

Constructing Fourier series

First, for convenience, let's define `sn` and `cn` as the sin and cos basis functions

```
In[1]:= sn = Sin[n π x / L]
```

```
Out[1]= Sin[ $\frac{n \pi x}{L}$ ]
```

```
In[2]:= cn = Cos[n π x / L]
```

```
Out[2]= Cos[ $\frac{n \pi x}{L}$ ]
```

The next line sets the font type and size for graphics; it will be more legible.

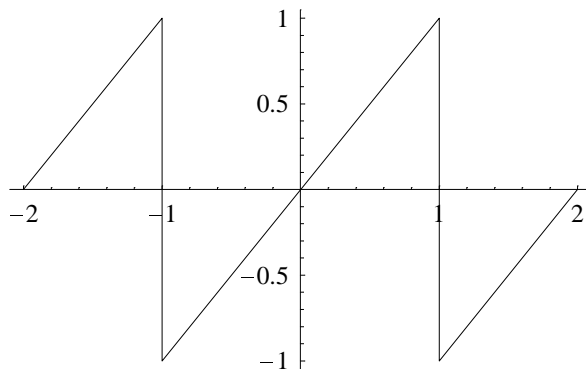
```
In[3]:= $TextStyle = {FontFamily -> "Times", FontSize -> 12}
```

```
Out[3]= {FontFamily -> Times, FontSize -> 12}
```

Sawtooth wave

First we will plot the wave to see what it looks like. In all the plots we will set $L = 1$. The function is defined over the interval $-L \leq x \leq L$. We extend the plot range to exhibit the periodicity.

```
ListPlot[{{-2, 0}, {-1, 1}, {-1, -1}, {1, 1}, {1, -1}, {2, 0}}, PlotJoined -> True]
```



- Graphics -

We use the Euler formulae for the Fourier components a_n and b_n :

```
an = Integrate[x cn, {x, -L, L}] / L^2
```

```
0
```

This is zero because the sawtooth is an odd function of x

```
bn = Integrate[x sn, {x, -L, L}] / L^2
```

$$\frac{-\frac{2L^2 \cos[n\pi]}{n\pi} + \frac{2L^2 \sin[n\pi]}{n^2\pi^2}}{L^2}$$

This looks a bit complicated, so let's try and simplify it

```
FullSimplify[bn]
```

$$\frac{2 (-n \pi \cos [n \pi] + \sin [n \pi])}{n^2 \pi^2}$$

We know that n is an integer variable. This enables the above expression to be further simplified. The following line shows how the supplementary condition is incorporated into the simplification commands. You must use the correct \in symbol; this is not the Greek epsilon ϵ , but the 'element of' symbol \in in the palette containing π , e , i , ∞ etc.

```
FullSimplify[bn, n \in Integers]
```

$$-\frac{2 (-1)^n}{n \pi}$$

Next I have defined the function $b[n]$ to give the Fourier sin coefficients. The copy / paste facility can be used to insert the right hand side from the expression above.

$$b[n_] := -\frac{2 (-1)^n}{n \pi}$$

Let's try it for $b[1]$:

```
b[1]
```

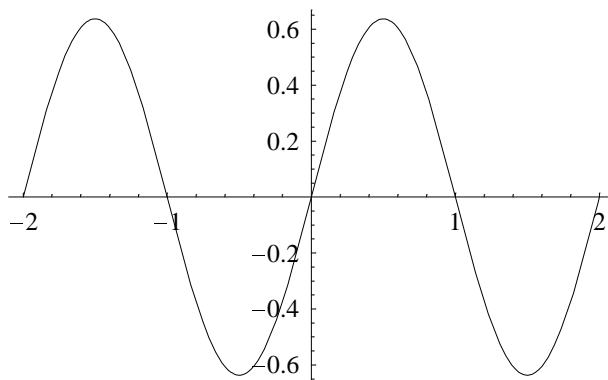
$$\frac{2}{\pi}$$

Now we will define the sum of the first m Fourier coefficients, so we can make comparisons for different m .

```
s[m_] := Sum[b[n] Sin[n \pi x], {n, 1, m}]
```

Now let's plot just the first Fourier coefficient -- the "fundamental"

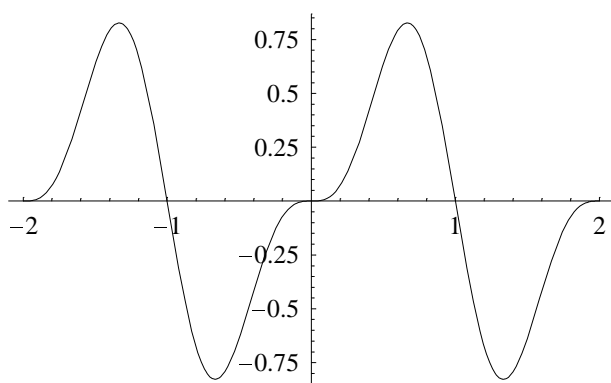
```
Plot[s[1], {x, -2, 2}]
```



- Graphics -

The first two terms look like this:

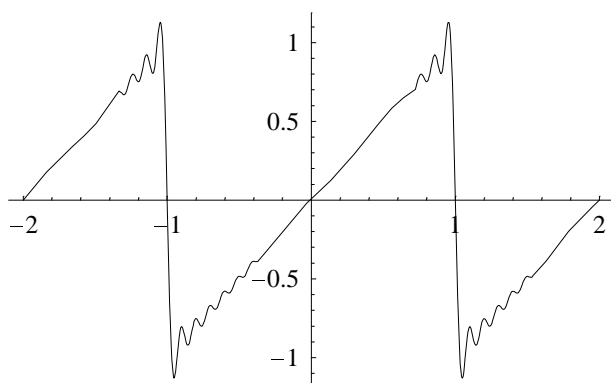
```
Plot[s[2], {x, -2, 2}]
```



- Graphics -

Extending the sum to 20 terms gives:

```
Plot[s[20], {x, -2, 2}]
```

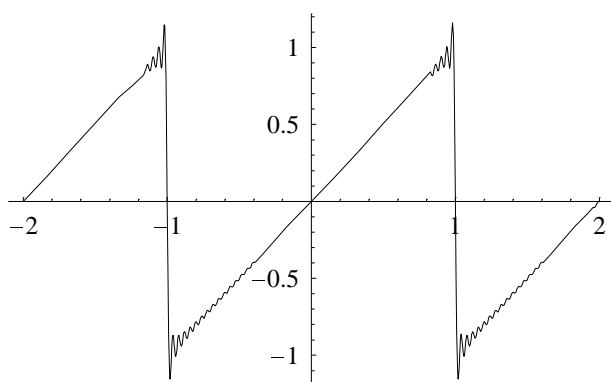


- Graphics -

Observe the Gibbs phenomenon.

