# Fourier Transforms

## Introduction

The important discovery of Fourier was that an arbitrary waveform could be constructed as a superposition of sine (and cosine) functions. We have already considered a first generalization of that principle – that the waveform can be constructed from a superposition of any appropriate complete orthonormal set of functions. These superpositions construct waveforms over a finite and predefined interval. Alternatively we may consider functions of infinite extent that are periodic on the interval. In this section we consider a different generalization of Fourier's principle where the waveform is non-repetitive and specified over an infinite range. In this case, as we shall see, the Fourier sum of sinusoids will become a Fourier *integral*. There is, of course, the final generalization where the sinusoids are replaced by other orthonormal functions; that is not treated in this course.

## Fourier series – summary of relevant knowledge

The Fourier series on the interval $-L \le x \le L$ is given by

$$F(x) = \sum_{n=0}^{\infty} \left\{ a_n \cos\left(\frac{n\pi}{L} x\right) + b_n \sin\left(\frac{n\pi}{L} x\right) \right\}$$

where the $n = 0$ term is subsumed in the sum. The Fourier components $a_n$ and $b_n$ are given by the Euler formulae

$$a_n = \frac{1}{L} \int_{-L}^{L} F(x) \cos\left(\frac{n\pi}{L} x\right) \mathrm{d}x$$

$$b_n = \frac{1}{L} \int_{-L}^{L} F(x) \sin\left(\frac{n\pi}{L} x\right) \mathrm{d}x .$$

The Fourier components are determined by the function $F(x)$ only over the interval $-L \le x \le L$. Outside this interval we can say either that the function is undefined or we can say that there is periodic behavior $F(x + 2L) = F(x)$; this is a matter of choice.

A slight change of notation will make closer connection with physical applications. In the argument of the sin and cos, we can interpret $n\pi/L$ as a wave number. Accordingly we write the $n^{\text{th}}$ wave number as

$$k_n = n\frac{\pi}{L} .$$

And then we can write the Fourier series and the inversion formulae as

$$F(x) = \sum_{k=k_0}^{k_\infty} \left\{ a_k \cos(kx) + b_k \sin(kx) \right\} \tag{1.1}$$

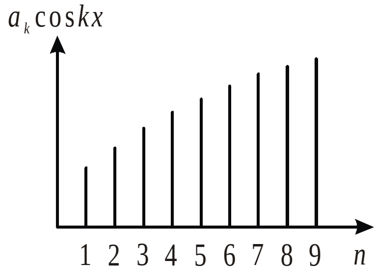
and

$$a_k = \frac{1}{L}\int_{-L}^{L} F(x)\cos(kx)\,\mathrm{d}x$$
$$b_k = \frac{1}{L}\int_{-L}^{L} F(x)\sin(kx)\,\mathrm{d}x\,.$$

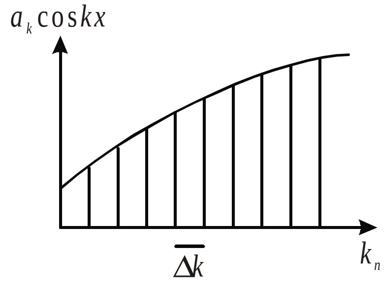
## The Fourier Integral

We now consider the case where interval over which the function is specified becomes indefinitely large – there is no periodicity and the function extends over all *x*. In other words, we want to take the limit $L \to \infty$.

This is not quite so simply done since the various factors $1/L$ will become problematic. The key to the solution is to approximate the Fourier sum, Eq. (1.1), by an integral.




