.

Example 1 Cos segment



Quarter of a cosine cycle

The function is defined on the interval $0 \le r \le R$. It is specified by

$$f(r) = \cos\left(\frac{\pi}{2R} r\right).$$

We want to find the Fourier-Bessel series

$$f(r) = \sum_{m=1}^{\infty} a_m J_0\left(\frac{\alpha_m}{R} r\right)$$

for this function. This will be done with the formula we have obtained for the Bessel coefficients

$$a_n = \frac{2}{R^2 J_1^2(\alpha_n)} \int_0^R r\, f(r) J_0\left(\frac{\alpha_n}{R} r\right) \mathrm{d}r\ .$$

In this case the integral for the coefficients is

$$a_n = \frac{2}{R^2 J_1^2(\alpha_n)} \int_0^R r \cos\left(\frac{\pi}{2R} r\right) J_0\left(\frac{\alpha_n}{R} r\right) \mathrm{d}r\ .$$

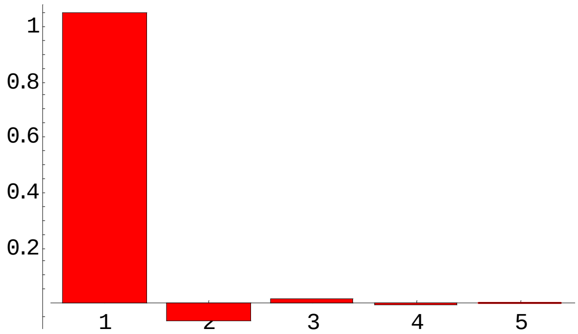
The integral is best simplified through the substitution $\rho = r / R$, giving

$$a_n = \frac{2}{J_1^2(\alpha_n)} \int_0^1 \rho \cos(\pi\rho / 2) J_0(\alpha_n \rho) \mathrm{d}\rho\ .$$

Mathematica can do these integrations numerically, for different *n*. The first few coefficients are

a_1 = 1.04952
a_2 = -0.0627929
a_3 = 0.0188382
a_4 = -0.00848181
a_5 = 0.00465136

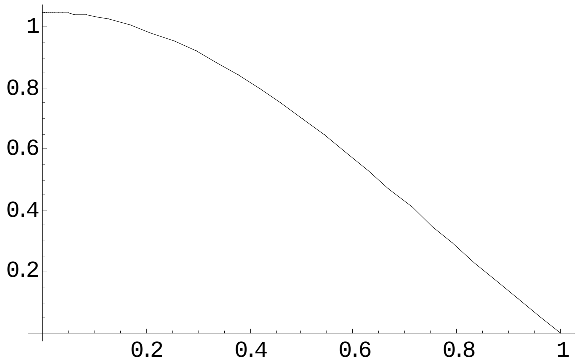
We can represent the coefficients in a bar chart:



Bessel coefficients of cosine quarter cycle

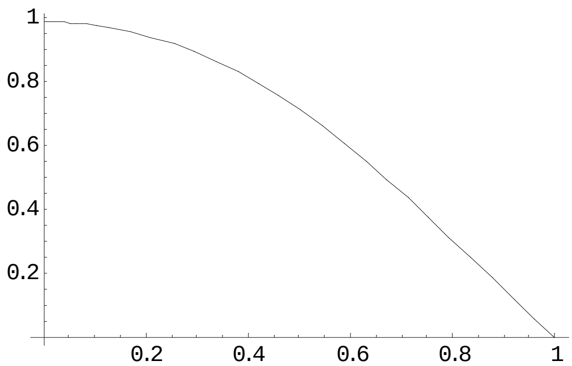
Convergence of the Fourier-Bessel series