Extending the sum to 50 terms gives

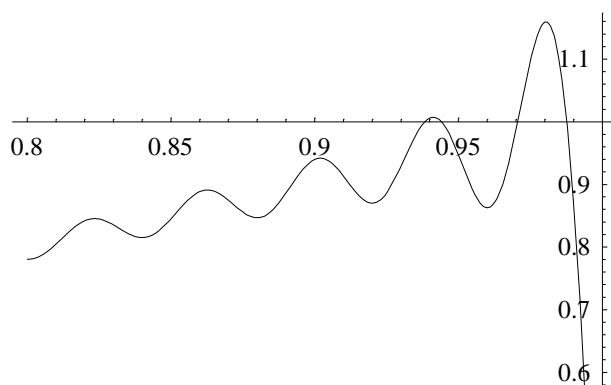
```
Plot[s[50], {x, -2, 2}]
```



- Graphics -

We can get a closer look at the Gibbs phenomenon

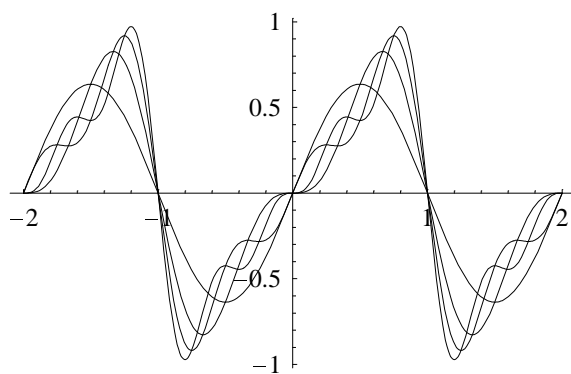
```
Plot[s[50], {x, 0.8, 1}]
```



- Graphics -

We can superimpose a number of the plots to see how adding more terms improves the fit

```
Plot[{s[1], s[2], s[3], s[4]}, {x, -2, 2}]
```



- Graphics -

The Table function will allow us to tabulate the Fourier coefficients

```
t = Table[b[n], {n, 1, 8}]
```

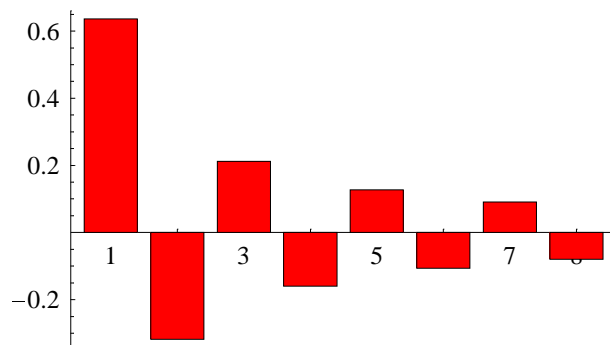
```
{ 2/π, -1/π, 2/3π, -1/2π, 2/5π, -1/3π, 2/7π, -1/4π }
```

The "Graphics" package includes a command to plot bar charts. This is a convenient way to represent the Fourier components. The next command loads the package.

```
In[4]:= << Graphics`
```

And now we ask for the bar chart. Note the bars are labelled from 1 automatically.

```
BarChart[t, PlotRange -> All]
```

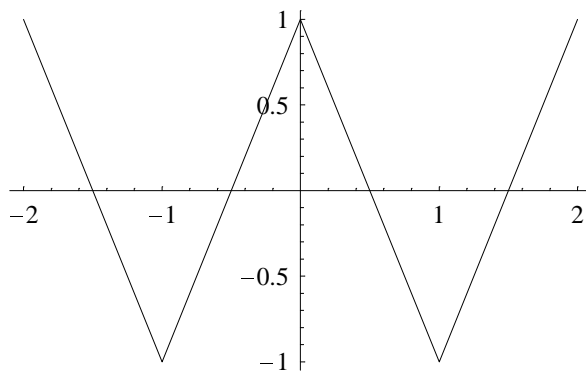


- Graphics -

Triangular wave

First plot the function.

```
ListPlot[{{-2, 1}, {-1, -1}, {0, 1}, {1, -1}, {2, 1}}, PlotJoined -> True]
```



- Graphics -

We will again use a_n and b_n to denote the Fourier components. These variables don't actually need to be cleared since they will be re-defined.

$$a_n = \frac{2}{L^2} \int_{-L}^0 (L/2 + x) \cos(n\pi x/L) dx + \frac{2}{L^2} \int_0^L (L/2 - x) \cos(n\pi x/L) dx$$

$$= \frac{4}{L^2} \left(\frac{L^2}{n^2 \pi^2} - \frac{L^2 \cos[n\pi]}{n^2 \pi^2} - \frac{L^2 \sin[n\pi]}{2 n \pi} \right)$$

Looks a bit complicated, so we try and simplify it

```
FullSimplify[%]
```

$$\frac{4 - 4 \cos[n\pi] - 2 n \pi \sin[n\pi]}{n^2 \pi^2}$$

We know that n is an integer variable. This enables the above expression to be further simplified.

```
FullSimplify[an, n ∈ Integers]
```

$$-\frac{4(-1 + (-1)^n)}{n^2 \pi^2}$$

Now for the bn

```
bn = 2 Integrate[(L/2 + x) sn, {x, -L, 0}] / L^2 +
      2 Integrate[(L/2 - x) sn, {x, 0, L}] / L^2
```

$$\frac{2 \left(-\frac{L^2}{2n\pi} + \frac{L^2 \cos[n\pi]}{2n\pi} - \frac{L^2 \sin[n\pi]}{n^2 \pi^2} \right)}{L^2} + \frac{2 \left(-\frac{L^2}{2n\pi} - \frac{L^2 \cos[n\pi]}{2n\pi} + \frac{L^2 \sin[n\pi]}{n^2 \pi^2} \right)}{L^2}$$

This looks messy, so try and simplify:

```
FullSimplify[bn]
```

```
0
```

This is zero, as expected since we are now dealing with an even function; it only has cos terms in its Fourier series.