Fourier sum: sum of heights




Integral approximation

The Fourier sum is given by the sum of the heights of the lines. But the sum of heights multiplied by the spacing is the area under the curve:

$$\sum \text{heights} = \frac{\text{area}}{\text{width of slices}}$$

or

$$\sum_k \quad = \frac{\int \quad \mathrm{d}k}{\Delta k}\,.$$

And the width of the slices is

$$\Delta k = k_n - k_{n-1} = \frac{\pi}{L}\,.$$

The Fourier sum is then approximated by the integral:

$$F(x) \approx \frac{L}{\pi}\int_0^\infty \left\{a_k \cos(kx) + b_k \sin(kx)\right\}\mathrm{d}k$$

as $L \to \infty$.

As $L \to \infty$ we see that the width of the slices tends to zero so in that limit the integral approximation becomes more accurate. But of course in this limit the Fourier components

$$a_k = \frac{1}{L}\int_{-L}^{L} F(x)\cos(kx)\,\mathrm{d}x$$
$$b_k = \frac{1}{L}\int_{-L}^{L} F(x)\sin(kx)\,\mathrm{d}x$$

tend to zero because of the $1/L$ pre-factor. In this case we define the Fourier sine and cosine *amplitudes* $f_\mathrm{s}(k)$ and $f_\mathrm{c}(k)$

$$f_\mathrm{c}(k) = La_k, \qquad f_\mathrm{s}(k) = Lb_k .$$

Thus $F(x)$ is expressed as a Fourier integral

$$F(x) = \frac{1}{\pi}\int_0^\infty \{ f_\mathrm{c}(k)\cos(kx) + f_\mathrm{s}(k)\sin(kx)\}\mathrm{d}k \qquad (1.2)$$

and the Fourier amplitudes are then given by

$$f_\mathrm{c}(k) = \int_{-\infty}^{\infty} F(x)\cos(kx)\,\mathrm{d}x$$
$$f_\mathrm{s}(k) = \int_{-\infty}^{\infty} F(x)\sin(kx)\,\mathrm{d}x .$$

## Complex form of the Fourier Integral

It is often convenient to express the Fourier integral relations in complex form. We define the complex Fourier amplitude $f(k)$ as

$$f(k) = f_\mathrm{c}(k) + \mathrm{i}f_\mathrm{s}(k)$$

so that

$$f(k) = \int_{-\infty}^{\infty} F(x)e^{\mathrm{i}kx}\mathrm{d}x$$

and then

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k)e^{-\mathrm{i}kx}\mathrm{d}k .$$

This result is obtained from Eq. (1.2) by substituting $\cos kx = \left(e^{\mathrm{i}kx} + e^{-\mathrm{i}kx}\right)/2$ and $\sin kx = \left(e^{\mathrm{i}kx} - e^{-\mathrm{i}kx}\right)/2\mathrm{i}$ .

We observe here the emergence of a symmetry between the Fourier transform relations for $F(x)$ in terms of $f(k)$ and for $f(k)$ in terms of $F(x)$. The position of the $2\pi$ factor is arbitrary; by a redefinition of $f(k)$ one can obtain the more symmetric form of the Fourier integrals with a pre-factor $1/\sqrt{2\pi}$ accompanying both integrals. We have adopted the convention most used in Physics applications. The position of the minus sign multiplying i is also arbitrary; there is a minus in one integral and not the other.

The Fourier pair

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k)e^{-ikx}\mathrm{d}k$$

$$f(k) = \int_{-\infty}^{\infty} F(x)e^{ikx}\mathrm{d}x$$

may be interpreted as telling us that a given function has a description in *x*-space as $F(x)$ and a description in *k*-space as $f(k)$. These are equally valid *representations* of the function.