Using just the first term of the series gives the "fundamental" component



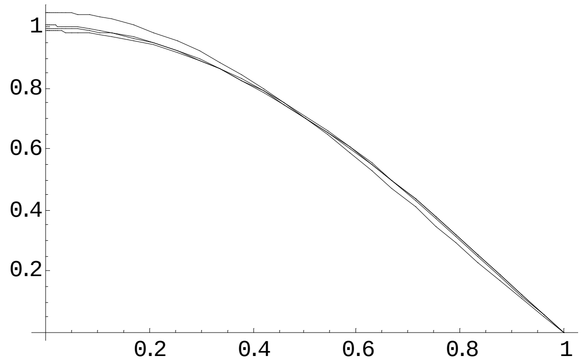
First term of the Fourier-Bessel series

This respects the boundary condition at $r/R = 1$. Adding the next term gives a slight improvement



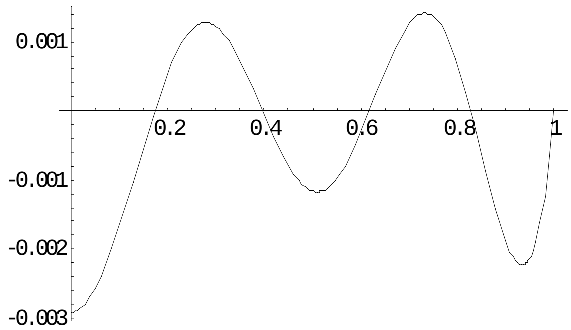
First two terms of the Fourier-Bessel series

We can see how the terms gradually build up by looking at the first few partial sums



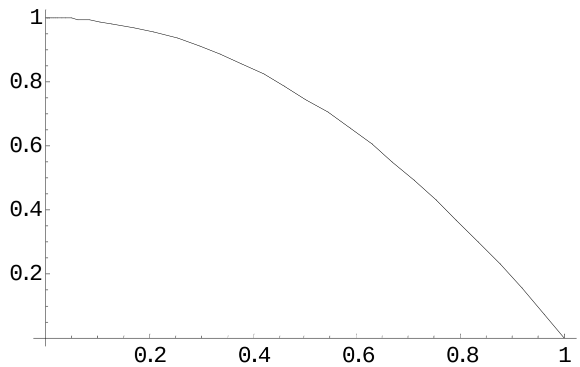
Sums of one, two, three and four terms of the Fourier-Bessel series

Finally it is instructive to examine the error resulting after using just four terms of the series. The graph below shows the difference between this approximation and the original function



Error in the Fourier-Bessel series

Example 2 Parabola



Parabola segment

The function is specified by

$$f(r) = 1 - \left(\frac{r}{R}\right)^2 \qquad 0 \le r \le R$$

so the expression for the Bessel coefficients is

$$a_n = \frac{2}{R^2 J_1^2(\alpha_n)} \int_0^R r\left(1 - \frac{r}{R}\right)^2 J_0\left(\frac{\alpha_n}{R} r\right) \mathrm{d}r \ .$$

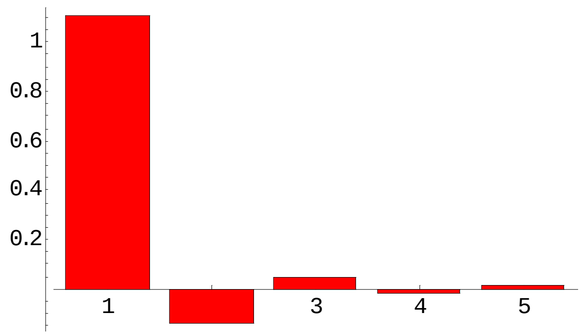
The integral is best simplified through the substitution $\rho = r / R$, giving

$$a_n = \frac{2}{J_1^2(\alpha_n)} \int_0^1 \rho(1-\rho)^2 J_0(\alpha_n \rho) \mathrm{d}\rho \, .$$

Mathematica can do these integrations numerically, for different *n*. The first few coefficients are

