PH2130 Mathematical Methods
Lab 2

Motion of a Pendulum

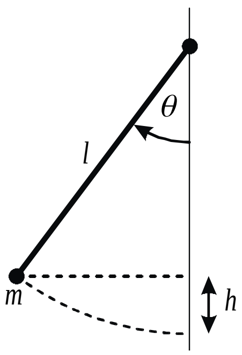
Introduction

You will (or at least you certainly should be) familiar with *small oscillations* of a pendulum as a good example of simple harmonic motion. You will have learned that one of the important properties of the simple harmonic oscillator is that the period of oscillation of an SHO is independent of the amplitude of the motion. And in fact Galileo, in his investigations of dynamics, used a pendulum for measuring time by counting cycles.

The purpose of this exercise is to investigate the motion of a pendulum when the oscillation amplitude is no longer restricted to being small. The general equation of motion for a pendulum is complicated. The approximation of small motion converts the equation to the SHO equation, which is readily soluble. The utility of *Mathematica*, here, is that it can solve the more complicated, general, equation for the pendulum. The solution is in terms of the so-called elliptic functions. *Mathematica* 'knows' all about these functions; it knows their properties and it can plot them. This allows you to study such complex systems without being an expert in 'special functions'. Let the computer do the hard work, while you concentrate on the physical understanding!

Some of the following exercises should be done with pen and paper. Usually you will be guided when to use *Mathematica*. Your completed assignment will include hand-written notes and calculations together with pieces of *Mathematica* print-out.

1 Setting up the pendulum equations of motion



The figure shows a mass m attached to the end of a light rigid rod of length l. The other end of the rod is held at a friction-free pivot. The equations of motion could be obtained by equating the force experienced by the mass to its acceleration times mass. However this problem is much easier to tackle from a consideration of the energy.

The first thing to decide is what variables will be used. In particular we need to choose what coordinate is best to use in describing the position of the mass. We shall use θ, the inclination of the rod to the vertical. It is useful to note that this coordinate, an angle, is dimensionless.

1 Show that the kinetic energy is given by

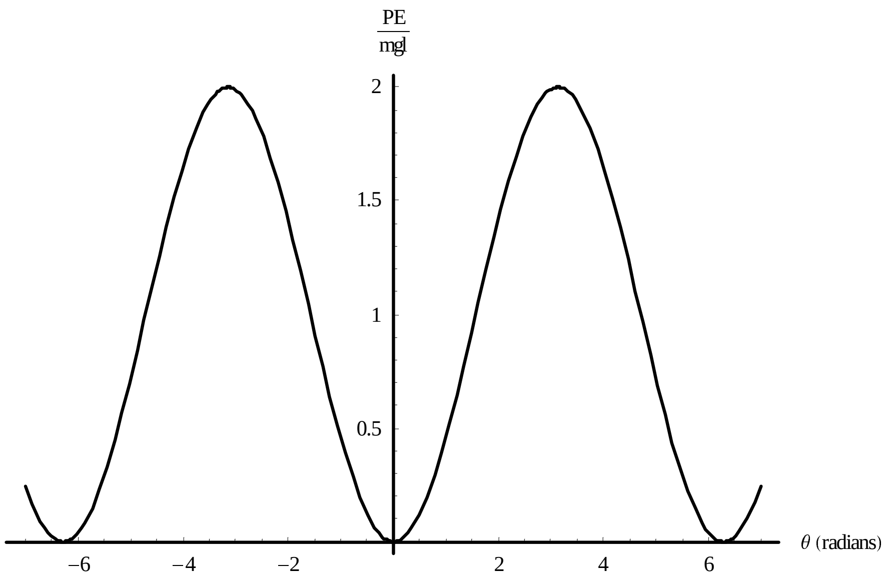
$$KE = \frac{1}{2} m l^2 \dot{\theta}^2 .$$

2 Show that the potential energy may be written as

$$PE = mgl\left(1 - \cos\theta\right).$$

3 You appreciate that there is always a choice of where the zero of potential energy is specified. With the above expression for *PE*, where is the zero?

4 Use *Mathematica* to plot out the potential energy as a function of angle θ.



Your picture should look something like the above. Investigate how to embellish your plot with an appropriate font and place sensible axis labels on the plot. Try and do better than I did. Include an account of the *Mathematica* commands in your report.