Thus far we have considered spatial functions. The same considerations apply to temporal functions; an arbitrary function of time may be expressed as a superposition (integral) of different frequencies. Thus replacing *x* by *t* and *k* by $\omega$ we obtain the Fourier pair

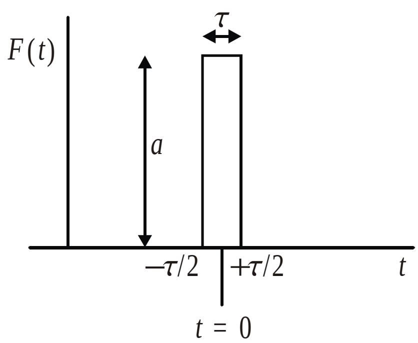
$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\omega)e^{-i\omega t}\mathrm{d}\omega$$

$$f(\omega) = \int_{-\infty}^{\infty} F(t)e^{i\omega t}\mathrm{d}t\,.$$

An example of this Fourier transform pair is in the area of acoustics; any sound can be represented as a superposition (integral) of pure tones of different frequencies.

### A gun shot

A gun shot is an example of a sharp impulse. This may be approximated by a high rectangular pulse of very short duration. We specify the magnitude of the pulse by *a* and its duration by $\tau$. We assume the pulse occurs at time $t = 0$.




*rectangular pulse approximating a gun shot*

The mathematical specification of the pulse, the function $F(t)$ is

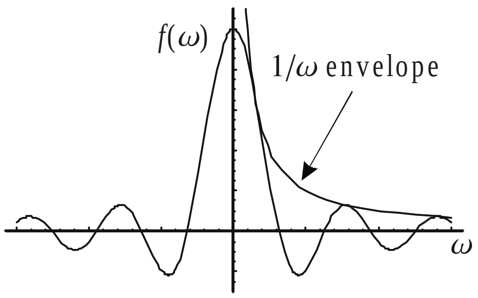
$$F(t) = a \qquad -\tau/2 \le t \le \tau/2$$
$$= 0 \qquad \text{otherwise}\,.$$

Thus the Fourier integral expression for the spectrum $f(\omega)$ is given by

$$f(\omega) = \int_{-\infty}^{\infty} F(t) e^{\mathrm{i}\omega t} \mathrm{d}t$$

$$= a \int_{-\tau/2}^{\tau/2} e^{\mathrm{i}\omega t} \mathrm{d}t = \left. \frac{a e^{\mathrm{i}\omega t}}{\mathrm{i}\omega} \right|_{-\tau/2}^{\tau/2} = \frac{2a}{\omega} \sin\left( \frac{\omega\tau}{2} \right).$$

This is shown in the figure.




*spectrum of a rectangular pulse*

The function $\sin(x)/x$ is known as the sinc function.

When the duration of the impulse gets shorter and shorter the spectrum gets broader and broader – a manifestation of the Uncertainty Principle. You can investigate this by making plots using *Mathematica*. be e limit $\tau \to 0$ will be treated in a later section.

## Idealization of the gun shot – the delta function

The duration of the gun shot, $\tau$, is very short and its instantaneous magnitude, $a$, is very large. As an idealization we may consider the limit where the duration tends to zero, while the instantaneous amplitude tends to infinity. At the same time we shall specify a finite value for the 'intensity' of the pulse, the area $a\tau$. For simplicity we will consider a shot of unit intensity: $a\tau = 1$. This is a minimalist description; we strip the system down to its bare essentials.

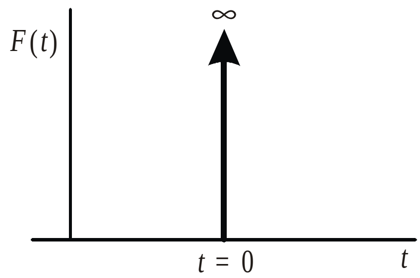
This idealized gun shot is thus defined as the limit of the box function $F(x)$ above, when

$$\tau \to 0$$

$$a \to \infty$$

$$\text{but } a\tau = 1.$$

In this limit the box becomes a 'spike', known as Dirac's delta function with symbol $\delta(t)$.




*delta function idealization of a gun shot*

The delta function is very useful, with many interesting properties. It should be mentioned, however that it is not a mathematically respectable function! $\delta(t)$ should be understood, rather, as the limit of a sequence of functions.