We shall simplify the expression for the cos components using the fact that n is an integer variable. We shall define the function a[n] to give the Fourier cos coefficients. We can use copy and paste for this:

$$a[n_] := -\frac{4(-1 + (-1)^n)}{n^2 \pi^2}$$

Now try an example to test it:

```
a[1]
```

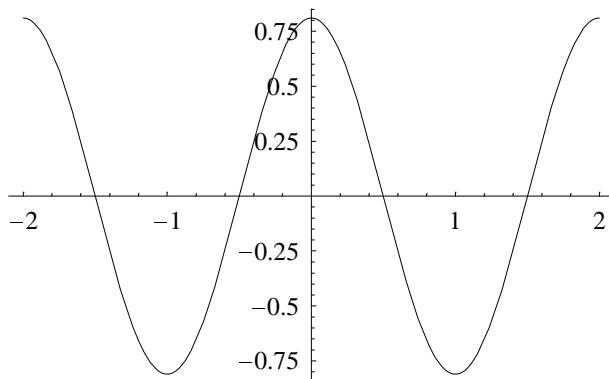
$$\frac{8}{\pi^2}$$

Now we will define the sum of the first m Fourier coefficients, so we can make comparisons for different m.

```
s[m_] := Sum[a[n] Cos[n π x], {n, 1, m}]
```

Here is the fundamental.

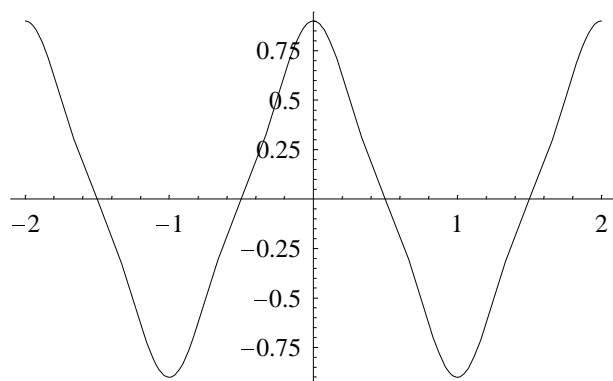
```
Plot[s[1], {x, -2, 2}]
```



```
- Graphics -
```

There is no second harmonic, but adding the third harmonic gives a very good improvement.

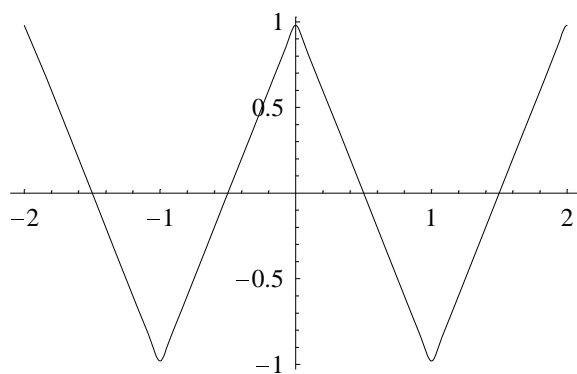
```
Plot[s[3], {x, -2, 2}]
```



- Graphics -

And when we have terms up to the 20th it looks very good indeed.

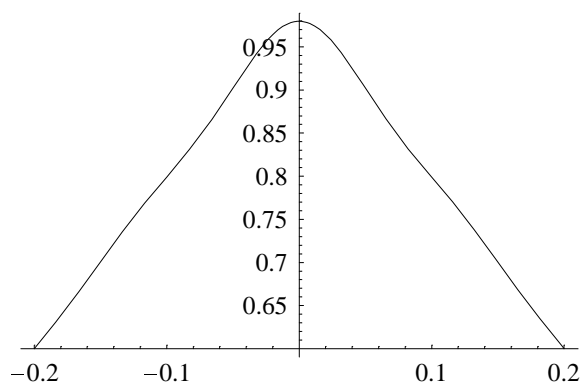
```
Plot[s[20], {x, -2, 2}]
```



- Graphics -

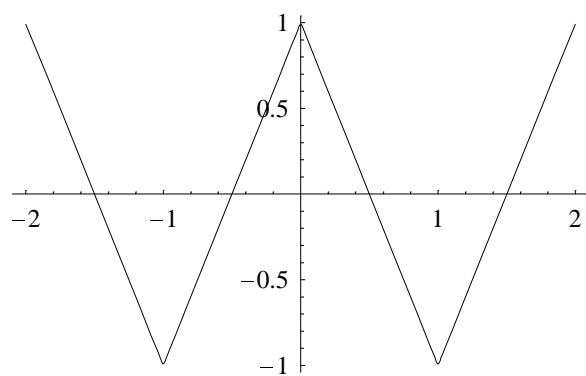
We shall look closely in the vicinity of the corner at $x = 0$ to see how good the fit is in this potentially problematic region.

```
Plot[s[20], {x, -0.2, 0.2}]
```



