PH2130 Mathematical Methods
Lab 6

This script should keep you busy for the next week. You should aim to create a tidy and well-structured *Mathematica* Notebook. Do include plentiful annotations to show that you know what you are doing, where you have experimented, and what you have learned. You should then find your notebooks useful to you later on when you are working on other problems.

This exercise is concerned with investigating approximations to the factorial function (recall the Gamma function and Stirling's approximation from the lectures).

1 Simple approximation procedure

The logarithm of the factorial may be expressed by the sum

$$\ln(n!) = \sum_{x=1}^{n} \ln x .$$

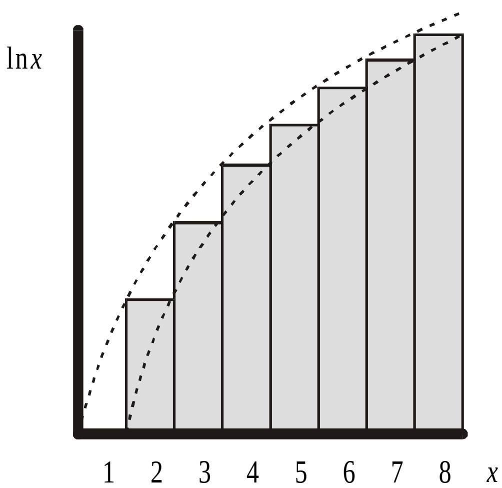
Be sure you are happy with this expression.

Approximations to the factorial function may be found by approximating the sum above by an integral. In the figure below the upper dotted line is an overestimate of the sum while the lower dotted line is an underestimate of the sum. For a general function $f(x)$ show that the upper estimate is given by

$$\int_1^{n+1} f(x)\mathrm{d}x$$

and the lower estimate is given by

$$\int_0^{n} f(x)\mathrm{d}x .$$



You should write this out on paper and present your working as clearly as possible.

Using the above result you should be able to show that the expression for ln(n!) satisfies the inequalities

$$n\ln n - n < \ln n! < (n+1)\ln(n+1) - n\,.$$

Your first task is to investigate, using *Mathematica*, how these upper and lower bounds approximate ln(n!) as n increases. You should do this by plotting the approximations together with the true values. You will find that *Mathematica* will give you the factorial function (even for non-integer values). You should think about the best way of representing the errors in the approximations.

The lower limit, which can be written as

$$n! \sim \left(\frac{n}{e}\right)^n$$

is often used in physical applications when n is very large.

2. Approximation via the integral formula

For a second look at approximating the factorial function we shall start from the integral expression we studied in the lectures. You should recall that the function

$$F(n) = \int_0^\infty t^n e^{-t}\mathrm{d}t$$

has all the properties of the factorial function when n is an integer. In this section we shall (or at least you will) investigate how to make a reasonable approximation to this integral.

If you want to see about approximating an integral, the first thing to do is to see what the function to be integrated looks like. It is easy to plot the function using *Mathematica*. Plot the $f_n(t) = t^n e^{-t}$ as a function of t for a few values of n, say $n = 10,\ 20,\ 30,\ 40,\ 50$. (You will see that it is better to make separate plots rather than a single multiple plot.) The problem then is to find/approximate the area under these curves.

You should observe two things about these curves. Firstly they are bell-shaped curves, and secondly each curve is peaked at $t = n$. You should prove this second point; probably it is easiest using *Mathematica*.

The key to this method of approximating the factorial function is to approximate the function $f_n(t) = t^n e^{-t}$ by a gaussian curve; you should appreciate that for large n the curves do indeed look rather gaussian. This may be done easiest by expanding the logarithm of $f_n(t)$ in powers of $t - n$, that is in powers of the distance away from the peak. There is no first power (why?) and the series is terminated at the second power. This is our approximation to the logarithm of $f_n(t)$, so the approximation to $f_n(t)$ is the exponential of this. In this way you should find the approximation to be

$$f_n(t) \approx n^n e^{-n} e^{-\frac{(t-n)^2}{2n}}.$$

For $n = 20$, say, plot the original $f_n(t)$ and its approximation to compare them.

Think carefully about what you are doing. This approximation to $f_n(t)$ might appear to be more complicated than the original $t^n e^{-t}$. But the crucial point is that the approximate function can be integrated to give a simple answer whereas the original function can't.

You might find it a bit tedious to evaluate the gaussian integral. But using Mathematica, and integrating from $-\infty$ to $+\infty$ you should find this approximation to the factorial function to be

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

This is better than the approximation found in the previous section.

3. Stirling's series for the factorial

This last expression for approximating *n*! is the first term in Stirling's asymptotic series for the factorial function:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left\{1 + \frac{1}{12n} + \frac{1}{288n^2} + \ldots\right\}.$$

The full series is obtained by an extension of the method of the previous section. Your third task is to investigate this series expression. You should examine how this compares with the previous approximations you obtained and you should consider the error involved in neglecting the square root factor (which does not appear in the first approximation).

This is very much an open-ended exercise. By all means experiment using *Mathematica* in your investigations, but remember that ultimately you should produce a well-structured and annotated *Mathematica* notebook.