The spectrum of the spike may be found from the spectrum of the box function upon taking the appropriate limit. We found

$$f(\omega) = \frac{2a}{\omega}\sin\left(\frac{\omega\tau}{2}\right).$$

Since $\tau$ is small the argument of the sine is small and so a series expansion is appropriate:

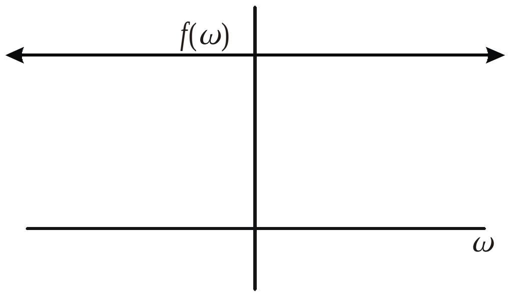
$$\begin{aligned} f(\omega) &\sim \frac{2a}{\omega}\left(\frac{\omega\tau}{2} - \frac{(\omega\tau/2)^3}{3!} + \ldots\right) \\ &= a\tau - \frac{a\omega^2\tau^3}{24} + \ldots \;. \end{aligned}$$

But we have the normalization condition $a\tau = 1$, which then leads to

$$f(\omega) \sim 1 - \frac{\omega^2\tau^2}{24} + \ldots$$

and now taking the limit $\tau \to 0$, this gives

$$f(\omega) = 1.$$



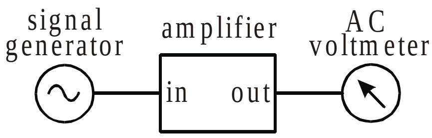

*spectrum of delta function*

This is a remarkable result. It is telling us that the spectrum of a delta function spike is a uniform superposition of sinusoids of all frequencies. This is known as a ‘white’ spectrum by analogy with light, where white light is the result of a superposition of colours from all regions of the visible spectrum.

When we looked at the finite-width box function we saw that as $F(t)$ became narrower, that $f(\omega)$ became broader. This, it was stated, was an example of the Uncertainty Principle. We now have a limiting example of this: when the width of $F(t)$ goes to zero, then the width of $f(\omega)$ becomes infinite.

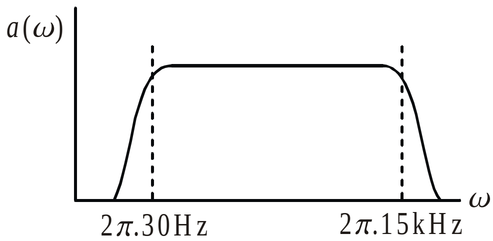
## Frequency response of an amplifier

The human ear responds to frequencies from about 30 Hz to about 15 kHz. An audio amplifier needs to cover this frequency range for faithful sound reproduction. In order to measure the frequency response of such an amplifier one would connect a signal generator to the amplifier's input and an AC voltmeter to the output. The amplitude of the generator is kept constant. The generator's frequency is varied and the voltmeter reading is recorded as a function of the frequency.




*measuring the frequency response of an amplifier*

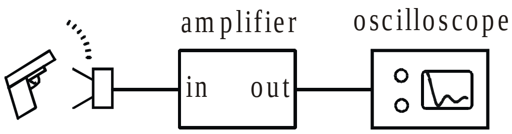
When the gain is plotted as a function of frequency there will be a flat region corresponding to the pass band and the gain will drop at higher and lower frequencies.




*amplifier frequency response*

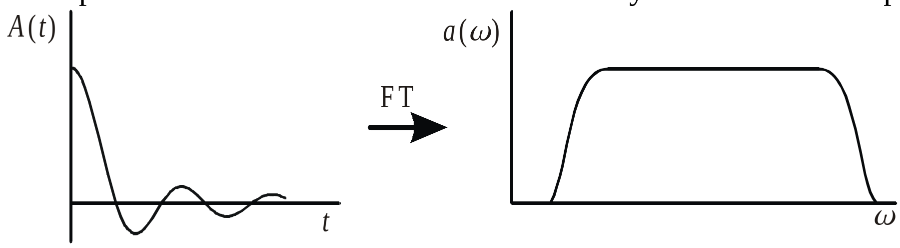
It takes time to change the generator frequency, let it settle down and then measure the amplifier output. All the data points must be collected and then plotted. Could the process be done quicker? Perhaps we could apply and measure all frequencies at the same time!