5 Conservation of energy allows you to write down the equation

$$E = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\left(1 - \cos\theta\right).$$

This equation relates the coordinate θ and its time derivative. Thus it may be regarded as an equation of motion, the solution of which is the time evolution $\theta(t)$. Show that the equation of motion may be written in the 'standard' form:

$$\dot{\theta}^2 - \frac{2g}{l}\cos\theta + \frac{2g}{l}\left(1 - \frac{E}{mgl}\right) = 0.$$

6 This equation is a *nonlinear*, homogeneous, first order, ordinary differential equation, with constant coefficients. There are two reasons why the equation is nonlinear; what are these?

7 By differentiation with respect to time, show that the equation of motion may be written as

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0.$$

This is the 'force equation' since it is the (angular) equivalent of Newton's second law equation $F = ma$.

8 The above equation is now nonlinear in only *one* respect. What is that?

2 Qualitative features of pendulum motion.

Depending on the energy of the pendulum, there are two distinct types of motion. For small energies there is the conventional oscillatory motion where the mass swings backwards and forwards. There is, however, a critical energy $E_c = 2mgl$. For energies larger than E_c the pendulum travels round and round the pivot; the motion is *rotation* rather than *oscillation*. For rotational motion the coordinate θ increases continuously whereas for vibration the coordinate θ varies only between two extreme values.

1 Show that when $E > E_c$ the motion is rotational.

2 Show that when $E < E_c$ the motion is oscillatory and find the extreme values, θ_c, taken by the coordinate.

3 Make rough plots, by hand, of the qualitative behaviour of θ as a function of time for these two cases.

4 Describe what happens in the borderline case when $E = E_c$.

3 Small oscillations

When the amplitude of the oscillations is small, then $\cos\theta$ in the energy equation and $\sin\theta$ in the force equation can be expanded to leading order in θ. The resultant force equation is then

$$\ddot{\theta} + \frac{g}{l}\theta = 0 .$$

1 Show that the solution of this equation may be written

$$\theta(t) = \theta_0 \cos\omega t$$

where θ_0 is the amplitude of the oscillations and the angular frequency ω is given by

$$\omega = \sqrt{g/l} .$$

This result tells us that we can replace g/l by ω^2 in all our equations, where ω is the angular frequency of *small* oscillations.

2 Now let us turn attention to the energy equation. Show that this equation, in the small amplitude limit, may be expressed as

$$\dot{\theta}^2 + \omega^2\theta^2 - \omega^2\theta_0^2 = 0 .$$

3 We investigate integrating up this equation, so that we can then use the same method for the general equation (not restricted to small amplitudes). Show that the equation may be written as

$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{1}{\omega}\frac{1}{\sqrt{\theta_0^2 - \theta^2}}.$$

4 This may be integrated, formally, as

$$t = \frac{1}{\omega}\int \frac{\mathrm{d}\theta}{\sqrt{\theta_0^2 - \theta^2}}.$$

By changing the variable of integration to $\theta = x\theta_0$ show that the time is given by

$$t = \frac{1}{\omega}\int^{\theta/\theta_0} \frac{\mathrm{d}x}{\sqrt{1 - x^2}}.$$

5 Use your knowledge of integration methods, or use *Mathematica* to do the integral, and show that a solution of the problem is

$$\theta(t) = \theta_0 \sin \omega t.$$

6 Can you make some comment about the constant of integration?

4 Large amplitude oscillations

When the energy of the pendulum is less than the critical value E_c, the motion is oscillatory. The amplitude of the oscillation, θ_0, is related to the critical energy.

1 Show that the amplitude of the oscillation, θ_0, is given by

$$\cos\theta_0 = 1 - 2E/E_c.$$

2 By integrating up the energy equation, show that the time is given by

$$t = \frac{1}{\omega\sqrt{2}}\int \frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_0}}.$$