- Graphics -

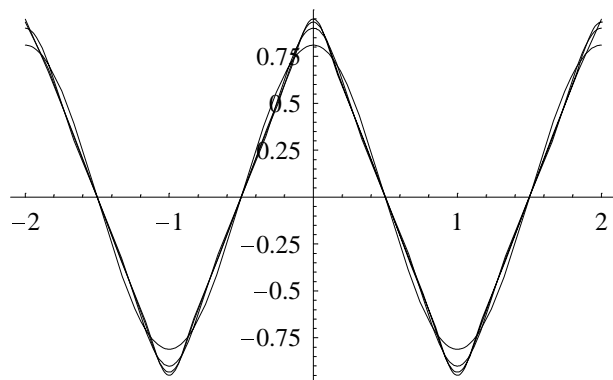
```
Plot[s[50], {x, -2, 2}]
```



- Graphics -

We can plot a number of the partial sums to see how things improve.

```
Plot[{s[1], s[3], s[5], s[7]}, {x, -2, 2}]
```



- Graphics -

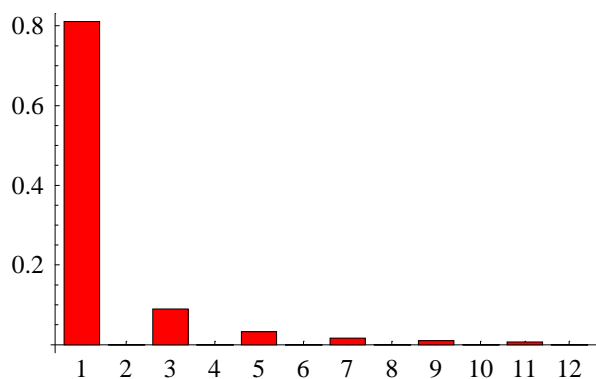
We now tabulate the Fourier coefficients,

```
t = Table[a[n], {n, 1, 12}]
```

$$\left\{ \frac{8}{\pi^2}, 0, \frac{8}{9\pi^2}, 0, \frac{8}{25\pi^2}, 0, \frac{8}{49\pi^2}, 0, \frac{8}{81\pi^2}, 0, \frac{8}{121\pi^2}, 0 \right\}$$

and plot them.


```
BarChart[t, PlotRange -> All]
```

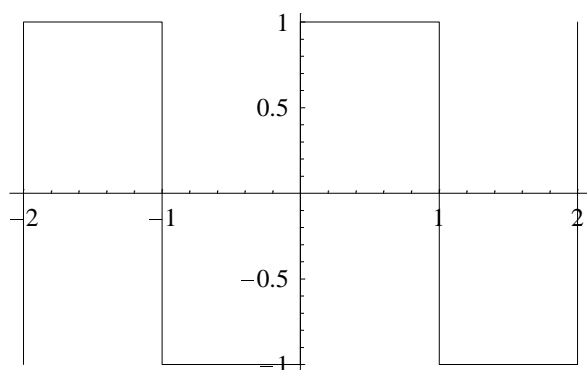


- Graphics -

Square wave

First plot the function.

```
ListPlot[{{-2, -1}, {-2, 1}, {-1, 1}, {-1, -1}, {0, -1},
          {0, 1}, {1, 1}, {1, -1}, {2, -1}, {2, 1}}, PlotJoined -> True]
```



- Graphics -

Here are the b_n components.

```
bn = (2 / L) Integrate[sn, {x, 0, L}]
```

$$\frac{2}{L} \left(\frac{L}{n\pi} - \frac{L \cos[n\pi]}{n\pi} \right)$$

We won't even try to find the a_n coefficients; since the function, as defined, is odd, there are only sin terms in the Fourier series.

We shall simplify the b_n :

```
Simplify[bn, n ∈ Integers]
```

$$\frac{2 - 2 \cos[n\pi]}{n\pi}$$

We shall define the function for the sin coefficients using straightforward 'cut and paste':

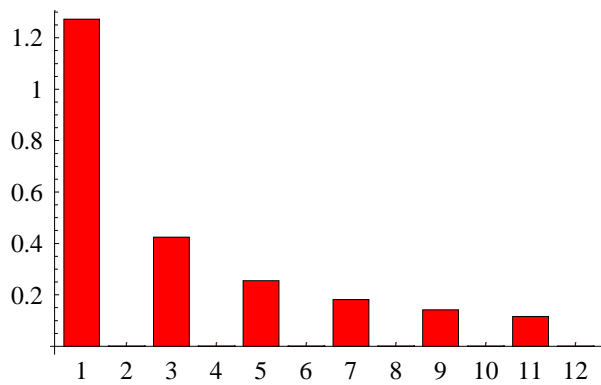
$$b[n_] := \frac{2 - 2 \cos[n\pi]}{n\pi}$$

And we shall tabulate and plot the coefficients directly.

```
t = Table[b[n], {n, 1, 12}]
{4/π, 0, 4/3π, 0, 4/5π, 0, 4/7π, 0, 4/9π, 0, 4/11π, 0}
```

The graphics package has been loaded already.

```
BarChart[t, PlotRange -> All]
```



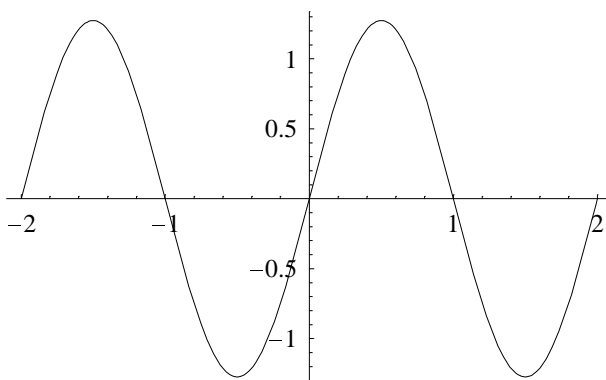
- Graphics -

Now we will define the sum of the first m Fourier coefficients, so we can make comparisons for different m.

```
s[m_] := Sum[b[n] Sin[nπ x], {n, 1, m}]
```

Here is the fundamental:

