PH2130 Mathematical Methods
Lab 5

This script should keep you busy for the next week. You should aim to create a tidy and well-structured *Mathematica* Notebook. Do include plentiful annotations to show that you know what you are doing, where you have experimented, and what you have learned. You should then find your notebooks useful to you later on when you are working on other problems.

You should previously have received a handout on Orthogonal Functions, and *Mathematica* Notebooks on Fourier Series and on Fourier-Bessel Series. The handout on Orthogonal functions summarizes what you should have learned in lectures, and it also contains plentiful examples of these series. It includes some diagrams of functions and bar charts of expansion coefficients.

The *Mathematica* Notebooks show how all the calculations were done and how the graphics were produced. You should be familiar with most of these.

Your assignment for the next week is to investigate Legendre polynomials. *Mathematica* 'knows' about Legendre polynomials; the *Mathematica* instruction for $P_n(x)$ is LegendreP[n, x].

A. Printing and plotting Legendre polynomials

Using the *Mathematica* instructions for $P_n(x)$ print out the first few Legendre polynomials. Make a plot of the first few Legendre polynomials. Try and label them sensibly.

B. Orthogonality of the Legendre polynomials

In discussing the orthogonality of any functions we must know the *interval* and the *weight function*. The interval for orthogonality of $P_n(x)$ is $-1 \le x \le 1$ and the weight function is unity.

To show that the $P_n(x)$ are orthogonal you need to show that the integral

$$\int_{-1}^{1} P_n(x) P_m(x) \mathrm{d}x$$

is zero when $n \ne m$. It is not possible (easily) to demonstrate this in the general case using *Mathematica*. You should, however generate a two dimensional table of the elements

$$t[n,m] = \int_{-1}^{1} P_n(x) P_m(x) \mathrm{d}x$$

using the Mathematica commands

TableForm[Table[t[n,m], {n,0,10}, {m,0,10}]].

You should be sure you understand the structure of this compound command.

By inspection of the table you should be able to make some inferences about orthogonality. Also you should be able to discern the general form for the diagonal terms $t[n,n]$.

C. Expression for the expansion coefficients
Any function $f(x)$ specified on the interval $-1 \le x \le 1$ can be expressed as a sum of Legendre polynomials with appropriate coefficients

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x).$$

You should use the orthogonality properties of $P_n(x)$ that you have demonstrated to show that the expansion coefficients a_n are given by

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) \mathrm{d}x .$$

This expression should be worked out on paper: not using *Mathematica*. Either write the derivation on paper or type it up using a word processor (but you will need to use an equation editor as well).

D. Constructing Legendre series
Your challenge is to investigate the Legendre series to approximate the semicircle $f(x) = \sqrt{1-x^2}$. You should follow the same procedure as you used in the previous lab, for Fourier-Bessel series. First you should evaluate the coefficients a_n by integration. You will need about the first ten coefficients. It will be instructive to plot a bar chart of the coefficients.

Next you should define a function giving the various partial sums approximating the function.

Finally you should plot the various approximations and examine the remnant error. You will find the *Mathematica* command `Evaluate[…]` to be useful.