If we put all frequencies together then we will produce a delta function. This could be implemented as a gun shooting in the vicinity of a microphone. Then the output, transient response, of the amplifier contains the superposition of the responses to the constituent sinusoids of the gun shot – all frequencies in equal measure.




*transient response of amplifier*

The frequency response of the amplifier is thus given from the spectrum of sinusoids in the transient response. To find this we need to Fourier analyze the transient response.

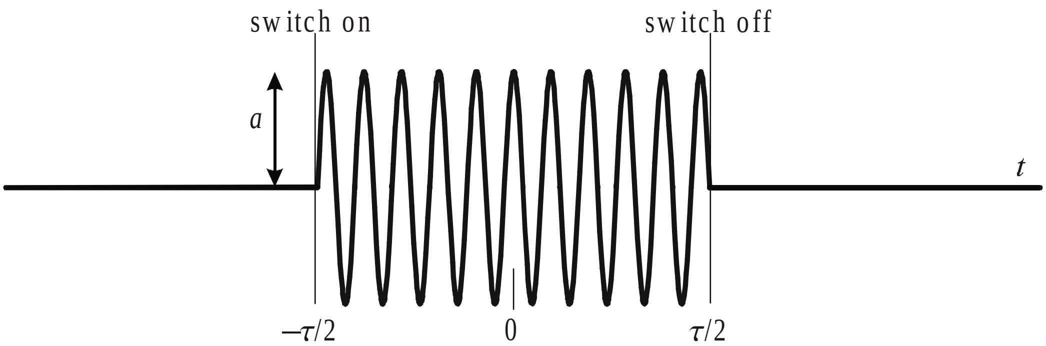



*Fourier analysis of the transient response gives the frequency response*

This demonstrates an alternative way of measuring the frequency response of an amplifier. Rather than painstakingly sweeping the input frequency and recording the output signal, we may apply a delta function input and Fourier analyze the output. In other words we apply all sine waves together and sort them at the output.

## Spectrum of a finite sine wave

A sine wave of infinite duration has a spectrum comprising a single frequency. The lesson of Fourier theory is that a sine wave of finite duration may be synthesized by combining a range of (infinite duration) sine waves over a range of frequencies.




*finite duration cosine wave*

We shall represent the wave form as

$$F(t) = ae^{i\omega_0 t} \qquad -\tau/2 < t < \tau/2$$
$$= 0 \qquad \text{otherwise .}$$

Here $a$ is the amplitude of the cosine burst and we have chosen the complex form for mathematical simplicity. Because of the linearity of the Fourier transform relations, the spectrum of the cosine is found from the real part and that of the sine from the imaginary part of the complex spectrum.

The Fourier spectrum is easily found:

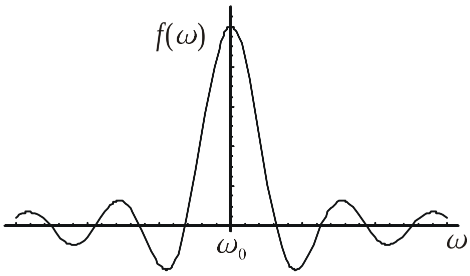
$$f(\omega) = \int_{-\infty}^{\infty} F(t) e^{i\omega t} dt$$

$$= a \int_{-\tau/2}^{\tau/2} e^{i(\omega-\omega_0)t} dt .$$

And this is integrated to give

$$f(\omega) = \frac{2a}{\omega-\omega_0} \sin(\omega-\omega_0)\tau/2 .$$

This is the sinc function we have encountered before. Now however, it is centred on and peaked at $\omega = \omega_0$ .