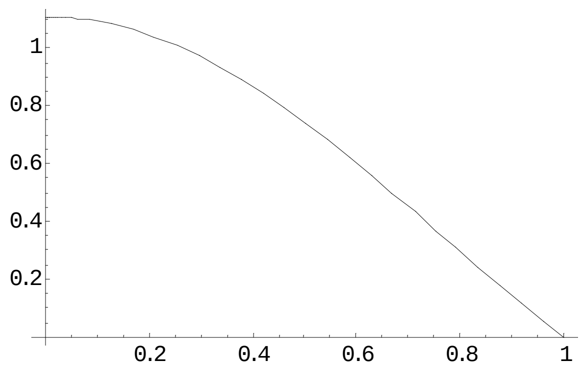
a_1 = 1.10802226121862812
a_2 = -0.139777505298383196
a_3 = 0.0454764707273626722
a_4 = -0.0209909018240905798
a_5 = 0.0116362429798040678

We can represent the coefficients in a bar chart



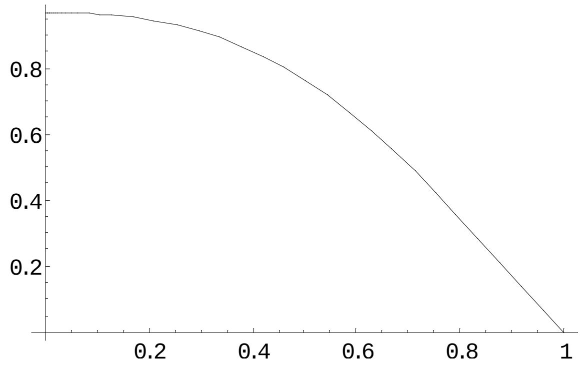
Bessel coefficients of a parabolic segment

Convergence of the Fourier-Bessel series
Using just the first term of the series gives the "fundamental" component



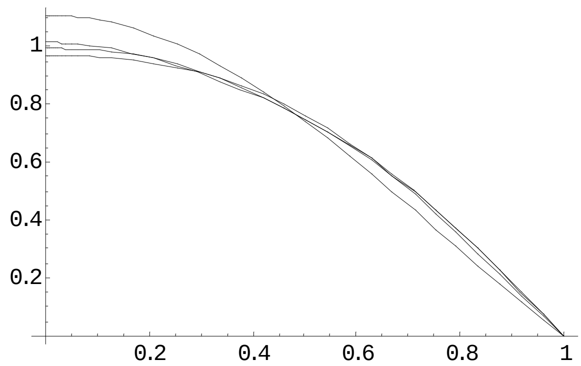
First term of the Fourier-Bessel series

As in the cosine case, this is a fairly good approximation already. The boundary condition at $r/R = 1$ is obeyed (as it must be). Adding the next term gives a slight improvement



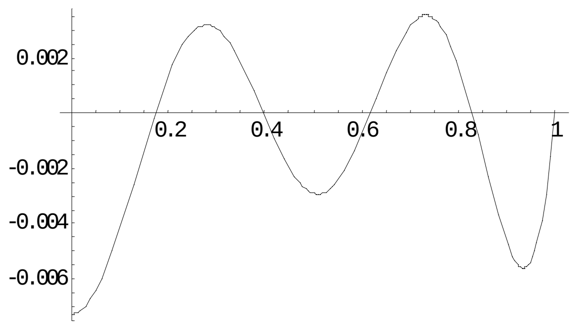
First two terms of the Fourier-Bessel series

We can see how the terms gradually build up by looking at the first few partial sums



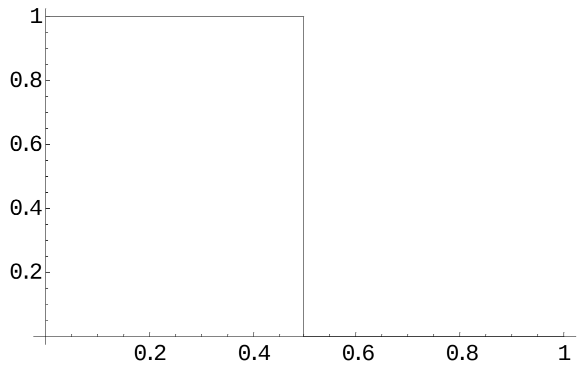
Sums of one, two, three and four terms of the Fourier-Bessel series

