UNIVERSITY OF LONDON

BSc and MSci EXAMINATION 2000

For Internal Students of

Royal Holloway

DO NOT TURN OVER UNTIL TOLD TO BEGIN

PH2130B: MATHEMATICAL METHODS

Time Allowed: TWO hours

Answer QUESTION ONE and TWO other questions

No credit will be given for attempting any further questions

Approximate part-marks for questions are given in the right-hand margin

Calculators ARE permitted

GENERAL PHYSICAL CONSTANTS

Permeability of vacuum	μ_0	=	$4\pi \times 10^{-7}$	H m^{-1}
Permittivity of vacuum	ε_0	=	8.85×10^{-12}	F m^{-1}
	$1/4\pi\varepsilon_0$	=	9.0×10^{9}	m F^{-1}
Speed of light in vacuum	c	=	3.00×10^{8}	m s^{-1}
Elementary charge	e	=	1.60×10^{-19}	C
Electron (rest) mass	m_e	=	9.11×10^{-31}	kg
Unified atomic mass constant	m_u	=	1.66×10^{-27}	kg
Proton rest mass	m_p	=	1.67×10^{-27}	kg
Neutron rest mass	m_n	=	1.67×10^{-27}	kg
Ratio of electronic charge to mass	e/m_e	=	1.76×10^{11}	C kg^{-1}
Planck constant	h	=	6.63×10^{-34}	J s
	$\hbar = h/2\pi$	=	1.05×10^{-34}	J s
Boltzmann constant	k	=	1.38×10^{-23}	J K^{-1}
Stefan-Boltzmann constant	σ	=	5.67×10^{-8}	W m^{-2} K^{-4}
Gas constant	R	=	8.31	J mol^{-1} K^{-1}
Avogadro constant	N_A	=	6.02×10^{23}	mol^{-1}
Gravitational constant	G	=	6.67×10^{-11}	N m^{2} kg^{-2}
Acceleration due to gravity	g	=	9.81	m s^{-2}
Volume of one mole of an ideal gas at STP		=	2.24×10^{-2}	m^{3}
One standard atmosphere	P_0	=	1.01×10^{5}	N m^{-2}

MATHEMATICAL CONSTANTS

$e \cong 2.718$ $\pi \cong 3.142$ $\log_e 10 \cong 2.303$

PART MARKS

ANSWER **ONLY FIVE** sections of ***Question One***.

You are advised not to spend more than **40 minutes** answering ***Question One***.

1. (a) Write down the diffusion equation and the wave equation. What is the principal difference between these two equations and their solutions? **[4]**

(b) In the context of differential equations, explain the meaning of the term *quantisation*. Explain how boundary conditions can lead to quantisation. **[4]**

(c) Write down an orthogonality integral for a set of functions $\varphi_n(x)$. What is meant by the *domain* and the *weight function* of the orthogonality integral? **[4]**

(d) The Fourier transform relation between the functions $f(t)$ and $F(\omega)$ is given by

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{i\omega t} \mathrm{d}\omega .$$

Express $F(\omega)$ in terms of $f(t)$ and describe briefly how differential equations may be solved using Fourier transforms. **[4]**

(e) With the aid of sketches, discuss briefly how the circularly symmetric vibrations of a drum head are related to the $J_0(r)$ Bessel function. **[4]**

(f) Give an example of a first-order second-degree homogeneous ordinary differential equation. Is this a linear equation? **[4]**

2. Two conducting concentric spherical shells are maintained at electric potentials V_{in} and V_{out}. In the space between the shells the potential obeys the Laplace equation.

(a) In rectangular Cartesian coordinates write down the differential equation obeyed by the potential $V(x, y, z)$ between the shells. **[2]**

(b) The problem of determining the potential in the region between the shells is simplified considerably through the use of spherical polar coordinates. Explain this in terms of symmetry. **[3]**

(c) In spherical polar coordinates r, ϑ, φ the laplacian operator is given by

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial}{\partial \vartheta}\right) + \frac{1}{r^2 \sin^2\vartheta}\frac{\partial^2}{\partial \varphi^2}.$$

Show that in this coordinate system the potential between the spheres obeys the *ordinary differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}V}{\mathrm{d}r}\right) = 0.$$ **[3]**

(d) Find two linearly independent solutions to this equation. **[4]**

(e) Using this result or otherwise, show that the potential between the plates may be expressed as

$$V(r) = \frac{1}{r_{\text{in}} - r_{\text{out}}}\left\{\left(r_{\text{in}}V_{\text{in}} - r_{\text{out}}V_{\text{out}}\right) - \frac{r_{\text{in}}r_{\text{out}}}{r}\left(V_{\text{in}} - V_{\text{out}}\right)\right\}$$ **[5]**

where r_{in} and r_{out} are the radii of the two shells.