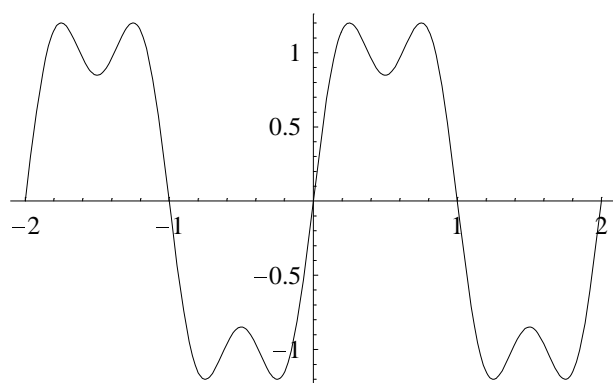
```
Plot[s[1], {x, -2, 2}]
```



- Graphics -

Including the next term gives:

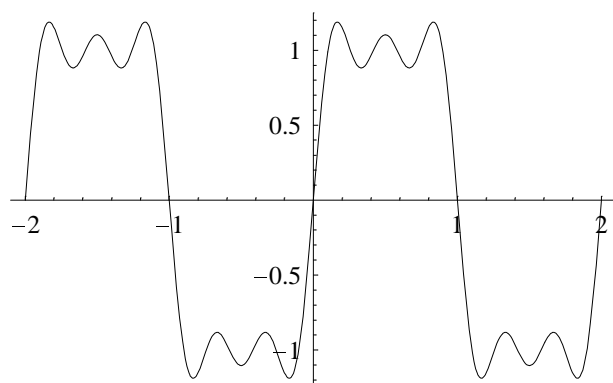
```
Plot[s[3], {x, -2, 2}]
```



- Graphics -

And it gradually gets better:

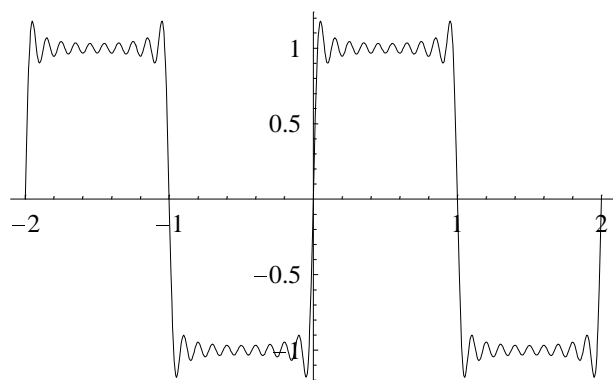
```
Plot[s[5], {x, -2, 2}]
```



- Graphics -

With 20 terms things look good but the Gibbs phenomenon is apparent.

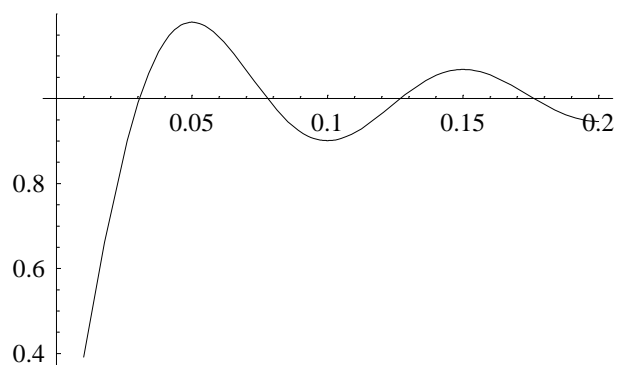
```
Plot[s[20], {x, -2, 2}]
```



- Graphics -

Looking in the vicinity of a step shows the Gibbs phenomenon clearly.

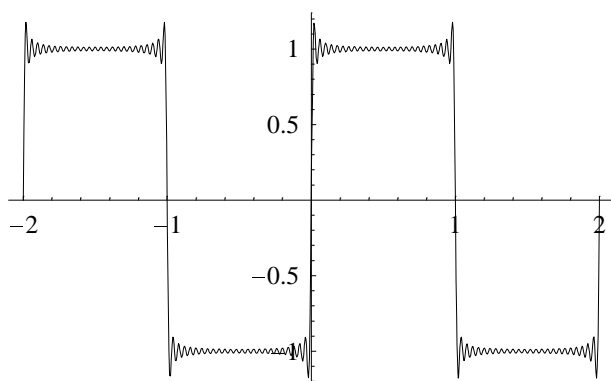
```
Plot[s[20], {x, 0.01, 0.2}]
```



- Graphics -

Here is the sum to the 50th term.

```
Plot[s[50], {x, -2, 2}]
```

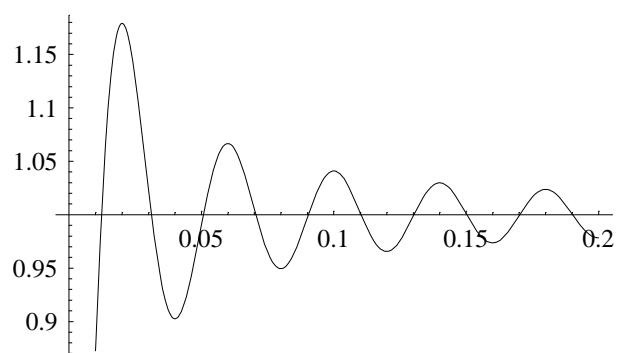


- Graphics -

The important point is that the Gibbs phenomenon remains - even with a large number of terms.

Here is the Gibbs phenomenon.