*spectrum of a finite sinusoid*

The value of $f(\omega)$ at its peak at $\omega = \omega_0$ is found by expanding the sin before setting $\omega = \omega_0$; this gives

$$f(\omega_0) = a\tau .$$

*Very long sinusoid*

Now $a$ the amplitude of the sine wave is a finite quantity. It then follows that as the sinusoid is of longer and longer duration, the peak $f(\omega_0)$ becomes infinitely large, but at the same time it gets narrower and narrower. In particular, when the train is of infinite duration the height becomes infinite as the width tends to zero. But this is reminiscent of the delta function. Let us check on the area of $f(\omega)$ to see if it remains finite. We can find the area of $f(\omega)$ by using a trick. We start with the inverse Fourier transform, giving $F(t)$ in terms of $f(\omega)$:

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega) e^{-i\omega t} d\omega .$$

The area is given by the integral with $t = 0$. Thus we have

$$\int_{-\infty}^{\infty} f(\omega) d\omega = 2\pi F(0) .$$

But $F(0) = a$, so that the area is indeed finite: $2\pi a$.

Since the area under the delta function is unity, by definition, it follows that the spectrum for an infinite sinusoid of (angular) frequency $\omega_0$ and amplitude $a$ is given by

$$f(\omega) = 2\pi a\delta(\omega - \omega_0).$$

Of course writing the spectrum as a delta function is equivalent, physically, to saying that the waveform comprises a sinusoid at a single frequency only.

Incidentally, this gives an alternative mathematical representation of the delta function as

$$\delta(\omega) = \lim_{\tau\to 0}\frac{1}{\pi\omega}\sin\omega\tau/2.$$

*A paradox*

Let us return to a sine train of finite duration. We have seen how its spectrum is described by the sinc function

$$f(\omega) = \frac{2a}{\omega - \omega_0}\sin(\omega - \omega_0)\tau/2.$$

In particular, the $t$ here indicates the duration of the train.

There is a laboratory instrument known as a *spectrum analyzer*. Its function is to display the spectrum of its input signal. Now consider the following:

You have a spectrum analyzer set up in the laboratory. At a given instant you turn on a sine wave generator connected to the analyzer. If you look at the analyzer display you will see the signal's spectrum. Then by studying the shape of the spectrum you can extract the value of $t$. So you will know how ling the train will last, and when the signal generator will be switched off. In other words a spectrum analyzer can enable you to see into the future!

The paradox is resolved by appreciating that the spectrum analyzer has a frequency resolution determined by the length of time it receives its input. An infinite duration is required for infinite resolution. And when the signal is observed for a time $t$ only there is a corresponding uncertainty in the frequency resolution of $\Delta\omega \sim 1/t$, meaning that a delta function spike is broadened by this amount. Thus the analyzer is not able to tell you when to turn the generator off.

## The Gaussian curve

One function is the Fourier transform of itself. This is the Gaussian function

$$F(t) = e^{-t^2/2}.$$

The Fourier transform is calculated as

$$f(\omega) = \int_{-\infty}^{\infty} F(t) e^{i\omega t} dt$$
$$= \int_{-\infty}^{\infty} e^{i\omega t - t^2/2} dt.$$

This can be evaluated by hand, by completing the square in the exponent to give

$$f(\omega) = \sqrt{2\pi}\ e^{-\omega^2/2}.$$

This property is unique to the Gaussian function. You shouldn't be concerned with the factors of $2\pi$ which are dotted around; their precise positions are purely a matter of convention.