Finally it is instructive to examine the error resulting after using just four terms of the series. The graph below shows the difference between this approximation and the original function



Error in the Fourier-Bessel series

Example 3 Box function



Box function

This function is defined by

$$\begin{aligned} f(r) &= 1 \qquad 0 \le r \le R/2 \\ &= 0 \qquad R/2 \le r \le R \end{aligned}.$$

In this case the integral for the Bessel coefficients is

$$a_n = \frac{2}{R^2 J_1^2(\alpha_n)} \int_0^{R/2} r\, J_0\!\left(\frac{\alpha_n}{R} r\right) \mathrm{d}r .$$

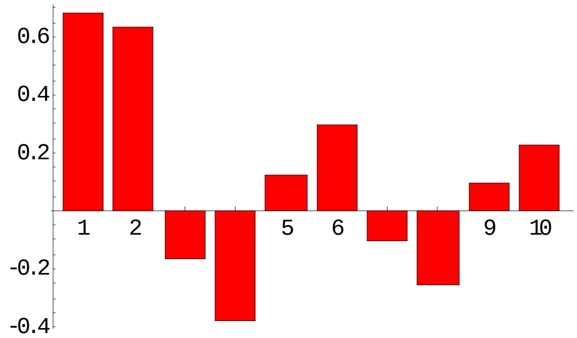
The integral is best simplified through the substitution $\rho = r/R$, giving

$$a_n = \frac{2}{J_1^2(\alpha_n)} \int_0^{1/2} \rho\, J_0(\alpha_n \rho)\, \mathrm{d}\rho .$$

Mathematica can do these integrations numerically, for different *n*. The first few coefficients are

a_1 = 0.679913,
a_2 = 0.630669,
a_3 = -0.162771,
a_4 = -0.377686,
a_5 = 0.124119,
a_6 = 0.297036,
a_7 = -0.105339,
a_8 = -0.253308,
a_9 = 0.0933174,
a_{10} = 0.224716

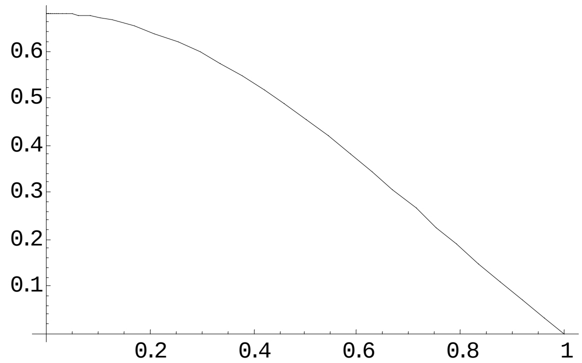
We can represent the coefficients in a bar chart



Bessel coefficients of box function

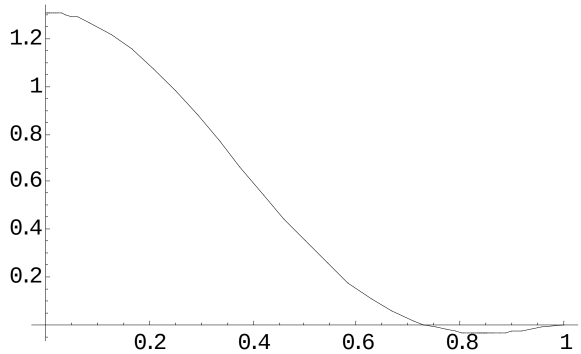
Convergence of the Fourier-Bessel series