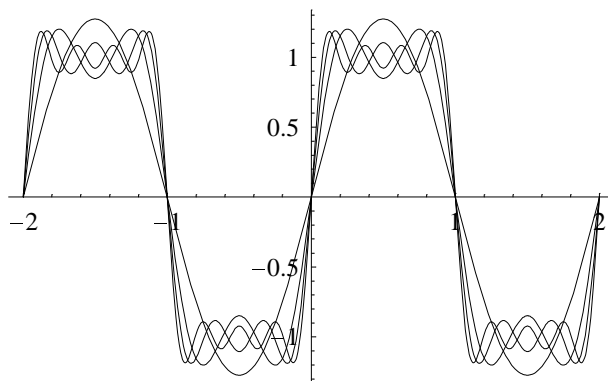
```
Plot[s[50], {x, 0.01, 0.2}]
```



- Graphics -

We can plot a number of the partial sums to see how things gradually improve.

```
Plot[{s[1], s[3], s[5], s[7]}, {x, -2, 2}]
```

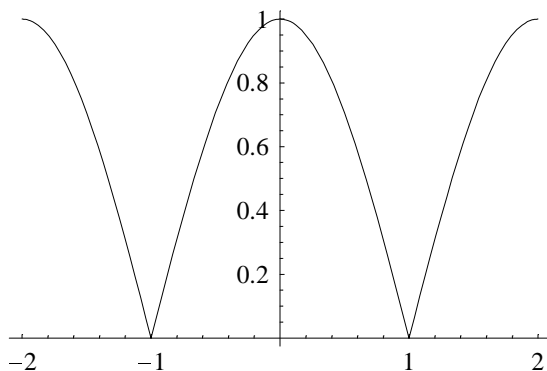


- Graphics -

Full wave rectified sine curve

First we plot the function.

```
Show[Plot[Cos[x π / 2], {x, -1, 1}],
Plot[-Cos[x π / 2], {x, -2, -1}], Plot[-Cos[x π / 2], {x, 1, 2}]]
```



- Graphics -

Here is the an integral.. There are no sin terms in the series as we have specified the function to be even in x.

```
an = (1 / L) Integrate[Cos[π x / (2 L)] cn, {x, -L, L}]
```

$$\frac{2L \operatorname{Sin}\left[\frac{1}{2}(-1+2n)\pi\right]}{(-1+2n)\pi} + \frac{2L \operatorname{Sin}\left[\frac{1}{2}(1+2n)\pi\right]}{(1+2n)\pi}$$

Simplifying this gives:

```
FullSimplify[an, n ∈ Integers]
```

$$-\frac{4(-1)^n}{-\pi + 4n^2\pi}$$

We define the a[n] as a function of n

$$\mathbf{a[n_]} := -\frac{4(-1)^n}{-\pi + 4n^2\pi}$$

and we tabulate the cos coefficients

```
t = Table[a[n], {n, 1, 12}]
```

$$\left\{ \frac{4}{3\pi}, -\frac{4}{15\pi}, \frac{4}{35\pi}, -\frac{4}{63\pi}, \frac{4}{99\pi}, -\frac{4}{143\pi}, \right.$$

$$\left. \frac{4}{195\pi}, -\frac{4}{255\pi}, \frac{4}{323\pi}, -\frac{4}{399\pi}, \frac{4}{483\pi}, -\frac{4}{575\pi} \right\}$$

In this case there is an $n = 0$ coefficient since the function does not average to zero.

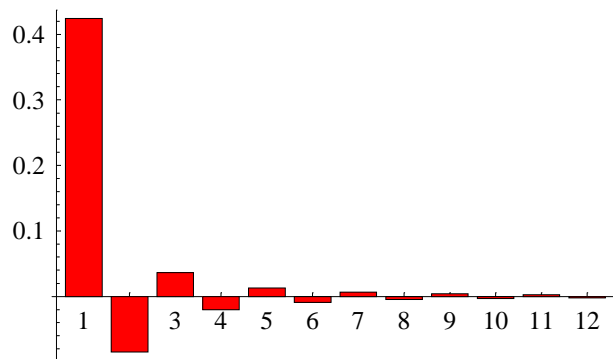
We don't include it in the table of coefficients since the bar chart automatically labels the first bar as $n = 1$.

Here is the $n = 0$ term

```
a[0]
```

$$\frac{4}{\pi}$$

```
BarChart[t, PlotRange -> All]
```



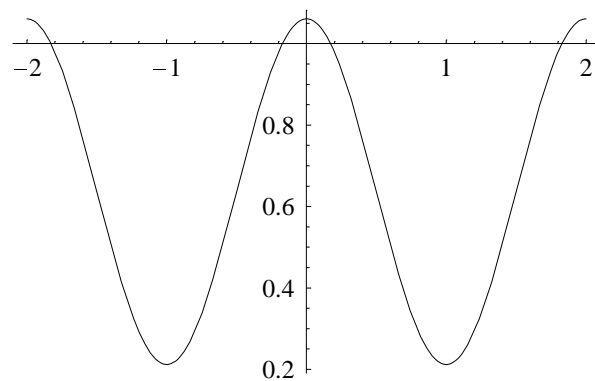
- Graphics -

Now we will define the sum of the first m Fourier coefficients, so we can make comparisons for different m .

```
s[m_] := a[0] / 2 + Sum[a[n] Cos[n π x], {n, 1, m}]
```

Here is the fundamental.

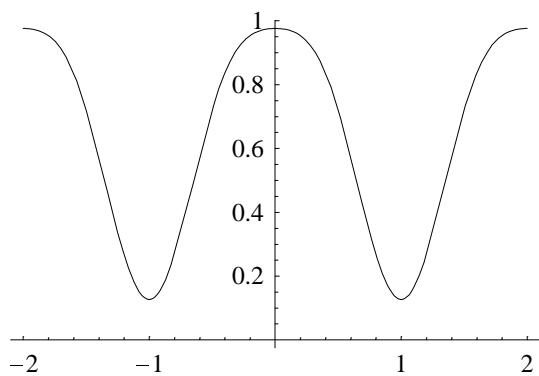
```
Plot[s[1], {x, -2, 2}]
```



- Graphics -

With the first harmonic the top is flattening and the bottom sharpening.

