UNIVERSITY OF LONDON

BSc EXAMINATION 1999

For Internal Students of

Royal Holloway

DO NOT TURN OVER UNTIL TOLD TO BEGIN

PH2130B: MATHEMATICAL METHODS

Time Allowed: TWO hours

Answer QUESTION ONE and TWO other questions

No credit will be given for attempting any further questions

Approximate part-marks for questions are given in the right-hand margin

Calculators ARE permitted

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GENERAL PHYSICAL CONSTANTS

Permeability of vacuum	μ_0	=	$4\pi \times 10^{-7}$	H m^{-1}
Permittivity of vacuum	ε_0	=	8.85×10^{-12}	F m^{-1}
	$1/4\pi\varepsilon_0$	=	9.0×10^{9}	m F^{-1}
Speed of light in vacuum	c	=	3.00×10^{8}	m s^{-1}
Elementary charge	e	=	1.60×10^{-19}	C
Electron (rest) mass	m_e	=	9.11×10^{-31}	kg
Unified atomic mass constant	m_u	=	1.66×10^{-27}	kg
Proton rest mass	m_p	=	1.67×10^{-27}	kg
Neutron rest mass	m_n	=	1.67×10^{-27}	kg
Ratio of electronic charge to mass	e/m_e	=	1.76×10^{11}	C kg^{-1}
Planck constant	h	=	6.63×10^{-34}	J s
	$\hbar = h/2\pi$	=	1.05×10^{-34}	J s
Boltzmann constant	k	=	1.38×10^{-23}	J K^{-1}
Stefan-Boltzmann constant	σ	=	5.67×10^{-8}	W m^{-2} K^{-4}
Gas constant	R	=	8.31	J mol^{-1} K^{-1}
Avogadro constant	N_A	=	6.02×10^{23}	mol^{-1}
Gravitational constant	G	=	6.67×10^{-11}	N m^2 kg^{-2}
Acceleration due to gravity	g	=	9.81	m s^{-2}
Volume of one mole of an ideal gas at STP		=	2.24×10^{-2}	m^3
One standard atmosphere	P_0	=	1.01×10^{5}	N m^{-2}

MATHEMATICAL CONSTANTS

e = 2.718 π = 3.142 $\log_e 10$ = 2.303

PART MARKS

ANSWER **ONLY FIVE** sections of ***Question One***

You are advised not to spend more than **40 minutes** answering ***Question One***

1. (a) The equation

$$\int_a^b \varphi_m(x)\varphi_n(x)\mathrm{d}x = \delta_{mn}$$

is known as an *orthogonality integral* for the functions $\varphi_n(x)$. How may an arbitrary function $f(x)$ be expressed in terms of the functions $\varphi_n(x)$ in the range $a \le x \le b$? **[4]**

(b) The Laplacian operator ∇^2 often appears in differential equations representing physical systems. Write down the form this operator takes in rectangular Cartesian coordinates and explain briefly the physical information that ∇^2 gives about a scalar field. **[4]**

(c) Write down the Fourier transform relations for the Fourier pair $f(t)$ and $F(\omega)$.

Outline briefly the procedure by which a differential equation may be solved using Fourier transfoms. **[4]**

(d) Explain the terms *order*, *degree*, *linear*, *nonlinear*, *homogeneous*, and *inhomogeneous* as applied to differential equations. Why are linear equations so much easier than nonlinear equations to solve? **[4]**

(e) Describe concisely how the power series method gives solutions to the equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = 0$$

of the form

$$y(x) = a_0\left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)$$

By examining the derivatives of the series in the brackets discuss the identification of the first and second brackets with the cosine and sine functions. **[4]**

(f) In the context of differential equations, explain the meaning of the term *quantisation*. Explain how boundary conditions can lead to quantisation. **[4]**

2. A metal bar of length L has both ends held at a fixed temperature θ_0. The bar is warmer than this towards the middle; at time $t = 0$ the temperature distribution is

$$\theta(x) = \theta_0 + \theta_1 \sin\frac{\pi x}{L}.$$

(a) Sketch this initial temperature variation. **[3]**

The field variable $\Psi(x,t) = \theta(x,t) - \theta_0$ obeys the diffusion equation

$$\frac{\partial^2 \Psi(x,t)}{\partial x^2} - \frac{1}{D}\frac{\partial \Psi(x,t)}{\partial t} = 0$$

where D is the thermal diffusion coefficient.

(b) Solve the differential equation by the method of separation of variables, to show that the temperature variation follows

$$\theta(x,t) = \theta_0 + \theta_1 \sin\left(\frac{\pi x}{L}\right) e^{-D\pi^2 t/L^2}.$$ **[14]**

(c) What is the mathematical form for the temperature variation of the centre point of the rod? Sketch this. **[3]**

3. The solution to the Bessel equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + \left(1 - \frac{m^2}{x^2}\right) y = 0$$

may be expressed as a power series

$$y(x) = \sum_{r=0}^{\infty} c_r x^{r+\rho}$$

where ρ is a constant.

Show that two solutions can be found corresponding to the values $\rho = +m$ and $\rho = -m$. **[4]**

Determine the first four terms of each solution. **[11]**

For the case where $m = -½$ find a relationship between one solution and the common trigonometric functions. **[5]**

PART MARKS

4. The sine and cosine functions satisfy the following integral relations:

$$\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\mathrm{d}x = L\delta_{mn}$$
$$\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\mathrm{d}x = L\delta_{mn}$$
$$\int_{-L}^{L} \cos\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\mathrm{d}x = 0$$

where
$$\begin{aligned}\delta_{mn} &= 1 \quad \text{when} \quad m = n \\ &= 0 \quad \text{when} \quad m \neq n\end{aligned}$$

and the case $m = n = 0$ is specifically excluded.

(a) The function $f(x)$ may be expressed as the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left\{a_m \cos\left(\frac{m\pi}{L}x\right) + b_m \sin\left(\frac{m\pi}{L}x\right)\right\}$$

over the interval $-L \le x \le L$. Show how the above integral relations allow the coefficients a_m, b_m to be found from the Euler formulae: **[7]**

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x\,.$$

(b) Obtain an expression for the Fourier components of the sawtooth function specified by **[4]**

$$f(x) = \frac{x}{L}, \qquad -L \le x \le L$$

You should find that the a_n vanish; explain this.

(c) Sketch the Fourier approximation as more terms are added to the series. **[3]**

(d) Describe the behaviour of the Fourier series *outside* the interval $-L \le x \le L$. **[3]**

(e) Discuss the way a Fourier series approximates a function in the region of a discontinuity. **[3]**

5. The Schrödinger equation for a simple harmonic oscillator may be written, in terms of a dimensionless length variable y as

$$\frac{d^2\psi}{dy^2} + \left(\varepsilon - y^2\right)\psi = 0$$

where ε is a constant.

(a) The conventional method of solution for this differential equation is to start by making the substitution

$$\psi(y) = f(y)e^{-y^2/2}.$$

What is the purpose of this substitution? **[3]**

(b) By making the substitution show that the function $f(y)$ obeys the equation

$$\frac{d^2 f}{dy^2} - 2y\frac{df}{dy} + (\varepsilon - 1)f = 0.$$ **[6]**

(c) The next step in the quantum oscillator problem is to solve the preceding equation for the function $f(y)$. Explain clearly the arguments which require $f(y)$ to be a polynomial of finite degree. **[3]**

(d) Show how this leads to the energy quantisation expression

$$\varepsilon = 2n + 1$$

where $n = 0, 1, 2, 3, \cdots$ **[8]**

END