Factorials and Euler's Gamma function

Introduction

The is a self-contained part of the course dealing, essentially, with the factorial function and its generalization to: i) non-integer arguments, and ii) negative arguments. And as a bonus we will obtain a simple expression for the factorial of very large arguments.

You should recall that the terms of Taylor power series expansions involve factorials. In the spirit of the extension of series expansions to incorporate Frobenius series, a similar extension of the factorial function is useful. The terms of various series expansions are given in terms of non-integer factorials, so we must be able to handle this. It should also be pointed out that the factorial is somewhat anomalous, mathematically-speaking, in that it is defined for the integers but there is nothing "in between"; you certainly can't differentiate such a function. So you could regard this section as providing a smooth interpolation of the factorial between the discrete integer points.

In statistical mechanics/thermodynamics you will need to calculate probabilities and the number of ways particles can be rearranged etc. This involves considering factorials of very large numbers (of the order of Avogadro's number $\sim 6\times10^{23}$). A mathematical expression which can be differentiated and manipulated is necessary here. The last part of this section will consider this.

The factorial as an integral

We shall define a new function and examine its properties. Previously we have considered functions defined through the differential equations they satisfy. In this case the function will be defined (simply?) as an integral:

$$F(x) = \int_0^\infty t^x e^{-t}\mathrm{d}t\ .$$

This cannot be integrated (in terms of the elementary functions). It is instructive to integrate by parts, differentiating the first term and integrating the second. Thus we are setting

$$u = t^x, \quad \frac{\mathrm{d}v}{\mathrm{d}t} = e^{-t}$$

and we then have

$$\frac{\mathrm{d}u}{\mathrm{d}t} = xt^{x-1}, \quad v = -e^{-t}.$$

We then use the integration by parts rule

$$\int u\frac{\mathrm{d}v}{\mathrm{d}t}\mathrm{d}t = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}t}\mathrm{d}t$$

so that

$$F(x) = -t^x e^{-t}\Big|_0^\infty + x\int_0^\infty t^{x-1}e^{-t}\mathrm{d}t\ .$$

The first term vanishes since the function is zero at both end points. And the integral in the second term looks suspiciously like the original definition for $F(x)$, the only difference being that x has been replaced by $x-1$. Thus we conclude

$$F(x) = xF(x-1).$$

What we have here is a *recurrence relation* for the function $F(x)$; from a given initial value x_0 we can thus step up (or down) one at a time.

Integer values

As an example, we could start from $F(0)$ and step up, giving F for all integer x. And fortunately we can evaluate $F(0)$ by direct integration, since

$$\begin{aligned} F(0) &= \int_0^\infty e^{-t}\mathrm{d}t \\ &= 1. \end{aligned}$$

So using the recurrence relation we build up

$$\begin{array}{ll} x=1 & F(1) = F(0) = 1 \\ x=2 & F(2) = 2F(1) = 2 \\ x=3 & F(3) = 3F(2) = 3\times 2 \\ x=4 & F(4) = 4F(3) = 4\times 3\times 2 \\ :: & :::::::::::::::::::: \\ x=n & F(n) = n! \end{array}$$

We have obtained the factorial function. Thus we conclude that the factorial function may be specified by the integral

$$F(x) = \int_0^\infty t^x e^{-t}\mathrm{d}t$$

and of course this may be extended to non-integer arguments.

The recurrence relation also allows the extension to negative values of x.

The Gamma function

By convention mathematicians prefer to use the gamma function (Euler's gamma function) when extending the factorial idea to non-integer and negative arguments. There is a shift of 1 in the definition. They use

$$\Gamma(x) = (x-1)!$$

In other words, the gamma function is specified through the integral

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t$$

which then has the recurrence relation

$$\Gamma(x+1) = x\Gamma(x).$$

Fractional arguments

Integer plus a half, or integer minus a half values of $\Gamma(x)$ may be built up from $\Gamma(1/2)$. The integral expression for this is

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-1/2} e^{-t} \mathrm{d}t \,,$$

which may be transformed to the gaussian integral

$$2\int_0^\infty e^{-y^2} \mathrm{d}y = \sqrt{\pi}.$$

Thus we have the special value

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

and we can build up from this

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$$

$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{4}\sqrt{\pi}$$

etc.

Negative arguments

To treat negative arguments we may use the recurrence relation to step *down* from a given initial argument:

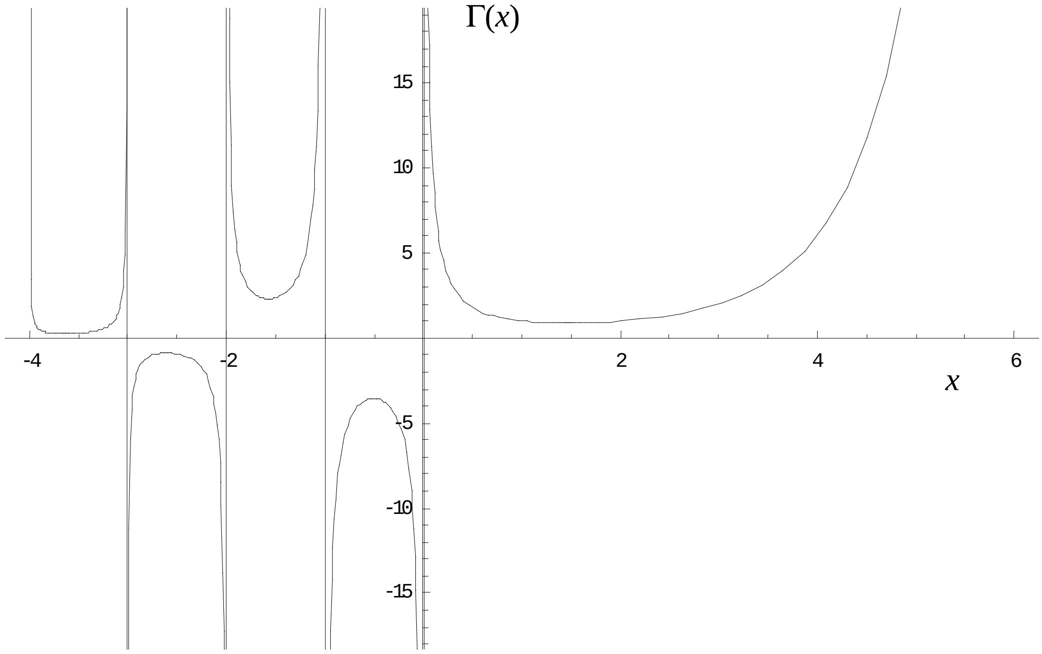
$$\Gamma(x) = \frac{1}{x}\Gamma(x+1).$$

This has the immediate consequence that $\Gamma(0)$ is infinite, and then by extension the gamma function will be infinite for all negative integer *x*.

But for non-integers there is no problem. Thus starting from ½ we would find, for example,

$$\begin{aligned}\Gamma\left(-\frac{1}{2}\right) &= \Gamma\left(\frac{1}{2}\right)\Big/ -\frac{1}{2} \\ &= -2\Gamma\left(\frac{1}{2}\right) \\ &= -2\sqrt{\pi}.\end{aligned}$$

The gamma function is plotted in the following figure.



Euler's gamma function

Observe the divergences for negative integer arguments.

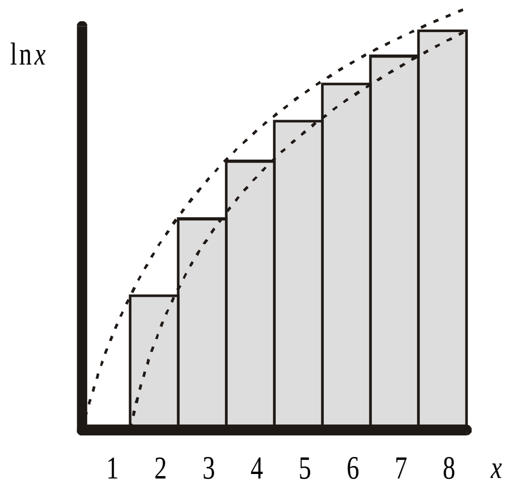
Large arguments – Stirling's formula
It is easy to *approximate* the factorial function for large argument. Since

$$n! = 1\times 2\times 3\times 4\times 6\times \ldots \times n\,,$$

on taking the logarithm we have

$$\begin{aligned}\ln(n!) &= \ln(1\times 2\times 3\times 4\times 6\times \ldots \times n)\\ &= \ln 1 + \ln 2 + \ln 3 + \ln 4 + \ldots + \ln n\\ &= \sum_{x=1}^{n} \ln x\,;\end{aligned}$$

we have converted the product into a sum. Now when n is large we make only a small error by converting this sum into an integral.



Integral approximation to sum

The area under the *lower* dotted curve is given by $\int_0^n \ln x \, \mathrm{d}x$, so that we adopt the approximation

$$\sum_{x=1}^{n} \ln x \approx \int_0^n \ln x \, \mathrm{d}x .$$

On performing the integral this gives the approximate expression

$$\ln n! \approx n \ln n - n .$$

We may take the exponential of this to find the approximation to $n!$ itself

$$\begin{aligned} n! &\approx \exp\left(n \ln n - n\right) \\ &= \exp\left(\ln n^n - n\right) \\ &= n^n e^{-n} \\ &= \left(\frac{n}{e}\right)^n . \end{aligned}$$

These approximate formulae,

$$\ln n! \approx n \ln n - n \qquad \text{and} \qquad n! \approx \left(\frac{n}{e}\right)^n$$

are called Stirling's approximation (to the factorial function).

A more systematic analysis, starting from the integral expression for the gamma function and expanding it term by term gives the series

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left\{1 + \frac{1}{12n} + \frac{1}{288n^2} + \ldots\right\}.$$

In applications where we are interested in the logarithm of $n!$ when n is large we may ignore the square root factor at the left hand side and the inverse series on the right hand side; only the middle part remains, in agreement with the original approximation.

Q: Why can you neglect the square root factor?

The important concepts of this section are:

- The factorial function is re-defined / represented as an integral.
- The recurrence relation is found through integrating by parts.
- The value for $n = 1$ is found directly by integrating
- The integral expression allows extension to non-integer and negative arguments.
- Euler's Gamma function is defined as $\Gamma(x) = (x-1)!$
- Special value $\Gamma(1/2) = \sqrt{\pi}$.
- Negative arguments accommodated by stepping down with the recurrence relation.
- Stirling's approximation for large x obtained by taking the *logarithm* of $n!$ and approximating the resultant sum by an integral.