3 Use *Mathematica* together with manual simplification to do the integration, to show that

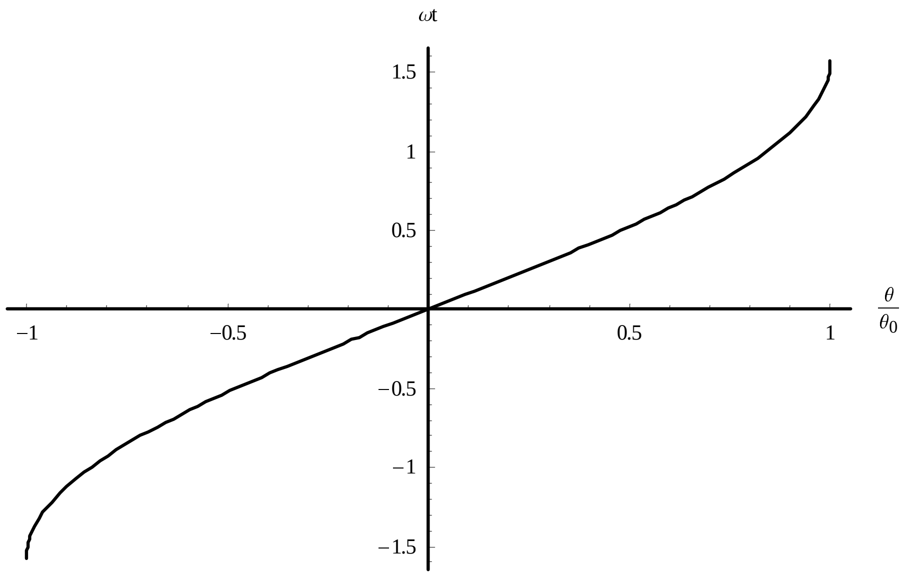
$$t = \frac{\sqrt{2}}{\omega\sqrt{1-\cos\theta_0}}\,\mathrm{EllipticF}\left[\frac{\theta}{2}, \frac{2}{1-\cos\theta_0}\right].$$

You don't need to know about the elliptic function EllipticF. It is a function of two arguments, and *Mathematica* knows all about it. So *Mathematica* can plot the behaviour of the system.

4 In order to plot the oscillations of this pendulum we need to remove all superfluous factors from the equation. Let us adopt a dimensionless time ωt and let us measure the magnitude of the oscillation in terms of the maximum displacement θ_0. In other words, we work in terms of the dimensionless variable $x = \theta/\theta_0$. Then the evolution expression becomes

$$\omega t = \frac{\sqrt{2}}{\sqrt{1-\cos\theta_0}}\,\text{EllipticF}\left[\frac{x\theta_0}{2}, \frac{2}{1-\cos\theta_0}\right].$$

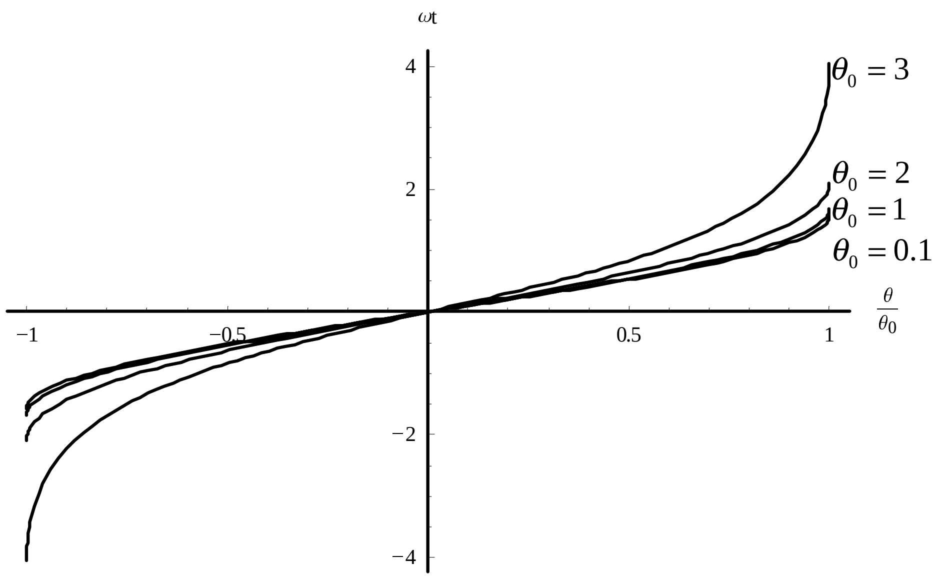
For small θ_0 we expect the evolution to be sinusoidal.



Evolution of pendulum for θ_0 = 0.1

The above figure shows the behaviour of the oscillating pendulum when $\theta_0 = 0.1$. Remember we are measuring angles in radians, so this is quite a small amplitude. Use Mathematica to plot this evolution and compare it with the small-amplitude limit (you will need the ArcSin[] function for this).

5 Next plot the evolution for the various amplitudes: $\theta_0 = 0.1, 1, 2, 3$. Your figure should look like this:



Evolution of pendulum for θ_0 = 0.1, 1,2 and 3

6 You see that as the amplitude approaches the maximum value of π, the period gets longer. Discuss physically why this is.