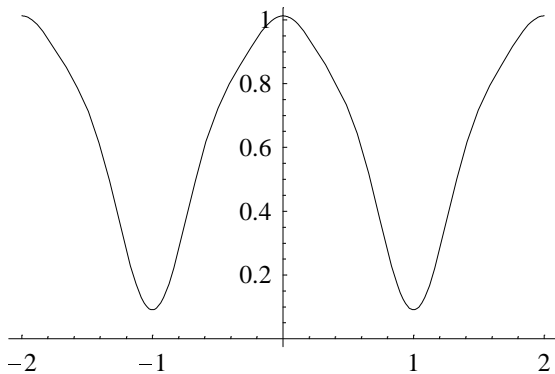
```
Plot[s[2], {x, -2, 2}]
```



- Graphics -

Here is the three term sum

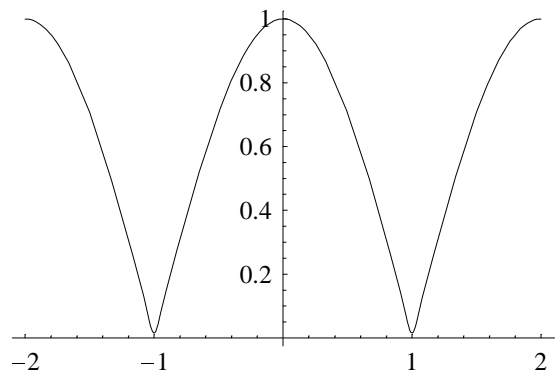
```
Plot[s[3], {x, -2, 2}]
```



- Graphics -

With 20 terms things look very good indeed.

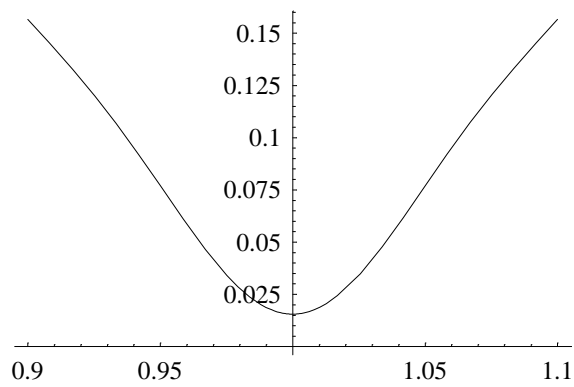
```
Plot[s[20], {x, -2, 2}]
```



- Graphics -

The next plot shows how the Fourier series behaves in the vicinity of the 'cusp'.

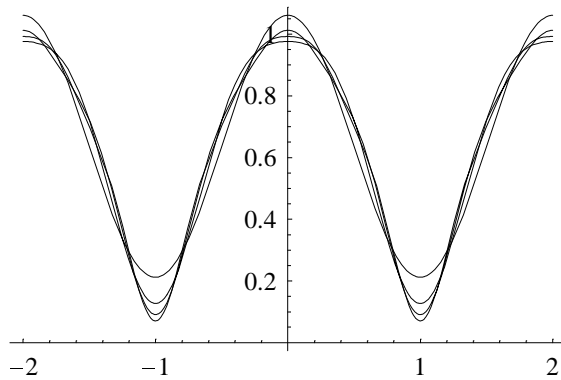
```
Plot[s[20], {x, 0.9, 1.1}]
```



- Graphics -

We can plot a number of the partial sums to see how things gradually improve.

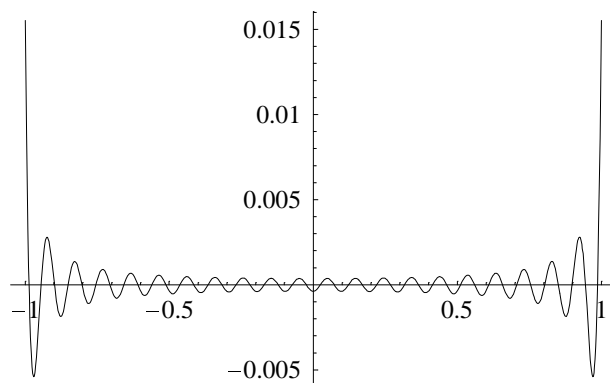
```
Plot[{s[1], s[2], s[3], s[4]}, {x, -2, 2}]
```



- Graphics -

Finally we can look at how the 20 term Fourier series differs from the original function.

```
Plot[s[20] - Cos[π x / 2], {x, -1, 1}, PlotRange -> All]
```



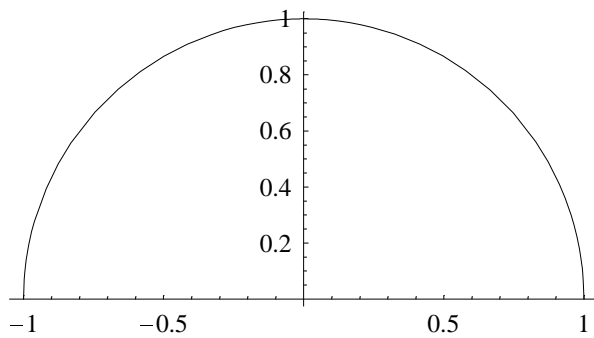
- Graphics -

Semicircle

```
In[16]:= y[x_] := Sqrt[1 - x^2]
```



```
Plot[y[x], {x, -1, 1}, AspectRatio -> 0.5]
```



```
- Graphics -
```

Only cos terms will appear since the semicircle is an even function of x . It is convenient to split the integral into two bits.

```
an1 = Integrate[y[x] Cos[n π x], {x, 0, 1}]
```

```
If[Im[n] == 0,  $\frac{\text{BesselJ}[1, n \pi]}{2 n}$ ,  $\int_0^1 \sqrt{1-x^2} \text{Cos}[n \pi x] dx$ ]
```

```
an2 = Integrate[y[x] Cos[n π x], {x, -1, 0}]
```

```
 $\int_{-1}^0 \sqrt{1-x^2} \text{Cos}[n \pi x] dx$ 
```

This is what *Mathematica* 4 gives us. Version 3 gives the correct answer. Since the semicircle is even, we know that a_{n2} must be equal to a_{n1} . Then the expression for the a coefficients must be:

```
In[5]:= a[n_] :=  $\frac{\text{BesselJ}[1, n \pi]}{n}$ 
```

First we find the amplitude of the constant term: the coefficient of $\text{Cos}[0]$

```
In[6]:= a[0] = Limit[ $\frac{\text{BesselJ}[1, n \pi]}{n}$ , n -> 0]
```