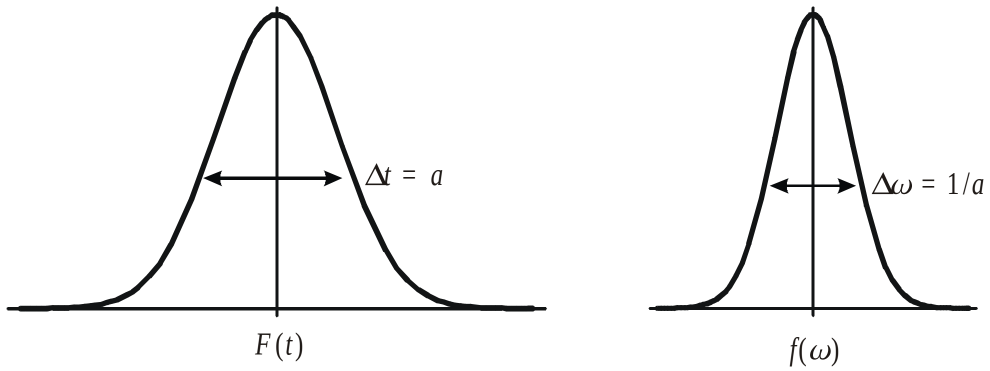
Things become more interesting when we specify the width of the Gaussian curve. If we write

$$F(t) = e^{-t^2/2a^2}$$

then $a$ is the rms width of the curve. The Fourier transform of this can be obtained from the previous result by a change of variables; the result is

$$f(\omega) = \sqrt{2\pi a^2}\ e^{-a^2\omega^2/2}.$$

So when the width of $F(t)$ is $a$, the corresponding width of $f(\omega)$ is $1/a$.




*Gaussian function and its Fourier transform*

Yet again we have encountered the result that the 'width' of a function and of its Fourier transform have an inverse relation between them. For the time-frequency Fourier pair we express this as

$$\Delta t \Delta\omega = 1$$

while for the position-wave number Fourier pair we have correspondingly

$$\Delta x \Delta k = 1.$$

These results hold for a Gaussian curve. Now the Gaussian has the important property that the uncertainty product is a minimum. For general functions the uncertainty product is greater than this. Thus in the general case we write the Fourier transform uncertainty relations as

$$\Delta t \Delta \omega \geq 1$$

$$\Delta x \Delta k \geq 1.$$

Physically, this is setting a fundamental limit of resolution in any spectroscopic-type of measurement.

### Connections with Quantum Mechanics

Central to quantum mechanics are the Einstein relation and the deBroglie relation:

$$E = \hbar \omega$$

$$p = \hbar k$$

where $E$ is energy, $p$ is momentum, $\omega$ is angular frequency and $k$ is wave number.

Very simply, we may substitute for $\omega$ and $k$ in the uncertainty relations using the Einstein and deBroglie relations to give

$$\Delta t \Delta E \geq \hbar$$

$$\Delta x \Delta p \geq \hbar.$$

These are the conventional quantum uncertainty relations.

We note that the wave function for a particle at a specified position $x_0$ is the delta function

$$\Psi(x) = \delta(x - x_0).$$

This follows from the probabilistic interpretation of the wave function – the uncertainty in $x$ is zero. If we then ask about a momentum measurement, we must express the wave function as a function of $p$. This is done using a Fourier transform.

$$\psi(p) = \int_{-\infty}^{\infty} \Psi(x) e^{\mathrm{i} px/\hbar} \mathrm{d}x.$$

So when $\Psi(x)$ is a delta function then $\psi(p)$ is a constant; the uncertainty in $p$ is infinite. On the other hand, if the particle has a well-defined momentum $p_0$ then

$$\psi(p) = \delta(p - p_0),$$

the position wave function is a constant and so the uncertainty in position is infinite.

The same arguments apply to time and energy. In particular, the time-energy uncertainty relation tells us that energy conservation may be violated – for short times. You may ‘borrow’ a large amount of energy for a short time or a small amount for a longer time. One also concludes that it takes a ‘long’ time to make an accurate measurement of energy.