7 In order to find the way the period of the motion varies with the amplitude of the oscillation, note that the period T will be four times the time it takes for the pendulum to travel from $\theta = 0$ to the turning point at $\theta = \theta_0$ Thus

$$T = \frac{4}{\omega\sqrt{2}} \int_0^{\theta_0} \frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_0}} .$$

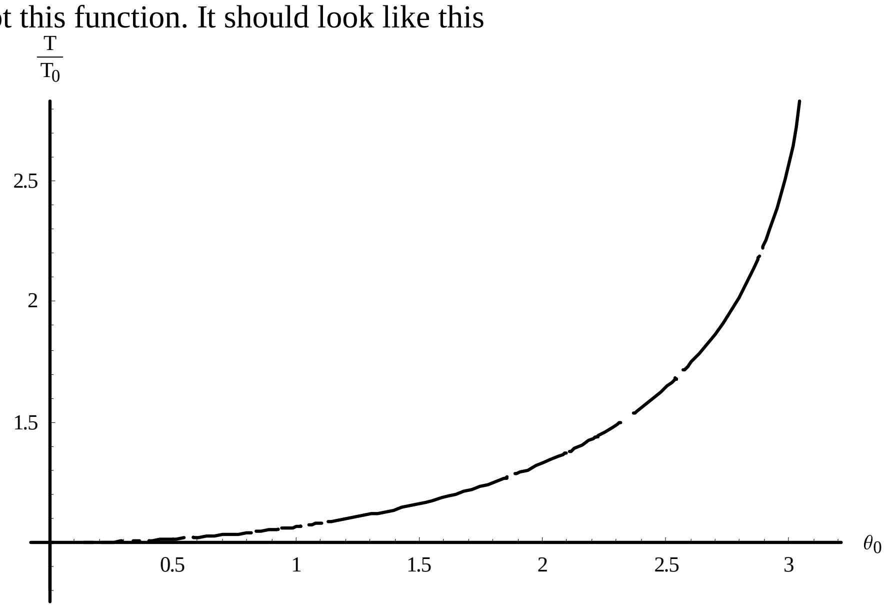
Note that the period for small oscillations T_0, is given by $T_0 = 2\pi/\omega$. Thus show that the period for amplitude θ_0 is given by

$$\frac{T}{T_0} = \frac{\sqrt{2}}{\pi} \int_0^{\theta_0} \frac{\mathrm{d}\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

8 Use *Mathematica*, together with manual simplification to show that the period depends on amplitude as:

$$\frac{T}{T_0} = \frac{2\sqrt{2}}{\pi\sqrt{1-\cos\theta_0}} \mathrm{EllipticF}\left[\frac{\theta_0}{2}, \frac{2}{1-\cos\theta_0}\right],$$

and plot this function. It should look like this



Period of pendulum as a function of amplitude

This shows the pendulum period increasing as the amplitude approaches the critical value of $\theta_c = \pi$.

You will see some gaps in the curve; I think these are regions where *Mathematica* has difficulty making evaluations of the elliptic function.

You might be inclined to investigate the period as a power series expansion in θ_0. I don't think *Mathematica* can do it. This is probably because the function is varying too slowly at the origin; recall trying to expand e^{-1/x^2} .

5 Rotational motion

When the energy of the pendulum is greater than the critical value E_c, the motion is rotational and the angular coordinate θ increases continuously with time. In this case the equation of motion is best cast in terms of the energy E and the critical energy E_c.