Using just the first term of the series gives the "fundamental" component



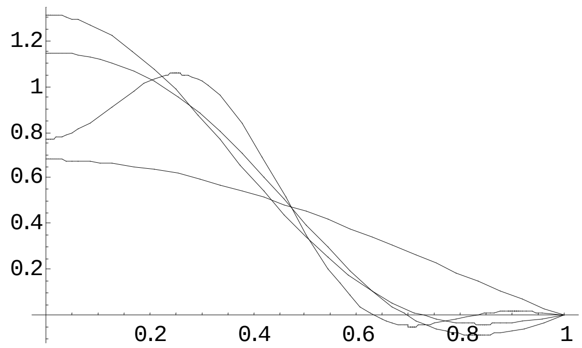
First term of the Fourier-Bessel series

This really does not look anything like the box function. Adding the second term makes a very slight improvement:



First two terms of the Fourier-Bessel series

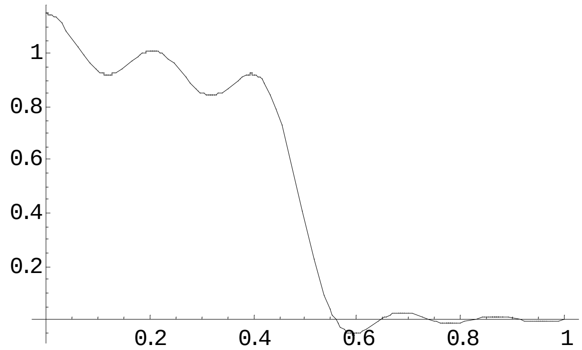
We see that adding the second term has caused the curve to go below zero, which helps with sharpening up the discontinuity at $r/R = ½$. If we liik at the first few partial sums together then we can see how the various terms contribute to making an approximation to the box function



Sums of one, two, three and four terms of the Fourier-Bessel series

It is clear that we need quite a few terms to make a realistic approximation to the box function. That is why we evaluated so many coefficients originally. This should also be apparent from the bar graph of coefficients; they do not fall off terribly fast, as the cos and parabola ones did.

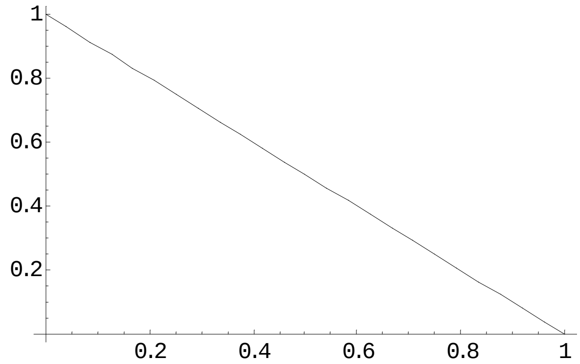
The sum of ten terms looks like it is going in the right direction:



Ten term Fourier-Bessel series for box function

The "wiggles" here are the Gibbs phenomenon, because of the discontinuity in the function.

Example 4 Triangle function



Triangle function

This function is defined by

$$f(r) = 1 - r/R \qquad 0 \le r \le R.$$

In this case the integral for the Bessel coefficients is

$$a_n = \frac{2}{R^2 J_1^2(\alpha_n)} \int_0^R r\left(1 - \frac{r}{R}\right) J_0\left(\frac{\alpha_n}{R} r\right) \mathrm{d}r.$$

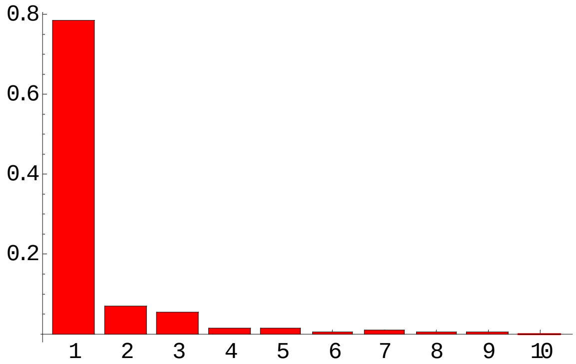
The integral is best simplified through the substitution $\rho = r/R$, giving

$$a_n = \frac{2}{J_1^2(\alpha_n)} \int_0^1 \rho(1-\rho) J_0(\alpha_n \rho) \mathrm{d}\rho.$$

Mathematica can do these integrations numerically, for different n. The first few coefficients are