(f) Sketch this result and comment on its physical interpretation. **[3]**

3. (a) The orthogonality of the Legendre polynomials $P_l(x)$ is expressed in the integral

$$\int_{-1}^{1} P_l(x) P_m(x) \mathrm{d}x = \frac{2}{2l+1}\delta_{lm}$$

where δ_{lm} is the Kroneker delta symbol. A function $f(x)$ may be expressed, over the interval $-1 \leq x \leq 1$, as a linear sum of Legendre polynomials:

$$f(x) = \sum_n a_n P_n(x).$$

Show that the coefficients a_n may be found from the orthogonality integral as

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x) P_n(x) \mathrm{d}x .$$ **[6]**

(b) The first few Legendre ploynomials are

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2 - 1),\ P_3(x) = \frac{1}{2}(5x^3 - 3x),$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3),\ P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x).$$

With the aid of sketches, discuss why odd functions $f(x)$ are expressed in terms of only odd-order Legendre polynomials, and even functions are expressed in terms of only even-order polynomials. **[3]**

(c) A parabolic cap is specified by the equation

$$y(x) = 1 - x^2, \quad -1 \leq x \leq 1.$$

Sketch this function. **[2]**

(d) Using the orthogonality relations above, show that the function $y(x)$ can be expressed in terms of the Legendre polynomials as

$$y(x) = \frac{2}{3}P_0(x) - \frac{2}{3}P_2(x).$$ **[7]**

(e) Discuss the possibility of representing the function $1/x^2$ in terms of Legendre polynomials. **[2]**

4. The differential equation

$$\frac{d^2 y}{dx^2} + \frac{1}{2x}\frac{dy}{dx} + \frac{1}{4x} y = 0$$

has a solution which may be expressed as a simple power series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \; .$$

(a) Show that the recurrence relation for the coefficients a_n may be written as **[6]**

$$a_n = -\frac{a_{n-1}}{2n(2n-1)} .$$

(b) Identify this solution with $\cos\sqrt{x}$. **[6]**

(c) A second order differential equation must have two independent solutions. However the simple series method has given only one solution. The other solution is actually $\sin\sqrt{x}$. Why has this solution not been found? **[3]**

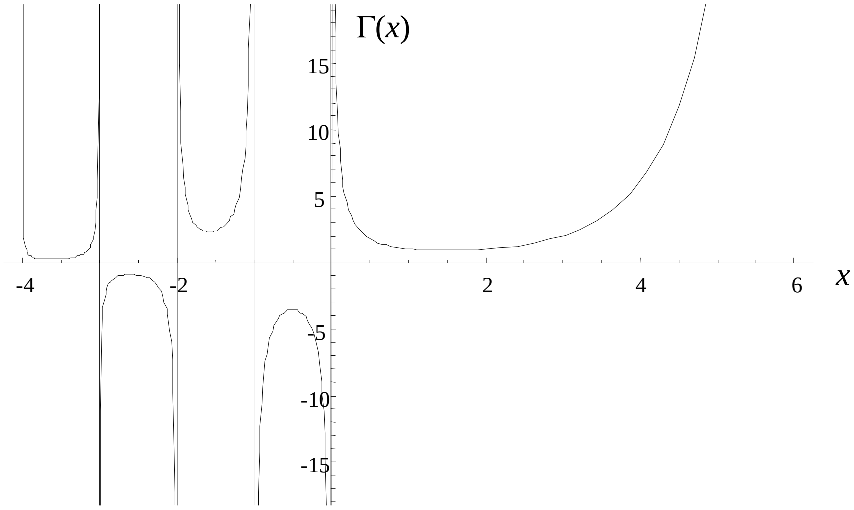
(d) Outline a way in which this solution could be found. **[5]**

PART MARKS

5. Euler's gamma function may be defined through the integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t$$

and its behaviour is shown in the graph.



(a) Demonstrate through integration by parts that the gamma function satisfies the recurrence relation

$$\Gamma(x) = (x-1)\Gamma(x-1).$$ **[4]**

(b) For integer n show that the gamma function is connected with the factorial function through

$$\Gamma(n) = (n-1)!$$ **[4]**

(c) By using the recurrence relation, discuss the behaviour of the gamma function for negative integer arguments. **[4]**

(d) Given that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, find values for $\Gamma\left(-\frac{1}{2}\right)$ and $\Gamma\left(\frac{3}{2}\right)$. **[4]**

(d) Outline briefly one way in which the gamma function for large arguments may be approximated. **[4]**

END