1 Show that the energy equation of motion can be written as

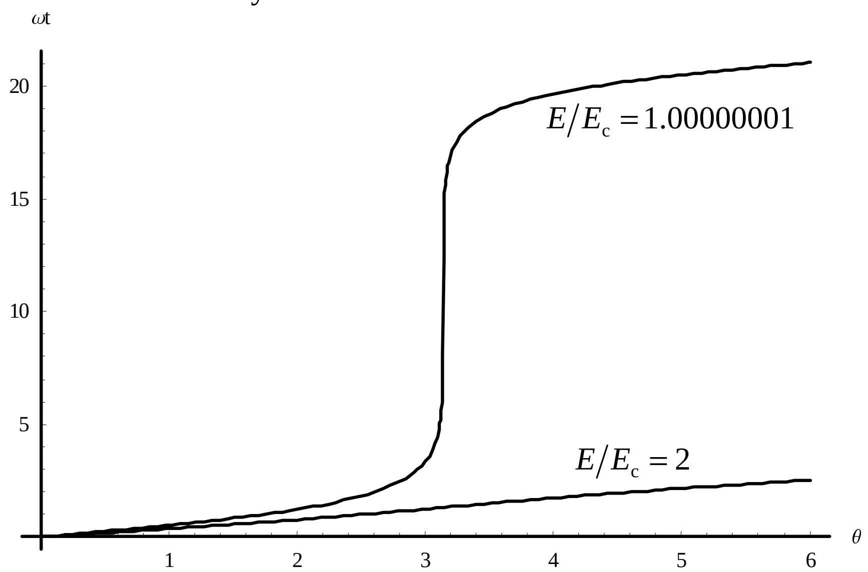
$$t = \frac{1}{\omega\sqrt{2}} \int \frac{\mathrm{d}\theta}{\sqrt{\cos\theta - (1 - 2E/E_c)}} .$$

2 Integrate this, using *Mathematica*, to obtain

$$\omega t = \frac{1}{\sqrt{e}} \text{EllipticF}\left[\frac{\theta}{2}, \frac{1}{e}\right]$$

where energy is measured in multiples of E_c, that is, $e = E/E_c$.

3 Now plot this function for different values of energy. When the energy is very close to (but greater than) the critical energy, we observe a step where the motion almost stops at the top of the motion. On the other hand, for large energies, the angle increases almost uniformly with time.



Rotational motion of pendulum

Your graph should look something like this.

4 As the energy tends to the critical energy, does the period tend to infinity or to a finite value? Think carefully about this and/or investigate using *Mathematica* as E/E_c tends closer to unity (from above).

5 For large energies the angular velocity tends to a constant, a function of the energy. This may be investigated by ignoring the potential energy in the energy equation. Derive an expression for the angular velocity as a function of energy in this limit.

6 Phase Plots

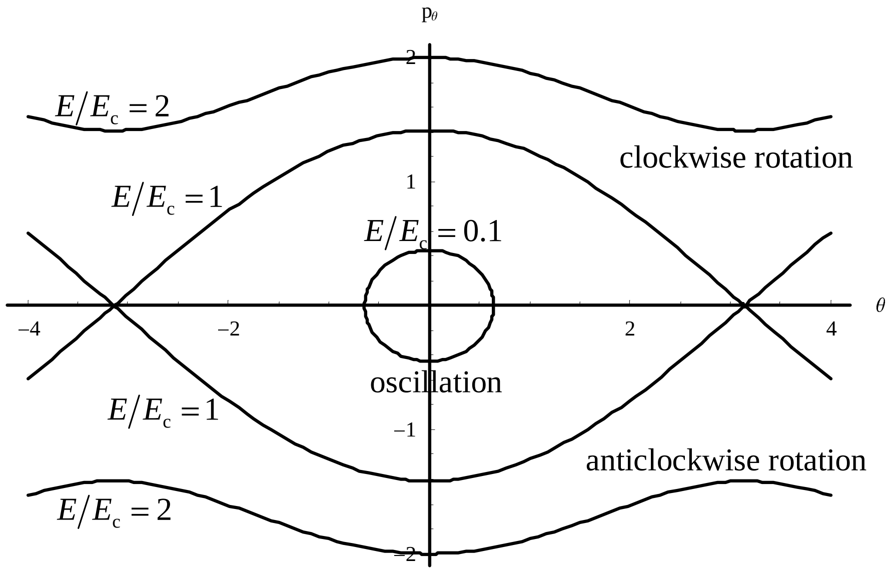
A convenient way of displaying the evolution of the pendulum (and other dynamical systems) is in terms of a *phase plot*, a plot of coordinate against momentum. The coordinate here is the angle θ, while the corresponding momentum is the angular momentum p_θ, where

$$p_\theta = ml^2\dot{\theta}\,.$$

In terms of this, the energy equation becomes

$$\frac{p_\theta^2}{2ml^2} + mgl\left(1-\cos\theta\right) = E\,.$$

1 Use *Mathematica* to make a phase plot for various values of energy. Your graph should look something like this.



Phase plot for pendulum

2 Explain in words why the upper and lower curves represent rotation while the inner, closed, curve represents oscillation.

3 The curves corresponding to $E/E_c = 1$ are of particular importance. The curves intersect. They intersect at points where the angular momentum, and thus the velocity, is zero. At what angles θ does this happen? Explain physically what is going on at these points.