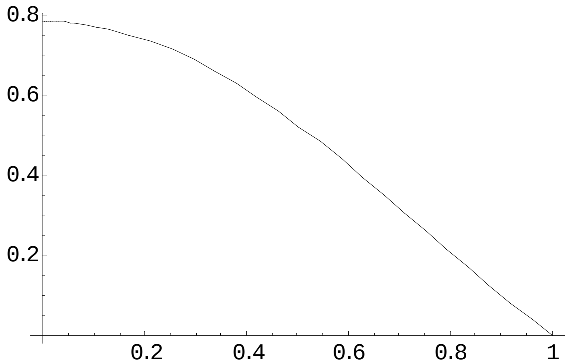
a_1 = 0.784519
a_2 = 0.0686889
a_3 = 0.0531141
a_4 = 0.0173627
a_5 = 0.0169808
a_6 = 0.00781659
a_7 = 0.00818725
a_8 = 0.00444123
a_9 = 0.00478526
a_{10} = 0.00292601.

We can represent the coefficients in a bar chart



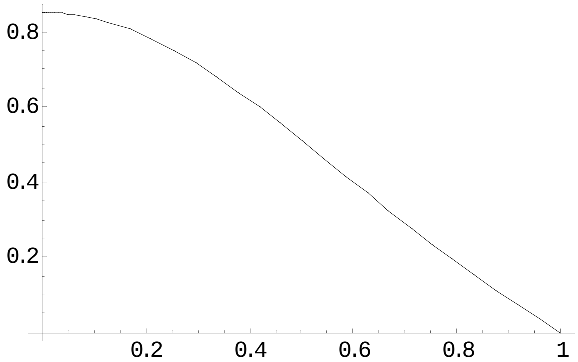
Bessel coefficients of triangle function

Convergence of the Fourier-Bessel series
Using just the first term of the series gives the “fundamental” component



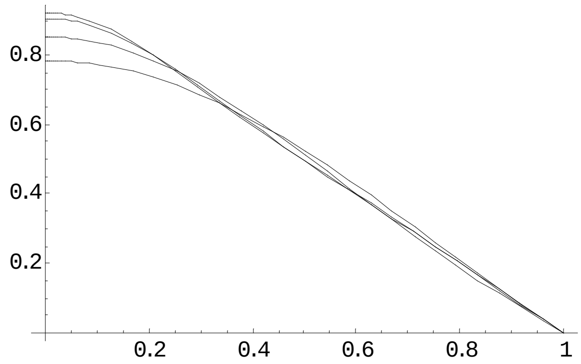
First term of the Fourier-Bessel series

The latter part of this curve looks reasonably straight but the beginning is all wrong.
Adding the second term makes a slight improvement:



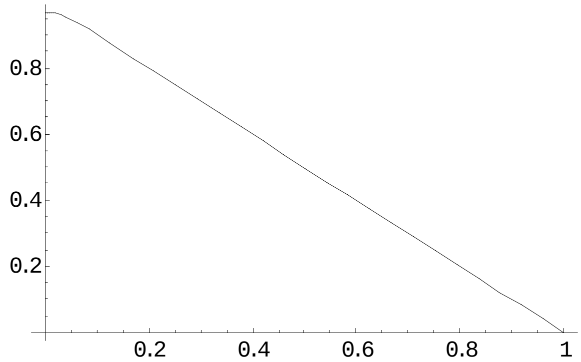
First two terms of the Fourier-Bessel series

We can see that adding more terms to the series improves the short – r behaviour. It is obvious that there will always be a problem at $r = 0$ since the Bessel functions are flat there, while we want a non-vanishing first derivative for the triangle.



