The Laplacian in different coordinate systems

The Laplacian

The Laplacian operator, operating on Ψ is represented by

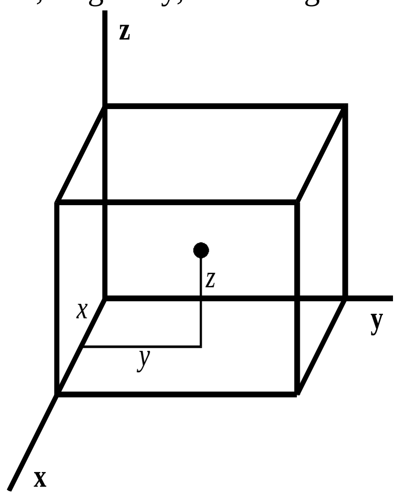
$$\nabla^2\Psi .$$

This operation yields a certain numerical property of the spatial variation of the field variable Ψ. Previously we have seen this property in terms of differentiation with respect to rectangular cartesian coordinates. But it is important to appreciate that the laplacian of Ψ is a *physical* property, independent of the particular coordinate system adopted. We know the *mathematical* form of ∇^2 in rectangular cartesian coordinates, and this can be used to find the mathematical expression for ∇^2 in other coordinate systems.

The fundamental point is that $\nabla^2\Psi$ gives a scalar quantity at a given point *independent of the coordinate system used*. In that sense we are saying that $\nabla^2\Psi$ is a physical property.

Rectangular cartesian coordinates

We encountered the Laplacian, originally, in rectangular cartesian coordinates.



rectangular cartesian coordinates

In rectangular cartesian coordinates the element of volume is given by

$$\mathrm{d}v = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

and the space is covered by letting the coordinates span the ranges

$$-\infty < x < \infty$$
$$-\infty < y < \infty$$
$$-\infty < z < \infty.$$

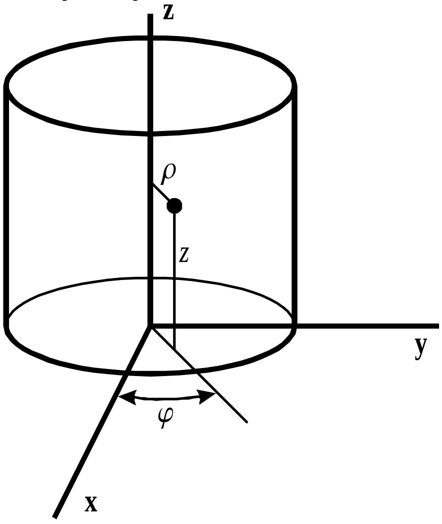
The Laplacian, operating on $\Psi(x, y, z)$ is given by

$$\nabla^2\Psi(x, y, z) = \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial z^2} .$$

If we were to transform to a different rectangular cartesian coordinate frame (by a rotation of the axes) then the discussion above requires that $\nabla^2\Psi$ would still have the same numerical. You can prove that this is so.

Cylindrical polar coordinates

The cylindrical polar coordinates ρ, φ, z are given, in terms of the rectangular cartesian coordinates x, y, z by



$$x = \rho\cos\varphi$$
$$y = \rho\sin\varphi$$
$$z = z.$$

cylindrical polar coordinates

In cylindrical polar coordinates the element of volume is given by

$$\mathrm{d}v = \rho\,\mathrm{d}\rho\,\mathrm{d}\varphi\,\mathrm{d}z\,.$$

The angle element $\mathrm{d}\varphi$ is the length of the circular arc subtended at the origin divided by the radius. And the volume element is the product of the arc length $\rho\,\mathrm{d}\varphi$ by the radial increment $\mathrm{d}\rho$ and the height increment $\mathrm{d}z$. The entire space is covered when the cylindrical polar coordinates span the ranges

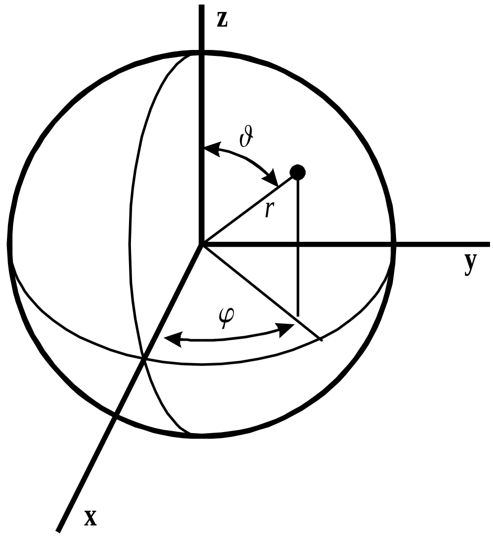
$$0 < \rho < \infty$$
$$0 < \varphi < 2\pi$$
$$-\infty < z < \infty.$$

The derivatives in the laplacian then transform, to give $\nabla^2\Psi$ in cylindrical polar coordinates as

$$\nabla^2\Psi(\rho,\varphi,z) = \frac{\partial^2\Psi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Psi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Psi}{\partial\varphi^2} + \frac{\partial^2\Psi}{\partial z^2}\,.$$

Spherical polar cordinates

The spherical polar coordinates r, ϑ, φ are given, in terms of the rectangular cartesian coordinates x, y, z by



$$x = r\sin\vartheta\cos\varphi$$
$$y = r\sin\vartheta\sin\varphi$$
$$z = r\cos\vartheta.$$

spherical polar coordinates

In spherical polar coordinates the element of volume is given by

$$\mathrm{d}v = r^2 \sin\vartheta\, \mathrm{d}r\, \mathrm{d}\vartheta\, \mathrm{d}\varphi \; .$$

The *solid angle* element $\mathrm{d}\mathbf{\Omega}$ is the area of spherical surface element subtended at the origin divided by the square of the radius:

$$d\mathbf{\Omega} = \sin\vartheta\, \mathrm{d}\vartheta\, \mathrm{d}\varphi \, .$$

And the volume element is the product of the spherical surface area element $r^2 \sin\vartheta\, \mathrm{d}\vartheta\, \mathrm{d}\varphi$ by the radial increment $\mathrm{d}r$. The entire space is covered when the cylindrical polar coordinates span the ranges

$$0 < r < \infty$$
$$0 < \vartheta < \pi$$
$$0 < \varphi < 2\pi.$$

The derivatives in the $\nabla^2\Psi$ then transform, to give the Laplacian in spherical polar coordinates as

$$\nabla^2\Psi(r,\vartheta,\varphi) = \frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\Psi}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2\Psi}{\partial\varphi^2} \, .$$