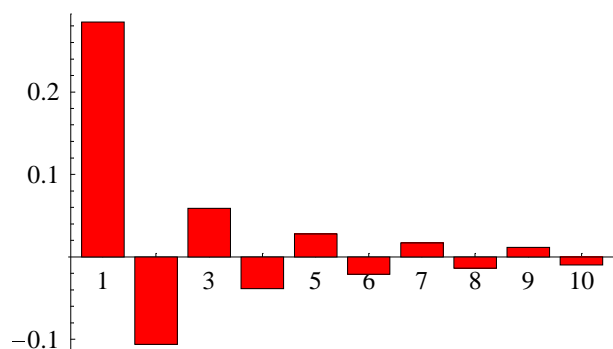
```
Out[6]=  $\frac{\pi}{2}$ 
```

And now we construct a table and `chsrt` of the higher-order terms.

```
In[7]:= A = Table[N[a[n]], {n, 1, 10}]
```

```
Out[7]= {0.284615, -0.106191, 0.0589084, -0.0386327, 0.027805,  
-0.0212348, 0.0168984, -0.0138602, 0.0116347, -0.00994692}
```

```
In[8]:= BarChart[A, PlotRange -> All]
```



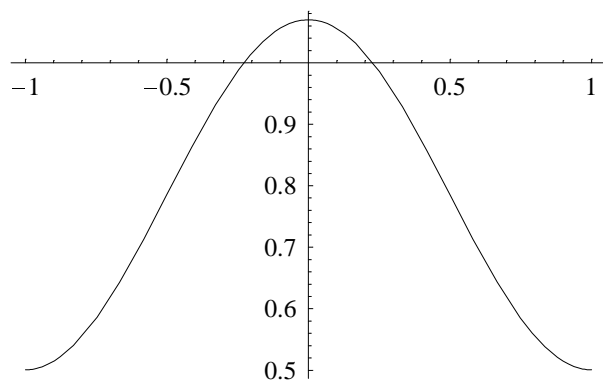
```
Out[8]= - Graphics -
```

Next we construct a function for partial sums.

```
In[9]:= s[m_] := a[0] / 2 + Sum[a[n] Cos[n π x], {n, 1, m}]
```

The following picture shows the constant term plus the first sinusoid.

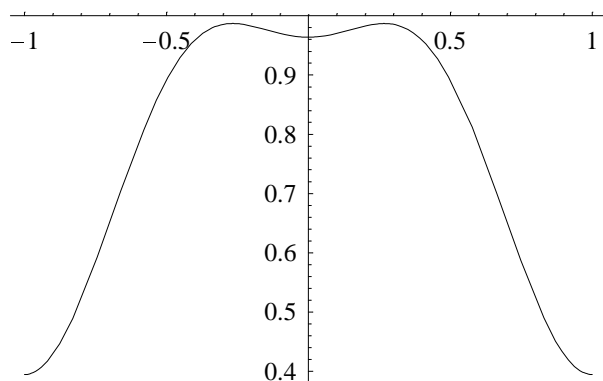
```
In[10]:= Plot[s[1], {x, -1, 1}]
```



```
Out[10]= - Graphics -
```

Adding another term gives:

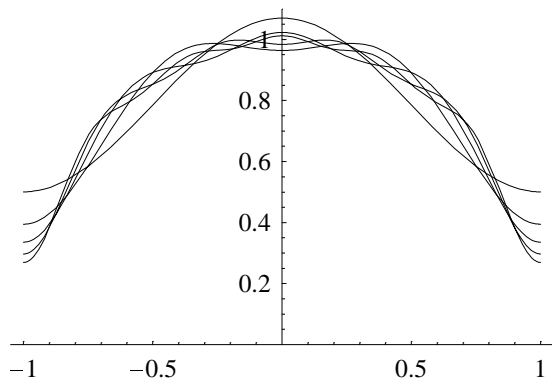
```
In[11]:= Plot[s[2], {x, -1, 1}]
```



```
Out[11]= - Graphics -
```

Here is a sequence of approximations

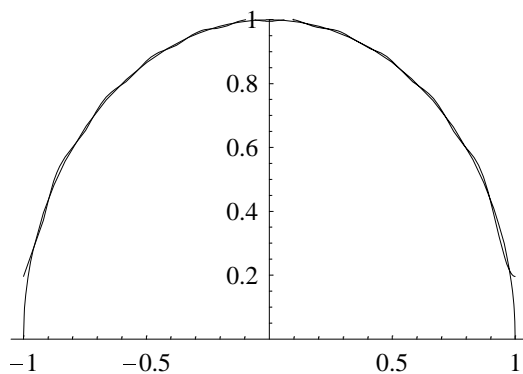
```
In[12]:= Plot[{s[1], s[2], s[3], s[4], s[5]}, {x, -1, 1}, PlotRange -> {0, 1.1}]
```



```
Out[12]= - Graphics -
```

This next curve comprises the first ten harmonics, compared with the original semicircle

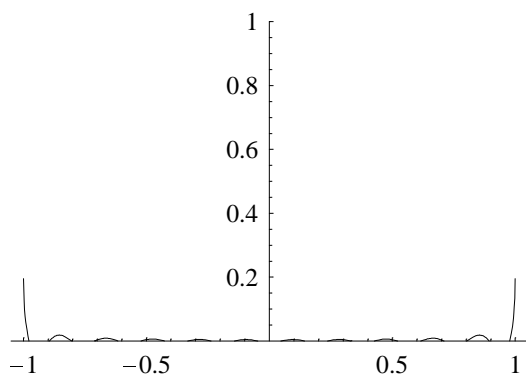
```
In[17]:= Plot[{s[10], y[x]}, {x, -1, 1}, PlotRange -> {0, 1}]
```



```
Out[17]= - Graphics -
```

Finally we examine the error in this approximation.

```
In[18]:= Plot[s[10] - y[x], {x, -1, 1}, PlotRange -> {0, 1}]
```



```
Out[18]= - Graphics -
```