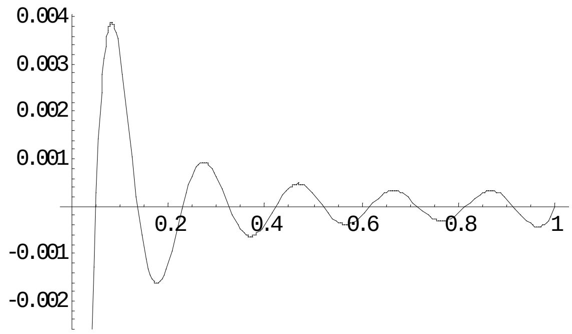
Sum of one, two, three and four terms of the Fourier-Bessel series

With ten terms to the series things are looking pretty good



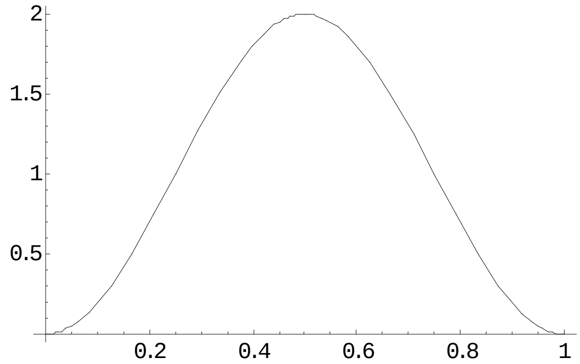
Ten term Fourier-Bessel series for the triangle function

Finally we show the error resulting after using ten terms of the series. The graph below shows the difference between this approximation and the original function.



Error in the Fourier-Bessel series

Example 5 Full cycle of a sine curve



Full cycle of a sine curve

The figure shows a full cycle of a sine curve, shifted so that the boundary condition $f(R)=0$ is satisfied. The function is defined by

$$f(r)=1-\cos(2\pi r/R) \qquad 0\le r\le R\,.$$

In this case the integral for the Bessel coefficients is

$$a_n=\frac{2}{R^2J_1^2(\alpha_n)}\int_0^R r\left(1-\cos\left(\frac{2\pi r}{R}\right)\right)J_0\left(\frac{\alpha_n}{R}r\right)\mathrm{d}r\,.$$

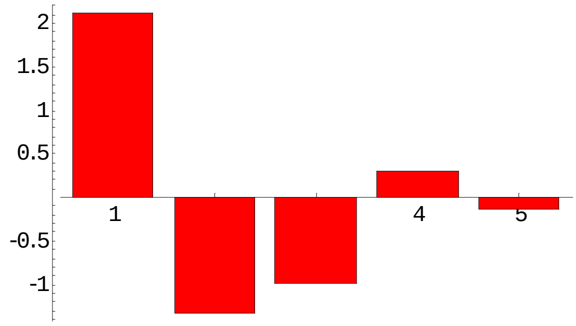
The integral is best simplified through the substitution $\rho=r/R$, giving

$$a_n=\frac{2}{J_1^2(\alpha_n)}\int_0^1\rho\left(1-\cos(2\pi\rho)\right)J_0(\alpha_n\rho)\,\mathrm{d}\rho\,.$$

Mathematica can do these integrations numerically, for different *n*. The first few coefficients are

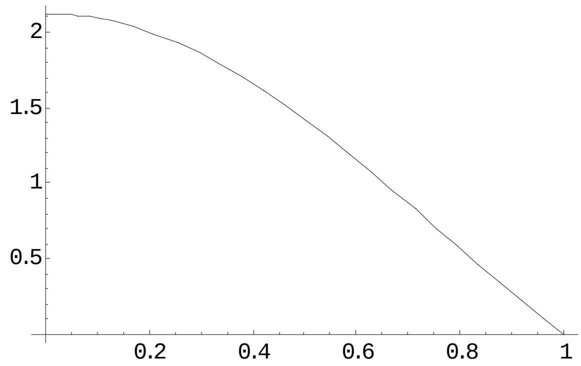
a_1 = 2.12341
a_2 = -1.33636
a_3 = -0.989625
a_4 = 0.292942
a_5 = -0.140432

We can represent the coefficients as a bar chart



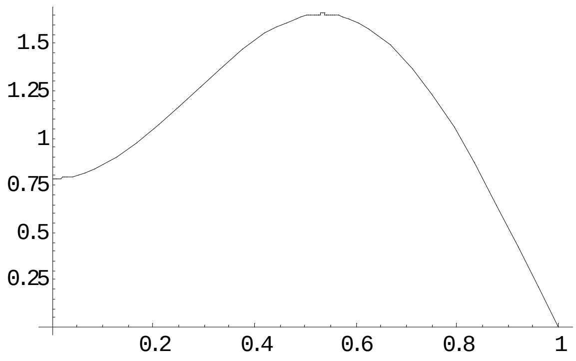
Fourier-Bessel coefficients of a full sine cycle

Convergence of the Fourier-Bessel series
Using just the first term of the series gives the "fundamental" component



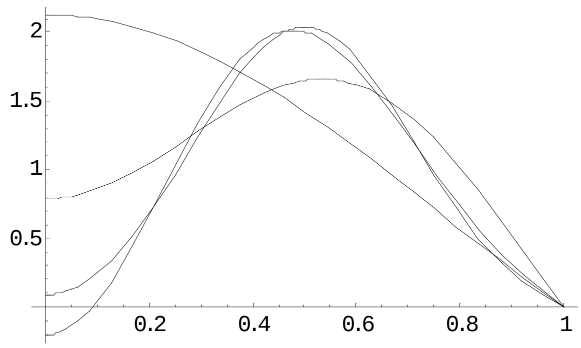
First term of the Fourier-Bessel series

This looks nothing like a cycle of a sine curve. Adding the next term shows that things are moving in the right direction:



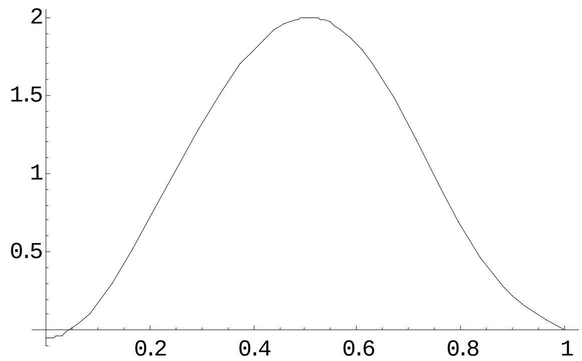
First two terms of the Fourier-Bessel series

The left hand side is coming down substantially, but there is a long way to go. If we add a few more terms then things improve. The next figure shows the behaviour for one, two, three and four terms. It is quite clear how the terms are contributing to obtaining the required function. Although the original function does not go negative, nevertheless we can see that some of the approximations do.



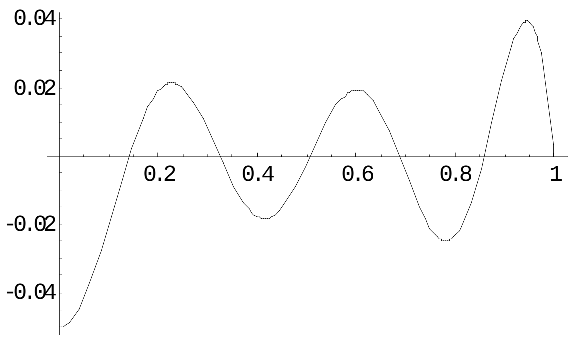
Sum of one, two, three and four terms of the Fourier-Bessel series

The fifth approximation is even better, although it still does not look quite right.



Five term Fourier-Bessel series for a sine wave cycle

Finally we show the error resulting after using five terms of the series. The graph below shows the difference between this approximation and the original function.



Error in the Fourier-Bessel series

The important concepts of this section are:

- The shape of the J_0 Bessel function
- Orthogonality of this function with different arguments
- Weight function in the orthogonality integral
- Zeros of the J_0 Bessel function
- Fourier-Bessel series: defined on interval $0 \le r \le R$, with boundary condition $f\left(R\right) = 0$.
- Formula for the Bessel coefficients
- Small number of terms needed when the coefficients decrease rapidly